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A MULTILINEARIZATION OF LITTLEWOOD-PALEY'S g-FUNCTION AND CARLESON MEASURES

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Introduction. Recently Coifman and Meyer [4] introduced a class of multilinear operators as a multilinearization of Littlewood-Paley's g-function. They studied L^2 estimates of such operators, using the notion of Carleson measures. In this note we shall develop their study further, by weakening their assumptions and obtain H^1, BMO and L^p estimates. Our techniques are essentially modifications of theirs, but we need many devices to make their ideas deeper at many points. Our main results are Theorems 1 and 2, and stated in Section 2. Notations and definitions are given in Section 1. There we introduce some classes of weight functions to state our theorems. In Section 3 we shall give preliminary lemmas and prove the main theorems in Section 4. In these sections Carleson measures play very important roles, but there we only quote lemmas giving relations between BMO and Carleson measures. We shall treat them systematically in Section 6, because we wish to treat many things related to BMO and Carleson measures. There, for example, we shall improve some recent results of Strichartz [11]. Some applications and examples of the main theorems are given in Section 5.

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1. Notations and Definitions. $\mathscr{D} = \mathscr{D}(\boldsymbol{R}^n) = C_0^\infty(\boldsymbol{R}^n)$ denotes the set of all infinitely differentiable functions with compact support on $\boldsymbol{R}^n$: the n-dimensional Euclidean space. $\mathscr{S} = \mathscr{S}(\boldsymbol{R}^n)$ is the set of all infinitely differentiable functions whose derivatives decrease rapidly. Recall that a locally integrable function f is said to be of bounded mean oscillation on $\boldsymbol{R}^n$ if the mean oscillation of f on any cube Q with sides parallel to the axes

$$\mathrm{MO}\,(f, Q) = \frac{1}{|Q|}\int_Q |f(x) - f_Q|dx$$

is uniformly bounded, where f_Q denotes the mean of f on Q

$$f_Q = \frac{1}{|Q|}\int_Q f(x)dx$$

and $|Q|$ is the Lebesgue measure of Q. The equivalence classes of functions of bounded mean oscillation modulo functions constant a.e. form a Banach space with norm $\|f\|_* = \sup_Q \mathrm{MO}(f, Q)$. We denote by BMO this Banach space or the space of all functions of bounded mean oscillation. $H^1 = H^1(\boldsymbol{R}^n)$ is the Hardy space H^1 of Stein and Weiss with norm $\|\cdot\|_{H^1}$, and H^1_{00} is the space of all $f \in \mathscr{S}$ such that the Fourier transform $\hat{f}$ has compact support bounded away from the origin (see [9, p. 231]).

A positive measure μ on $\boldsymbol{R}^{n+1}_+ = \boldsymbol{R}^n \times (0, \infty)$ is said to be a Carleson measure if there exists $C > 0$ such that

$$\int_{|x-y|<\varepsilon} \int_0^\varepsilon d\mu(x, t) \leqq C\varepsilon^n$$

for any $\varepsilon > 0$ and $y \in \boldsymbol{R}^n$. We denote by $\gamma(\mu)$ the infimum of such C.

Next we introduce some classes of weight functions related to the Dini condition. Let W be the set of all nondecreasing functions w on $(0, 1]$ with $0 \leqq w(t) \leqq 1$ on $(0, 1]$. We set for $a > 0$

$$W_0 = \left\{w \in W;\ \int_0^1 w(t)\frac{dt}{t} \leqq 1\right\},$$

$$W_1 = \left\{w \in W;\ \int_0^1 w(t)\log(e + 1/t)\frac{dt}{t} \leqq 1\right\},$$

$$W_2^a = \left\{w \in W;\ \int_0^1 w^{4/5}(t)\log^{1+a}(e + 1/t)\frac{dt}{t} \leqq 1\right\},$$

$$W_3^a = \left\{w \in W;\ \int_0^1 w^2(t)\log^{2+a}(e + 1/t)\frac{dt}{t} \leqq 1\right\},$$

$$W_j = \bigcup_{a>0} W_j^a \quad (j = 2, 3),$$

$$W_4 = \{w \in W;\ \text{there exists } C > 0 \text{ s.t. } bw(t) \leqq Cw(bt),\ 0 < b, t < 1\}.$$

Then we have essentially $W_0 \supset W_3 \supset W_1 \supset W_2$. In fact, we have $W_1 \supset W_2$. And if $w \in W_1$, we get by easy calculation $w(t)\log^2 1/t \leqq 2\int_0^1 w(t)\log 1/t(dt/t)$. Hence we get $\int_0^1 w^2(t)\log^3(e + 1/t)(dt/t) < 2(1 + \log 2)^2$. If $w \in W_3$, using the boundedness of w, we get easily $w(t)/t \in L^1(0, 1)$ by Hölder's inequality.

