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A PROBLEM ON ORDINARY FINE TOPOLOGY AND NORMAL FUNCTION

To the memory of Lu San Chen

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Abstract. Earlier in 1961, Doob proved that if $f(z)$ is a normal function in a disk, then every angular cluster value at a boundary point is also a fine cluster value at the point. He then asked whether or not the converse of this theorem is true. In this paper, we answer this question in the negative sense with respect to the ordinary fine topology of Brelot.

1. Introduction. Let $D(|z|<1)$ and $C(|z|=1)$ be the unit disk and circle respectively. Let $f(z)$ be a function defined in D. We say that the function f has an angular cluster value v at a boundary point $p \in C$, if there is an angle $A(p)$ lying in D with one vertex at p and a sequence $\{p_n\}$ of points $p_n \in A(p)$ such that

$$\lim_{n\to\infty} p_n = p \quad \text{and} \quad \lim_{n\to\infty} f(p_n) = v .$$

We shall now introduce the notion of fine topology in the sense of Brelot [2, p. 327]. Let E be a set and p a point. We say that E is thin at the point p, if either p is not a limit point of E or there exists a superharmonic function $s(z)$ such that

$$s(p) < \liminf_{z\to p} s(z) , \quad \text{where} \quad z \in E - p .$$

The first case is trivial and therefore only the second case will be considered in the sequel.

With the notion of thinness, we can now follow Doob [3 or 4] to define the fine cluster value. We say that the function f has a fine cluster value v at a point $p \in C$, if there is a set $T \subset D$ which is not thin at p and

$$\lim_{z\to p} f(z) = v , \quad \text{where} \quad z \in T .$$

In this case, the point p is called a fine limit point of the set T.

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It remains to introduce the notion of normal functions. We say that a function f meromorphic in D is normal in the sense of Lehto-Virtanen [6], if the family of transformations of f by the linear transformation taking D onto itself is a normal family in Montel's sense. In particular, if f is bounded holomorphic in D, then f is normal. In fact, our result is based on a construction of a Blaschke product which is bounded holomorphic in D. With those definitions, we can now state our result which answers Doob's question [3, p. 529].

THEOREM. *There is a function $f(z)$ normal in D such that f has a fine cluster value at a boundary point which is not an angular cluster value of f at the point.*

2. **Wiener criterion.** According to a theorem of Brelot [2, p. 327], we know that the notion of thinness is equivalent to that of irregularity. It follows from the Wiener criterion [8] (see also the book of Landkof [5, p. 298 and 308]) that a set E is thin at a point $p \in \bar{E}$ if and only if

$$\sum_{n=1}^{\infty} W(E_n) \log d_n^{-1} < \infty \ , \tag{1}$$

where $W(E_n)$ is the Wiener capacity of the set

$$E_n = E \cap \{z\colon d_{n+1} \leqq |z-p| < d_n\}\ , \quad 1 < a \leqq d_n/d_{n+1} \leqq b\ .$$

We notice that without loss of generality we may take the number $d_n = 2^{-n}$.

We shall now introduce the metric property of $W(E)$. To see this, we first observe that the relation between the Wiener capacity $W(E)$ and the logarithmic capacity $L(E)$ of a set E is the following (see for instance [5, p. 167]):

$$W(E) = 1/(\log 1/L(E))\ . \tag{2}$$

Moreover, if E is a line segment of length $|E|$, then the logarithmic capacity satisfies (see [5, p. 172])

$$L(E) = |E|/4\ . \tag{3}$$

3. **Proof of Theorem.** According to Section 1, we see that it is sufficient to construct a Blaschke product $B(z, a_n)$ whose zeros a_n tend tangentially to a boundary point, say, the point $z = 1$, such that the function B has the fine cluster value 0 at $z = 1$ and this value 0 is not an angular cluster value of B. In fact, the following Weierstrass product serves this property (see Seidel [7, p. 214])

$$B(z, a_n) = \frac{e^{(z+1)/(z-1)} - e^{-1}}{1 - e^{-1}\cdot e^{(z+1)/(z-1)}} = ze^{i\alpha} \prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{a_n - z}{1-\bar{a}_n z}\frac{|a_n|}{a_n} \tag{4}$$

with the zeros $a_n = \pi ni/(\pi ni + 1)$, $n = \pm 1, \pm 2, \cdots$.

It is easy to see that all zeros a_n are located on the horocycle

$$(x - 1/2)^2 + y^2 = 1/4 .$$

Thus the value 0 is not an angular cluster value of B at the point $z = 1$. Since

$$|B(e^{i\theta}, a_n)| = 1 , \qquad \text{for all} \quad \theta \neq 0 ,$$

this value 0 is neither an angular cluster value of B at any point on C. Moreover, the product B omits the value $-e^{-1}$ in D and therefore this value $-e^{-1}$ is the angular limit of B at $z = 1$.

