Tôhoku Math. Journ.
33 (1981), 561-565.

CERTAIN DECOMPOSITIONS OF BMO-MARTINGALES

YASUNOBU SHIOTA

(Received January 28, 1981)

1. **Introduction.** In [6] we considered a martingale version of the results in Coifman and Rochberg [1] under the condition that every martingale is continuous. This continuity condition made it possible to use the Varopoulos decomposition (see Varopoulos [7]) and to avoid some technical difficulties caused by jumps of sample paths. In this note, instead of the Varopoulos decomposition, we use the Herz-Lépingle representation of BMO-martingales (see Lemma 2 below), which, combined with the section theorem, enables us to remove the continuity condition.

The author would like to thank Dr. T. Sekiguchi for many helpful comments.

2. **Preliminaries.** Let $(\Omega, F, P; (F_t)_{t\in R^+})$ be a probability system which satisfies the usual conditions. We assume that the reader is familiar with the theory of general processes, especially the section theorem and the martingale theory. In the sequel T denotes the F_t-stopping time. Note that the constant C is not always the same in each occurrence.

DEFINITION 1. A uniformly integrable martingale $X = (X_t)$ is said to be a BMO-martingale if $\|X\|_{\mathrm{BMO}} = \sup_T \operatorname{ess.sup} E[|X_\infty - X_{T-}||F_T]$ is finite.

We denote by BMO the class of all BMO-martingales. BMO is a Banach space with the norm $\| \quad \|_{\mathrm{BMO}}$.

The following lemmas are well-known. For the proof, see Meyer [4] and [3] respectively.

LEMMA 1 (the inequality of John-Nirenberg's type). *Let X be a* BMO-*martingale. If $\alpha < 1/(8\|X\|_{\mathrm{BMO}})$, then $E[\exp \alpha|X_\infty - X_{T-}||F_T] < \infty$ a.s. for every T.*

LEMMA 2 (the Herz-Lépingle representation). *Let X be a* BMO-*martingale. Then there is a non-adapted process $B = (B_t)$ (not necessarily unique) such that* (a) $\int_0^\infty |dB_s| \leq C$ *for some constant C and* (b) $X_\infty = A_\infty$, *where A is the optional dual projection of B.*

DEFINITION 2. A uniformly integrable martingale $Y=(Y_t)$ is said to be a BLO-martingale if there is a positive constant C such that $Y_T - Y_\infty \leqq C$ and $|\Delta Y_T| \leqq C$ a.s. for every T.

BLO denotes the class of all BLO-martingales and BLO_+ the class of all positive BLO-martingales. If Y is in BLO with a constant C, then $\|Y\|_{\mathrm{BMO}} \leqq 3C$.

DEFINITION 3. A positive uniformly integrable martingale $W=(W_t)$ is said to be in the class A_1 (or satisfy the A_1-condition) if there is a positive constant C such that $W_T/W_\infty \leqq C$ a.s. for every T, and is said to be in the class S^+ (or satisfy the S^+-condition) if there is a positive constant C such that $W_T/W_{T-} \leqq C$ a.s. for every T.

The following lemma is due to Doléans-Dade and Meyer [2].

LEMMA 3 (the reverse Hölder inequality). *If W is in $A_1 \cap S^+$, then there are positive constants ε and C such that $E[W_\infty^{1+\varepsilon} | F_T] \leqq C W_T^{1+\varepsilon}$ a.s. for every T.*

3. Theorems.

THEOREM 1. *Any BMO-martingale X can be written in the form*

$$X = Y^1 - Y^2 ,$$

where Y^i $(i=1,2)$ is in BLO_+.

THEOREM 2. *X is in BMO if and only if there is a positive constant a_i $(i=1,2)$, a uniformly integrable martingale M_i $(\geqq 1)$ with $E[(M_i^*)^{\delta_i} | F_t] \in S^+$ for some $0<\delta_i<1$ $(i=1,2)$ and a bounded random variable K such that*

$$X_\infty = a_1 \log M_1^* - a_2 \log M_2^* + K$$

where $M_i^* = \sup_t |M_i(t)|$ $(i=1,2)$.