For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \boldsymbol{Z}^n$, ∂_ξ^α is the differential operator $(\partial^{\alpha_1}/\partial\xi_1^{\alpha_1})(\partial^{\alpha_2}/\partial\xi_2^{\alpha_2}) \cdots (\partial^{\alpha_n}/\partial\xi_n^{\alpha_n})$ and $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$. $\|f\|_p$ always denotes the usual L^p norm of f. Integration of f over the whole space $\boldsymbol{R}^n$ is often written as $\int f(x)dx$. The Fourier transform of f will be denoted by $\hat{f}$;

$$\hat{f}(\xi) = \int f(x)e^{-ix\cdot\xi}dx,$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$.

The letter C will always denote a constant and does not necessarily denote the same one. The letters j, k, m and r will always denote integers.

2. A class of multilinear operators: Statement of main results. For $k+1$ functions $\phi_0, \phi_1, \cdots, \phi_k$ on $\boldsymbol{R}^n$ and a function $u(t) \in L^\infty(\boldsymbol{R}_+)$ we define a $(k+1)$-linear operator T by

$$T(a_0, a_1, \cdots, a_k) = T(a_0, a_1, \cdots, a_k; \phi_0, \phi_1, \cdots, \phi_k) = \int_0^\infty \prod_{m=0}^{k} (\phi_{m,t} * a_m) u(t) \frac{dt}{t}$$

where $\phi_{m,t}(x) = \phi_m(x/t)t^{-n}$ and $(\phi * a)(x) = \int_{\boldsymbol{R}^n} \phi(x-y)a(y)dy$. This is a multilinearization of so-called Littlewood-Paley's g-function (Coifman-Meyer [4, p. 144]). What we will show in this paper is the following two theorems which generalize Coifman-Meyer's theorem 33 in [4, p. 144].

THEOREM 1. *Let* $|\phi_i(x)| \leqq (1+|x|)^{-n} w_i(1/1+|x|)$ *for some* $w_i \in W_2 \cap W_4$ $(i = 0, 1, \cdots, k)$. *Suppose there exist positive constants* $K_{\alpha,i}$, $C_{\alpha,i}$, A *and* B *such that*

$$|\partial_\xi^\alpha \hat{\phi}_i(\xi)| \leqq K_{\alpha,i}|\xi|^{-|\alpha|-1}, \quad |\xi| > B, \quad |\alpha| \leqq n+1, \quad i = 0, 1, \cdots, k,$$
$$|\partial_\xi^\alpha \hat{\phi}_i(\xi)| \leqq C_{\alpha,i}|\xi|^{-|\alpha|}, \quad |\xi| < A, \quad |\alpha| \leqq n+1, \quad i = 1, 2, \cdots, k,$$
$$|\partial_\xi^\alpha \hat{\phi}_0(\xi)| \leqq C_{\alpha,0}|\xi|^{-|\alpha|+1}, \quad |\alpha| < A, \quad |\alpha| \leqq n+1.$$

Then there exist $C_1, C_2, C_\infty > 0$ *such that*

(i) $\|T(a_0, \cdots, a_k)\|_2 \leqq C_2\|u\|_\infty \|a_0\|_* \prod_{j=2}^k \|a_j\|_\infty \|a_1\|_2$ *for* $a_1 \in L^2(\boldsymbol{R}^n)$, $a_0 \in \mathrm{BMO}$, $a_j \in L^\infty$ $(j = 2, \cdots, k)$,

(ii) $\|T(a_0, a_1, \cdots, a_k)\|_* \leqq C_\infty \|u\|_\infty \prod_{j=1}^k \|a_j\|_\infty \|a_0\|_*$ *for* $a_1 \in L^\infty \cap L^2$, $a_0 \in \mathrm{BMO}$, $a_j \in L^\infty$ $(j = 2, \cdots, k)$,

(iii) $\|T(a_0, a_1, \cdots, a_k)\|_1 \leqq C_1\|u\|_\infty \|a_0\|_* \prod_{j=2}^k \|a_j\|_\infty \|a_1\|_{H^1}$ *for* $a_1 \in H^1_{00}$, $a_0 \in \mathrm{BMO}$, $a_j \in L^\infty$ $(j = 2, \cdots, k)$.