To finish the proof, we need only show that the value 0 is a fine cluster value of B at $z = 1$. To do so, it suffices to consider the upper zeros a_n, $n = 1, 2, \cdots$. For convenience, we write

$$(5) \qquad \begin{cases} a_n = r_n e^{i\theta_n} , \quad \text{where} \quad r_n = \pi n(\pi^2 n^2 + 1)^{-1/2} , \\ d_n = |1 - a_n| = (\pi^2 n^2 + 1)^{-1/2} , \\ b_n = (r_n - d_n^q)e^{i\theta_n} , \quad \text{where} \quad q \geqq 3 , \quad \text{and} \\ t_n = \overline{a_n b_n} , \quad \text{with the length} \quad |t_n| = d_n^q . \end{cases}$$

The Theorem will be proved if we can show the following two properties:

(6) The set $T = \bigcup_1^\infty t_n$ is not thin at $z = 1$,

(7) The product $B(z, a_n) \to 0$, as $z \to 1$ and $z \in T$.

We begin by proving the property (6). Since t_n is a line segment, it follows from (3) and (5) that the logarithmic capacity of t_n satisfies

$$L(t_n) = 4^{-1}(\pi^2 n^2 + 1)^{-q/2} ,$$

and therefore by (2), we have

$$(8) \qquad \begin{aligned} W(t_n) &= (\log 4 + (q/2) \log (\pi^2 n^2 + 1))^{-1} , \\ &\geqq ((q + 1) \log n)^{-1} , \qquad \text{for} \quad n \geqq 4 . \end{aligned}$$

In order to apply the Wiener criterion (1), we need only choose a subsequence $\{n_j\}$ of $\{n\}$, say, $n_j = 2^j$, $j = 1, 2, \cdots$. We then have

$$d_{n_j} = (\pi^2 2^{2j} + 1)^{-1/2} ,$$

so that

$$1 < a \leqq d_{n_j}/d_{n_{j+1}} \leqq 2 , \qquad \text{for} \quad j = 1, 2, \cdots .$$

This together with (1) and (8) yields that for some constant $k > 0$,

$$(9) \qquad \sum_{j=1}^{\infty} W(t_{n_j}) \log d_{n_j}^{-1} \geqq k \sum_{j=1}^{\infty} \frac{1}{q+1} = \infty .$$

This shows that the subset $\bigcup_j t_{n_j}$ of T is not thin at $z = 1$, and therefore T itself can not be thin at $z = 1$. This proves (6).

It remains to prove (7). According to a theorem of Bagemihl and Seidel [1, Theorem 3], it suffices to show that the hyperbolic metric satisfies

$$(10) \qquad \rho(a_n, b_n) \to 0 , \quad \text{as} \quad n \to \infty ,$$

where $\rho(a_n, b_n) = (1/2) \log (|1 - \bar{a}_n b_n| + |a_n - b_n|)(|1 - \bar{a}_n b_n| - |a_n - b_n|)^{-1}$.

In view of (5), we see that

$$|1 - \bar{a}_n b_n| = 1 - r_n(r_n - d_n^q) \quad \text{and} \quad |a_n - b_n| = d_n^q .$$

It follows that

$$(11) \qquad \rho(a_n, b_n) = \frac{1}{2} \log \left(\frac{1 + r_n}{1 + r_n - d_n^q} \cdot \frac{1 - r_n + d_n^q}{1 - r_n} \right) .$$

Clearly, by (5), we have

$$r_n \to 1 \quad \text{and} \quad d_n^q \to 0 , \quad \text{as} \quad n \to \infty ,$$

so that

$$(12) \qquad \lim_{n\to\infty} \frac{1 + r_n}{1 + r_n - d_n^q} = 1 .$$

Moreover, by the restriction $q \geqq 3$ in (5), we find that

$$(13) \qquad \lim_{n\to\infty} \frac{1 - r_n + d_n^q}{1 - r_n} = 1 + \lim_{n\to\infty} \frac{(\pi^2 n^2 + 1)^{1/2} + \pi n}{(\pi^2 n^2 + 1)^{(q-1)/2}} = 1 .$$

By substituting (12) and (13) into (11), we obtain the desired result (10). This establishes (7) and therefore the value 0 is a fine cluster value of B at $z = 1$. The proof is complete.

REMARK. In view of the rapid divergence of (9), we see that the set T is in a sense "very thick". In fact, the divergence of (9) can be achieved even if the constant q is replaced by the order $O(j)$.

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