4. Proof of Theorems.

PROOF OF THEOREM 1. Take a process B in Lemma 2 corresponding to X and consider the Jordan decomposition of $B: B = B^1 - B^2$, where B^i $(i=1,2)$ is an increasing process. Denote by A^i the optional dual projection of B^i and put $Y_t^i = E[A_\infty^i | F_t]$. Clearly A^i is increasing, $X_\infty = Y_\infty^1 - Y_\infty^2$ and Y^i is a positive martingale. Now we will show that Y^1 and Y^2 are in BLO. From the definition of the optional dual projection, we can easily deduce $E[A_\infty^i - A_{T-}^i | F_T] = E[B_\infty^i - B_{T-}^i | F_T]$. Hence it follows that

$$Y_T^i - Y_\infty^i = E[A_\infty^i | F_T] - A_\infty^i = E[A_\infty^i - A_{T-}^i | F_T] - (A_\infty^i - A_{T-}^i)$$
$$\leqq E[A_\infty^i - A_{T-}^i | F_T] = E[B_\infty^i - B_{T-}^i | F_T]$$
$$= E[B_\infty^i | F_T] \leqq C \quad \text{a.s. for every } T .$$

Furthermore we have

$$Y_T^i = E[A_\infty^i | F_T] = A_{T-}^i + E[A_\infty^i - A_{T-}^i | F_T] = A_{T-}^i + E[B_\infty^i - B_{T-}^i | F_T]$$

and so $A_{T-}^i \leqq Y_T^i \leqq A_{T-}^i + C$. Thus by the section theorem, we have

(1) $$A_{\cdot-}^i \leqq Y_\cdot^i \leqq A_{\cdot-}^i + C .$$

Since $A_{\cdot-}^i$ is left continuous, we also have

(2) $$A_{\cdot-}^i \leqq Y_{\cdot-}^i \leqq A_{\cdot-}^i + C .$$

From (1) and (2), it follows that $|\Delta Y_T| \leqq C$ a.s. for every T. This completes the proof.

For the proof of Theorem 2, we need the following.

LEMMA 4. *Y is in* BLO$_+$ *if and only if* $W_t = E[\exp \alpha Y_\infty | F_t]$ $(\geqq 1)$ *is in* $A_1 \cap S^+$ *for some* $\alpha > 0$. *If we suppress the condition* $W \geqq 1$, *then Y is in* BLO.

PROOF. Let Y be in BLO$_+$. By Lemma 1, there are positive constants α and C such that $E[\exp \alpha |Y_\infty - Y_{T-}| | F_T] \leqq C$. Hence by the definition of BLO, $E[\exp \alpha Y_\infty | F_T] \leqq C \exp \alpha Y_\infty$, that is, W is in A_1. By Jensen's inequality, $\exp \alpha Y_T \leqq E[\exp \alpha Y_\infty | F_T]$. Then we apply the section theorem and take the left-hand limits: $\exp \alpha Y_{\cdot-} \leqq CE[\exp \alpha Y_\infty | F_\cdot]_-$. Hence $E[\exp \alpha Y_\infty | F_\cdot]/E[\exp \alpha Y_\infty | F_\cdot]_- \leqq E[\exp \alpha Y_\infty | F_\cdot]/\exp \alpha Y_{\cdot-}$. Since $E[\exp \alpha Y_\infty | F_T]/\exp \alpha Y_{T-} \leqq E[\exp \alpha |Y_\infty - Y_{T-}| | F_T] \leqq C$, we see that W is in S^+. It is clear that $W \geqq 1$.

Conversely assume that W is in $A_1 \cap S^+$ for some $\alpha > 0$. Since W is in A_1, by Sekiguchi [5, Lemma 1] and the section theorem, we have $E[\exp \alpha Y_\infty | F_\cdot] \leqq C \exp \alpha Y_\cdot$. Thus by taking the left-hand limits, we have

(3) $$E[\exp \alpha Y_\infty | F_\cdot]_- \leqq C \exp \alpha Y_{\cdot-} .$$

By the S^+-condition and the section theorem, we also have

(4) $$E[\exp \alpha Y_\infty | F_\cdot] \leqq CE[\exp \alpha Y_\infty | F_\cdot]_- .$$

From (3) and (4), it follows that $E[\exp \alpha Y_\infty | F_\cdot] \leqq C \exp \alpha Y_{\cdot-}$. Hence by Jensen's inequality,

$$\exp \alpha \Delta Y_T = \exp \alpha \Delta Y_T \exp \alpha E[Y_\infty - Y_T | F_T] \leqq E[\exp \alpha\{\Delta Y_T + (Y_\infty - Y_T)\} | F_T]$$
$$= [\exp \alpha (Y_\infty - Y_{T-}) | F_T] \leqq C .$$

Therefore $\Delta Y_T \leqq C$ a.s. for every T. On the other hand, by Jensen's inequality and the A_1-condition, we have

$$\exp \alpha Y_T \leqq E[\exp \alpha Y_\infty | F_T] \leqq C \exp \alpha Y_\infty .$$

Thus $\exp \alpha(Y_T - Y_\infty) \leqq C$, that is, $Y_T - Y_\infty \leqq C$ a.s. for every T. Furthermore by the right continuity of Y, $Y. - Y_\infty \leqq C$. Taking the left-hand limit and conditioning on $F.$, we obtain $-\Delta Y_T \leqq C$ a.s. for every T. If $W \geqq 1$, then Y is clearly positive. This completes the proof.