THEOREM 2. ϕ_i, u *be the same as in Theorem* 1. *Then there exist* $C_1, C_2, C_\infty > 0$ *such that*

(i) $\|T(a_0, a_1, \cdots, a_k)\|_2 \leqq C_2\|u\|_\infty \|a_0\|_2 \prod_{j=1}^k \|a_j\|_\infty$ *for* $a_0 \in L^2$, $a_j \in L^\infty$ $(j = 1, 2, \cdots, k)$,

(ii) $\|T(a_0, a_1, \cdots, a_k)\|_* \leqq C_\infty \|u\|_\infty \|a_0\|_* \prod_{j=1}^k \|a_j\|_\infty$ *for* $a_0 \in \mathrm{BMO} \cap L^2$, $a_j \in L^\infty$ $(j = 1, 2, \cdots, k)$,

(iii) $\|T(a_0, a_1, \cdots, a_k)\|_1 \leqq C_1\|u\|_\infty \|a_0\|_{H^1} \prod_{j=1}^k \|a_j\|_\infty$ *for* $a_0 \in H^1_{00}$, $a_j \in L^\infty$ $(j = 1, 2, \cdots, k)$.

We have as a consequence of Theorems 1 and 2 the following, using the multilinear interpolation theory of Calderón [2].

THEOREM 3. *Let ϕ_i, u be the same as in Theorem 1 and $1 \leqq p_i \leqq \infty$ $(i = 0, 1, \cdots, k)$ and $0 < 1/p = 1/p_0 + 1/p_1 + \cdots + 1/p_k \leqq 1$. Then there exists $C = C(p_i, k, n, C_{\alpha,i}, K_{\alpha,i}) > 0$ such that*

$$\| T(a_0, a_1, \cdots, a_k) \|_p \leqq C \| u \|_\infty \| a_0 \|_{\theta(p_0)} \prod_{j=1}^{k} \| a_j \|_{\eta(p_j)}$$

$$\text{for} \quad a_0 \in L^{\theta(p_0)}, \quad a_j \in L^{\eta(p_j)} \ (j = 1, \cdots, k),$$

where $L^{\theta(q)} = L^{\eta(q)} = H^1_{00}$ $(q = 1)$, $= L^q$ $(1 < q < \infty)$, $L^{\theta(\infty)} = \mathrm{BMO}$ and $L^{\eta(\infty)} = L^\infty$, while $\| a \|_{\theta(q)}$ and $\| a \|_{\eta(q)}$ are the corresponding norms of a.

REMARK 1. In the above three theorems, if $\int \phi_j(x)dx = 0$, then the assumption $a_j \in L^\infty$ (or $L^\infty \cap L^2$) can be replaced by $a_j \in \mathrm{BMO}$ (or $\mathrm{BMO} \cap L^2$, respectively).

REMARK 2. In order to prove (i) and (ii) of Theorems 1 and 2 we do not need $w_i \in W_4$ $(i = 0, 1, \cdots, k)$.

REMARK 3. In Theorems 1, 2 and 3, $w_i \in W_2$ can be replaced by $w_i \in W_3$ $(i = 1, 2, \cdots, k)$. And $w_0 \in W_2$ can be replaced by $w_0 \in W_3$ if we treat only the case $a_0 \in L^\infty$ instead of the case $a_0 \in \mathrm{BMO}$.

REMARK 4. In Theorem 1 (ii), $a_1 \in L^\infty \cap L^2$ cannot be replaced by $a_1 \in L^\infty$. Also in Theorem 2 (ii), $a_0 \in \mathrm{BMO} \cap L^2$ cannot be replaced by $a_0 \in \mathrm{BMO}$. One can easily give counterexamples.

3. Fundamental lemmas.

We begin with some elementary lemmas.

LEMMA 3.1. *Let $m \in \{0, 1, 2, 3\}$. Let $w_1, w_2 \in W_m$ and*

$$| f_j(x) | \leqq (1 + |x|)^{-n} w_j(1/1 + |x|) \qquad x \in \boldsymbol{R}^n, \quad j = 1, 2.$$

Then for any $\delta_0 > 0$ there exist $w \in W_m$ and $C > 0$ depending only on n and δ_0 such that for all $0 < \delta < \delta_0$

$$| f_{1,\delta} * f_2(x) | \leqq C(1 + |x|)^{-n} w(1/1 + |x|) \qquad x \in \boldsymbol{R}^n.$$

REMARK. The following proof shows that if $w_1, w_2 \in W_4$, we can choose $w \in W_4$, and if $a_1 > a_2 > 0$ and $w_j \in W_k^{a_j}$ $(j = 1, 2)$, $w \in W_k^{a_2}$ $(k = 2, 3)$.

PROOF. We have

$$
\begin{aligned}
(1) \quad & | f_{1,\delta} * f_2(x) | \\
& \leqq \int_{|x-y| > |x|/2} | f_{1,\delta}(x - y) f_2(y) | dy + \int_{|x-y| \leqq |x|/2} | f_{1,\delta}(x - y) f_2(y) | dy \\
& \leqq 2^n (2\delta + |x|)^{-n} w_1(2\delta / 2\delta + |x|) \int (1 + |y|)^{-n} w_2(1/1 + |y|) dy \\
& \quad + 2^n (2 + |x|)^{-n} w_2(2/2 + |x|) \int (1 + |y|)^{-n} w_1(1/1 + |y|) dy .
\end{aligned}
$$

Hence

$$|f_{1,\delta}*f_2(x)| \leqq C(2\delta+|x|)^{-n}w_1(2\delta/2\delta+|x|) + C(1+|x|)^{-n}w_2(2/2+|x|)\ . \tag{2}$$

Now, if $|x| \leqq 1$ we have clearly

$$|f_{1,\delta}*f_2(x)| \leqq w_2(1)\int|f_{1,\delta}(x)|dx = w_2(1)\int|f_1(x)|dx \leqq 1\ . \tag{3}$$

And if $|x| > 1$ we have $(2\delta+|x|)^n \geqq 2^{-n}(1+|x|)^n$, and so, by using the monotonicity of w_1

$$|f_{1,\delta}*f_2(x)| \leqq C_1(1+|x|)^{-n}w_1(2\delta_0/2\delta_0+|x|) + C_2(1+|x|)^{-n}w_2(2/2+|x|)\ . \tag{4}$$

Combining (3) and (4) we obtain the desired result.

LEMMA 3.2. *Let $g \in \mathcal{S}$ be such that $\hat{g}(\xi) = 1$ ($|\xi| < 1/4$), $= 0$ ($|\xi| \geqq 1/2$) and $m \in \{0, 1, 2, 3\}$. Let $w \in W_m$. Then, if $|f(x)| \leqq (1+|x|)^{-n}w(1/1+|x|)$ and* $\operatorname{supp}\hat{f} \subset \{1/2 < |\xi| < 2\}$, *there exist $w_1 \in W_m$ and $A, B > 0$ such that for any $\delta > 0$*

$$|f_\delta * g(x)| \leqq A(\delta+|x|)^{-n}w_1(\delta/\delta+|x|) \tag{5}$$

$$|(f_\delta - f_\delta * g)(x)| \leqq B(\delta+|x|)^{-n}w_1(\delta/\delta+|x|)\ . \tag{6}$$

PROOF. (5) follows from Lemma 3.1. (6) is rather easy.

LEMMA 3.3. *Let g and w be the same as in Lemma 3.2 and $\delta_0 > 0$. Then for any f with $|f(x)| \leqq (1+|x|)^{-n}w(1/1+|x|)$ and* $\operatorname{supp}\hat{f} \subset \{|\xi| < 2\}$, *there exist $w_1 \in W_m$ and $A, B > 0$ such that for any $\delta \geqq \delta_0$ the inequalities* (5) *and* (6) *hold.*

PROOF. Similar to the above proof.

LEMMA 3.4. *Let $w \in W_0$. Then there exists $C > 0$ such that for any ϕ with $|\phi(x)| \leqq (1+|x|)^{-n}w(1/1+|x|)$ and for any Carleson measure μ on $\boldsymbol{R}_+^{n+1}$ it holds*

$$\int_{\boldsymbol{R}_+^{n+1}} |f*\phi_t|^2 d\mu(x,t) \leqq C\gamma(\mu)\|f\|_2 \quad \textit{for} \quad f \in L^2(\boldsymbol{R}^n)\ .$$

PROOF. Since $(1+|x|)^{-n}w(1/1+|x|) \in L^1(\boldsymbol{R}^n)$ and is radial, the non-tangential maximal function of $f*\phi_t(x)$ is bounded by a constant multiple of Hardy-Littlewood's maximal function of $f(x)$ (Stein and Weiss [10, p. 59]). Hence we have the desired inequality by the Further result 4.4 in Stein [9, p. 236].

LEMMA 3.5. *Let $w_1, w_2 \in W_0$ and suppose $\phi(x) \in L^1(\boldsymbol{R}^n)$ satisfies*

$$|\phi(x)-\phi(y)| \leqq w_1(|x-y|) \quad \textit{for} \quad x, y \in \boldsymbol{R}^n ,$$
$$|\phi(x-y)-\phi(x)| \leqq (1+|x|)^{-n} w_2(|y|/1+|x|) \quad \textit{for} \quad 2|y| \leqq |x| .$$

Then for any $\alpha > 0$ *there exists* $C > 0$ *such that*

$$\int_{\boldsymbol{R}^n} \sup_{|x-y|<\alpha t} |f * \phi_t(y)| dx \leqq C\|f\|_{H^1} \quad \textit{for} \quad f \in H^1 .$$

This lemma was originally obtained by Fefferman-Stein [7, p. 152]. Our modification is due to M. Kaneko.