LEMMA 5. *If W ($\geqq 1$) is in $A_1 \cap S^+$, then there is a positive constant δ, $0 < \delta < 1$, a uniformly integrable martingale M ($\geqq 1$) with $E[(M^*)^\delta | F_t] \in S^+$ and a martingale H bounded above by 1 and bounded away from 0 such that $W_\infty = (M^*)^\delta H_\infty$. The converse, except that $W \geqq 1$, is also true.*

PROOF. By Lemma 3, there are two positive constants ε and C such that $E[W_\infty^{1+\varepsilon} | F_T] \leqq C W_T^{1+\varepsilon}$. Hence by the A_1-condition, $E[W_\infty^{1+\varepsilon} | F_T] \leqq C W_\infty^{1+\varepsilon}$. Put $M_t = E[W_\infty^{1+\varepsilon} | F_t]$ ($\geqq 1$). Then from the above inequality and the Hölder inequality, it follows that

$$(1/C)(M^*)^{1/(1+\varepsilon)} \leqq W_\infty \leqq (M^*)^{1/(1+\varepsilon)} .$$

Thus if we put $\delta = 1/(1+\varepsilon)$ and $H_t = E[(M^*)^{-\delta} W_\infty | F_t]$, then $W_\infty = (M^*)^\delta H_\infty$, $1/C \leqq H \leqq 1$ and $E[(M^*)^\delta | F_t] \in S^+$.

Conversely assume that $W_\infty = (M^*)^\delta H_\infty$, where M, δ and H satisfy the above conditions. It is easy to see that $W \in S^+$. To show that $W \in A_1$, we have only to treat the case when $W_\infty = (M^*)^\delta$. Now consider a uniformly integrable martingale N. Then we know that

$$E[(N^*)^\delta] \leqq C E[|N_\infty|]^\delta \tag{5}$$

(for the proof, see Shiota [6, Lemma 4]). We apply (5) to the new probability system $\Omega' = \{T < \infty\}$, $P' = P|_{\Omega'}/P(\Omega')$, $F'_t = F_{T+t}$ and the F'_t-martingale $M'_t = M_{T+t} - M_{T-}$ and then replace T by T_A ($A \in F_T$):

$$E[\sup_t |M_{T+t} - M_{T-}|^\delta | F_T] \leqq C E[|M_\infty - M_{T-}| | F_T]^\delta .$$

By this inequality, we have

$$E[\sup_t M_{T+t}^\delta | F_T] \leqq C (M_T^*)^\delta ,$$

where $M_T^* = \sup_{t \leqq T} |M_t|$, and so

$$E[(M^*)^\delta | F_T] \leqq E[(M_T^*)^\delta + \sup_t M_{T+t}^\delta | F_T] \leqq C (M_T^*)^\delta \leqq C (M^*)^\delta .$$

Therefore W is in A_1.

Combining Lemmas 4 and 5, we have the following.

LEMMA 6. *If Y is in* BLO_+, *then there is a positive constant a, a uniformly integrable martingale M ($\geqq 1$) with $E[(M^*)^\delta | F_t] \in S^+$ for some $0 < \delta < 1$ and a bounded random variable H such that $Y_\infty = a \log M^* + H$. Conversely if $Y_\infty = a \log M^* + H$, where a, M and H satisfy the above conditions, then Y is in* BLO.

Theorem 2 is clear from Theorem 1 and Lemma 6.

REFERENCES

[1] R. R. COIFMAN AND R. ROCHBERG, Another characterization of BMO, Proc. of Amer. Math. Soc. 79 (1980), 249-254.

[2] C. DOLÉANS-DADE AND P. A. MEYER, Inégalites de normes avec poids, Séminaire de Probabilités XIII, Lecture Notes in Math. 721, Springer-Verlag, Berlin-Heidelberg-New York, 1979, 313-331.

[3] P. A. MEYER, Notes sur les intégrales stochastiques III, Séminaire de Probabilités XI, Lecture Notes in Math. 581, Springer-Verlag, Berlin-Heidelberg-New York, 1977, 465-469.

[4] P. A. MEYER, Notes sur les intégrales stochastiques IV, Séminaire de Probabilités XI, Lecture Notes in Math. 581, Springer-Verlag, Berlin-Heidelberg-New York, 1977, 478-481.

[5] T. SEKIGUCHI, Weighted norm inequalities on the martingale theory, Math. Rep. Toyama Univ. 3 (1980), 37-100.

[6] Y. SHIOTA, On a decomposition of BMO-martingales, Tôhoku Math. J. this issue, 515-520.

[7] N. TH. VAROPOULOS, A probabilistic proof of the Garnett-Jones theorem on BMO, Pacific J. Math. 90 (1980), 201-221.

MATHEMATICAL INSTITUTE
TOHOKU UNIVERSITY
SENDAI, 980
JAPAN