LEMMA 3.6. *Let* $w_0 \in W_0 \cap W_4$ *and* $|\phi(x)| \leqq (1+|x|)^{-n} w(1/1+|x|)$ *and* $\operatorname{supp} \hat{\phi} \subset \{|\xi| < 1\}$. *Then there exists* $C > 0$ *such that for any Carleson measure* μ *on* $\boldsymbol{R}_+^{n+1}$ *it holds*

$$\int_{\boldsymbol{R}_+^{n+1}} |f * \phi_t| d\mu(x, t) \leqq C\gamma(\mu)\|f\|_{H^1} , \quad f \in H^1 .$$

PROOF. Let $h \in \mathscr{S}$ be such that $\hat{h}(\xi) = 1$ on $\{|\xi| < 1\}$. Since $\operatorname{supp} \hat{\phi} \subset \{|\xi| < 1\}$, we have then $\phi = \phi * h$. Hence $\partial\phi/\partial x_j = \phi * (\partial h/\partial x_j)$. Thus by Lemma 3.1 we have for $j = 1, \cdots, n$

$$\left|\frac{\partial \phi}{\partial x_j}(x)\right| \leqq C_j(1+|x|)^{-n} w_2(1/1+|x|) \quad \text{for some} \quad w_2 \in W_0 \cap W_4 .$$

We get $|\phi(x+y)-\phi(x)| \leqq C_0|y|$ for some $C_0 > 0$. There also exists $C_1 > 0$, by virtue of the mean value theorem and the monotonicity of w_2, such that

$$|\phi(x+y)-\phi(x)| \leqq C_1|y|(1+|x|)^{-n} w_2(2/2+|x|) , \quad 2|y| < |x| .$$

Hence if $|y| < 1$ and $2|y| < |x|$, we get, because of $w_2 \in W_4$,

$$|\phi(x+y)-\phi(x)| \leqq C_2(1+|x|)^{-n} w_2(|y|/1+|x|) ,$$

for another $C_2 > 0$. If $|y| \geqq 1$ and $2|y| < |x|$, using $w \in W_4$ and its monotonicity, we get

$$\begin{aligned}|\phi(x+y)-\phi(x)| &\leqq (1+|x|)^{-n}(w(2/2+|x|)+w(1/1+|x|)) \\ &\leqq C_3(1+|x|)^{-n} w(|y|/1+|x|) .\end{aligned}$$

Thus we can find $w_1 \in W_0 \cap W_4$ and $C > 0$ such that

$$|\phi(x+y)-\phi(x)| \leqq C(1+|x|)^{-n} w_1(|y|/1+|x|) , \quad 2|y| < |x| .$$

Therefore ϕ satisfies the assumption in Lemma 3.5, and hence by that lemma and the Further result 4.4 in Stein [9, p. 236] we obtain the desired result.

In the sequel, we shall use propositions, which will be proved in Section 6.

Lemma 3.7. *Let $w_1 \in W_0$ and $w_2 \in W_2$, and $|\psi_j(x)| \leqq (1+|x|)^{-n} w_j(1/1+|x|)$ $(j=1, 2)$ with* $\operatorname{supp} \hat{\psi}_1 \subset \{1/2 < |\xi| < 2\}$ *and* $\hat{\psi}_2(0) = 0$. *Then there exists $C > 0$ such that*

$$\left| \int_{R^n} \int_0^\infty (h * \psi_{1,t}(x))(a * \psi_{2,t}(x)) v(x, t) t^{-1} dt dx \right| \leqq C \|a\|_* \|v\|_\infty \|h\|_{H^1}$$

for all $h \in H^1_{00}(R^n)$, $v \in L^\infty(R^{n+1}_+)$ and $a \in \mathrm{BMO}(R^n)$.

Proof. Let $h \in H^1_{00}(R^n)$ and I be the above integral. Then since $\operatorname{supp} \hat{\psi}_1 \subset \{1/2 < |\xi| < 2\}$, there exists $g \in \mathscr{S}$ such that

$$\hat{\psi}_1(\xi) = -|\xi| e^{-|\xi|} \hat{g}(\xi) \hat{\psi}_1(\xi) .$$

Let $u = g * \psi_1$. Then by Lemma 3.1 there exist $C_1 > 0$ and $w_3 \in W_0$ such that

$$|u(x)| \leqq C_1 (1 + |x|)^{-n} w_3(1/1 + |x|) .$$

Let $P_t(x) = c_n t(t^2 + |x|^2)^{-(n+1)/2}$ be the Poisson kernel for R^{n+1}_+. Then, since $\hat{P}_t(\xi) = e^{-t|\xi|}$, we have $\psi_{1,t} = (t\,\partial P_t/\partial t) * u_t$. Hence we have

$$\begin{aligned} |I| &\leqq \int_{R^n} \int_0^\infty \left(\left| h * t \frac{\partial P_t}{\partial t} \right| * |u_t| \right) |a * \psi_{2,t}| t^{-1} dt dx \times \|v\|_\infty \\ &\leqq \|v\|_\infty \int_{R^n} \int_0^\infty \left| t \frac{\partial h(x, t)}{\partial t} \right| (|\phi_t| * |a * \psi_{2,t}|) t^{-1} dt dx \end{aligned}$$

where $\phi(x) = u(-x)$ and $h(x, t)$ is the Poisson integral of h. Now let $F = (h, h_1, \cdots, h_n)$ be the generalized Cauchy-Riemann system for h (Stein-Weiss [10, p. 231]). Then as is known (Stein [9, p. 217])

$$|\nabla F|^2 \leqq (n+1) |F| \Delta |F| .$$

Hence we get by Cauchy-Schwarz's inequality

$$|I| \leqq \|v\|_\infty \left(\int_{R^n} \int_0^\infty t \Delta |F| dt dx \right)^{1/2} \left(\int_{R^n} \int_0^\infty |F| (|\phi_t| * |a * \psi_{2,t}|^2) t^{-1} dt dx \right)^{1/2} . \tag{7}$$

Since $h \in H^1_{00}$ we have

$$\int_{R^n} \int_0^\infty t \Delta |F| dt dx = \int_{R^n} |F(x, 0)| dx \leqq C \|h\|_{H^1} .$$

Next, as is easily seen,

$$(|\phi_t| * |a * \psi_{2,t}|)^2 \leqq \|\phi\|_1 (|\phi_t| * |a * \psi_{2,t}|^2) .$$

Since $a \in \mathrm{BMO}$, $|\psi_2(x)| \leqq (1+|x|)^{-n} w_2(1/1+|x|)$ and $\int \psi_2(x) dx = 0$, we see by Proposition 6.1 that $d\mu = |a * \psi_{2,t}|^2 t^{-1} dt dx$ is a Carleson measure with $\gamma(\mu) \leqq C \|a\|_*$. By the lemma below, which we shall soon prove, we have that $|\phi_t| * d\mu$ is also a Carleson measure with $\gamma(|\phi_t| * d\mu) \leqq C_1\gamma(\mu)$. Therefore by the Further result 4.4 in Stein [9, p. 236] the second term in (7) is smaller than $C_2\|a\|_*\|h\|_{H^1}$. Thus we obtain the desired result.

REMARK. If $a \in L^\infty(\boldsymbol{R}^n)$, then $w_2 \in W_2$ can be replaced by $w_2 \in W_3$. One can use Proposition 6.2 in this case instead of Proposition 6.1.

LEMMA 3.8. *Let w be a nondecreasing function on $(0, 1)$ with $\int_0^1 w(t)t^{-1}dt \leqq 1$. Then there exists $C > 0$ such that if $\phi(x)$ is a nonnegative valued function with $|\phi(x)| \leqq (1 + |x|)^{-n}w(1/1 + |x|)$, $\phi_t * d\mu$ is a Carleson measure for any Carleson measure μ on $\boldsymbol{R}_+^{n+1}$ and*

$$\gamma(\phi_t * d\mu) \leqq C\gamma(\mu) ,$$

where the convolution is taken with respect to $x \in \boldsymbol{R}_n$.

PROOF (Suggested by A. Uchiyama). Let $s > 0$ and $x_0 \in \boldsymbol{R}^n$. Then

$$\begin{aligned} I_s(x_0) &= \int_{|x-x_0|<s}\int_0^s\int_{\boldsymbol{R}^n} \phi_t(x-y)d\mu(y, t)dx \\ &\leqq \int_{\boldsymbol{R}^n}\int_0^s\int_{|x-x_0|<s} (1 + |x-y|/t)^{-n}w(t/t + |x-y|)t^{-n}dxd\mu(y, t) . \end{aligned}$$

Dividing $\boldsymbol{R}^n$ into the meshes with side length s and center sk, $k \in \boldsymbol{Z}^n$, and using the monotonicity of w we have

$$\begin{aligned} I_s(x_0) &\leqq \sum_{|k|<4}\left(\int_{|y-x_0-sk|<s}\int_0^s d\mu(y, t)\right)\int_{\boldsymbol{R}^n}(1 + |x|)^{-n}w(1/1 + |x|)dx \\ &\quad + \sum_{|k|\geqq 4}\left(\int_{|y-x_0-sk|<s}\int_0^s d\mu(y, t)\right)(3/|k|)^n w(3/3 + |k|) \\ &\leqq C_1\gamma(\mu)s^n\int_{\boldsymbol{R}^n}(1 + |x|)^{-n}w(1/1 + |x|)dx . \end{aligned}$$

Since the last integral is equal to a constant multiple of $\int_0^1 w(t)t^{-1}dt$, we have established the lemma.

4. Proof of Theorems 1 and 2. First we shall give propositions fundamental to prove our main theorems. For $\delta = (\delta_0, \delta_1, \cdots, \delta_m)$ we denote

$$T_\delta(a_0, a_1, \cdots, a_m) = \int_0^\infty \prod_{j=0}^m (\phi_{j,\delta_j t} * a_j)u(t)\frac{dt}{t} ,$$

where $u(t) \in L^\infty(\boldsymbol{R}_+)$ and a_j, ϕ_j are appropriate functions.

PROPOSITION 4.1. *Let $w_j \in W_2$ $(j = 0, 1, \cdots, m)$, $w_0 \in W_4$ and $|\phi_j(x)| \leqq (1 + |x|)^{-n}w_j(1/1 + |x|)$ with* $\operatorname{supp}\hat{\phi}_j \subset \{1/2 < |\xi| < 2\}$ $(j = 0, 1, \cdots, r)$,

$\operatorname{supp} \hat{\phi}_j \subset \{|\xi| < 2\}$ $(j = r+1, \cdots, m)$. *Let* $\eta_{r+1}, \cdots, \eta_m > 0$. *Then for any* $\delta_j > 0$ $(j = 0, \cdots, r)$ *and any* $\delta_j \geqq \eta_j$ $(j = r+1, \cdots, m)$ *we have*

$$\| T_\delta(f, a_1, \cdots, a_m) \|_{\alpha(p)} \leqq C_p \| u \|_\infty \prod_{j=1}^{r} \| a_j \|_* \prod_{j=r+1}^{m} \| a_j \|_\infty \| f \|_{\beta(p)}$$

$$\text{for} \quad a_1, \cdots, a_r \in \mathrm{BMO}\,, \quad a_{r+1}, \cdots, a_m \in L^\infty \quad \text{and} \quad f \in L^{\beta(p)}\,,$$

where

(i) *if* $p = 2$, $\alpha(p) = \beta(p) = 2$,

(ii) *if* $p = \infty$, $\alpha(p) = \beta(p) = *$ *and* $L^{\beta(p)}$ *stands for* $\mathrm{BMO} \cap L^2$,

(iii) *if* $p = 1$, $\alpha(p) = 1$, $L^{\beta(p)}$ *stands for* H^1_{00} *and* $\beta(p)$ *for* H^1.

Here C_2, C_∞, C_1 *do not depend on* $\delta = (1, \delta_1, \cdots, \delta_m)$.

PROOF. Let $v \in \mathscr{S}$ be such that $\hat{v}(\xi) = 1$ on $|\xi| < 1/8m$, $= 0$ $(|\xi| > 1/4m)$ and $\theta_j = \phi_{j,\delta_j} * v$, $\psi_j = \phi_{j,\delta_j} - \theta_j$ $(j = 1, 2, \cdots, m)$. Then by Lemmas 3.2 and 3.3 we get for some $w'_j \in W_2$

$$|\psi_j(x)|, |\theta_j(x)| \leqq C(\delta_j + |x|)^{-n} w'_j(\delta_j/\delta_j + |x|)\,. \tag{8}$$

We have furthermore

$$\int \psi_j(x)dx = 0 \quad (j = 1, 2, \cdots, m) \quad \text{and} \tag{9}$$

$$\int \theta_j(x)dx = 0 \quad (j = 1, 2, \cdots, r)\,.$$

Hence we get by Lemma 6.4

$$\begin{aligned} &\| a * \theta_{j,t} \|_\infty, \| a * \psi_{j,t} \|_\infty \leqq C_1 \| a \|_* \quad t > 0\,, \quad a \in \mathrm{BMO} \quad (j = 1, \cdots, r)\,, \\ &\| a * \theta_{j,t} \|_\infty, \| a * \psi_{j,t} \|_\infty \leqq C_1 \| a \|_\infty \quad t > 0\,, \ a \in L^\infty \ (j = r+1, \cdots, m)\,, \end{aligned} \tag{10}$$

and by Proposition 6.1

$$\begin{aligned} &\gamma(|a * \psi_{j,t}(x)|^2 t^{-1} dt dx) \leqq C_1 \| a \|_*^2 \max(1, \delta_j^n) \\ &\gamma(|a * \psi_{j,t/\delta_j}(x)|^2 t^{-1} dt dx) \leqq C_1 \| a \|_*^2\,, \quad a \in \mathrm{BMO} \quad (j = 1, \cdots, m)\,. \end{aligned} \tag{11}$$

We note, if $\delta_j \geqq 16m$, then $\psi_j \equiv 0$, since $\operatorname{supp} \hat{\psi}_j \subset \operatorname{supp} \hat{\phi}_{j,\delta_j} \cap \operatorname{supp}(1 - \hat{v}) \subset \{1/2\delta_j < |\xi| < 2/\delta_j\} \cap \{|\xi| > 1/8m\} = \varnothing$.

Now $T_\delta(f, a_1, \cdots, a_m)$ can be written in the following form

$$\begin{aligned} T_\delta &= \int_0^\infty (f * \phi_{0,t}) \prod_{j=1}^{m} (a_j * \theta_{j,t}) \frac{u(t)}{t} dt \\ &\quad + \sum \int_0^\infty (f * \phi_{0,t}) \prod_{k=1}^{r_1} (a_{j_k} * \theta_{j_k,t}) \prod_{r_1+1}^{m} (a_{j_k} * \psi_{j_k,t}) u(t) \frac{dt}{t}\,. \end{aligned} \tag{12}$$

Proof of (i). The first term in (12) can be written in the form

$$g(x) = \int_0^\infty \psi_t * \left[(f * \phi_{0,t}) \prod_{j=1}^m (a_j * \theta_{j,t}) \right] u(t) \frac{dt}{t} \tag{13}$$

for some radial function $\psi \in \mathscr{S}$ with $\operatorname{supp} \hat{\psi} \subset \{1/4 < |\xi| < 4\}$. Then for any $h \in L^2(\boldsymbol{R}^n)$ we have via Fubini's theorem

$$I_1 = \int_{\boldsymbol{R}^n} h(x) g(x) dx = \int_0^\infty \int_{\boldsymbol{R}^n} (h * \psi_t)(f * \phi_{0,t}) \prod_{j=1}^m (a_j * \theta_{j,t}) u(t) \frac{dxdt}{t} . \tag{14}$$

Hence by Cauchy-Schwarz's inequality and by (10) and Lemma 6.6

$$\begin{aligned} |I_1| &\leqq C\Big(\int_{\boldsymbol{R}^n} \int_0^\infty |h * \psi_t|^2 t^{-1} dtdx\Big)^{1/2} \Big(\int_{\boldsymbol{R}^n} \int_0^\infty |f * \phi_{0,t}|^2 t^{-1} dtdx\Big)^{1/2} \\ &\quad \times \|u\|_\infty \prod_{j=1}^r \|a_j\|_* \prod_{r+1}^m \|a_j\|_\infty \\ &\leqq C_1 \|h\|_2 \|f\|_2 \prod_{j=1}^r \|a_j\|_* \prod_{r+1}^m \|a_j\|_\infty \|u\|_\infty . \end{aligned}$$

To estimate the second term, put

$$\begin{aligned} S(f) = \int_0^\infty (f * \phi_{0,t}) \prod_{k=1}^{r_1} (a_k * \theta_{k,t}) \prod_{r_1+1}^{r} (a_k * \psi_{k,t}) \prod_{r+1}^{r+r_2} (a_k * \theta_{k,t}) \\ \times \prod_{r+r_2+1}^{m} (a_k * \psi_{k,t}) u(t) t^{-1} dt , \end{aligned} \tag{15}$$

where $r_1 + r_2 < m - 1$. In the following for the sake of simplicity we denote the integrand in $S(f)$ by $A(t, x)t^{-1}$. Without loss of generality we may assume $16m \geqq \delta_{r_1+1} \geqq \cdots \geqq \delta_r$ and $16m \geqq \delta_{r+r_2+1} \geqq \cdots \geqq \delta_m$. Let $\eta = \min(\delta_r, \delta_m)$. Assume first $\eta = \delta_r$. Then the spectrum of the integrand is contained in $\{|\xi| < 32m(m+1)/\eta t\}$. Let $\phi \in \mathscr{S}$ be radial and $\hat{\phi}(\xi) = 1$ $(|\xi| < 32m(m+1))$, $= 0$ $(|\xi| > 64m^2)$. Then we get

$$S(f) = \int_0^\infty \phi_{\eta t} * A(t, x) \frac{dt}{t} .$$

For any $h \in L^2$ we put $I_2 = \int_{\boldsymbol{R}^n} hS(f) dx$. Then we have via Fubini's theorem

$$I_2 = \int_0^\infty \int_{\boldsymbol{R}^n} (h * \phi_{\eta t}) A(t, x) t^{-1} dxdt .$$

By (11) and Cauchy-Schwarz's inequality we have

$$\begin{aligned} |I_2| \leqq C \|u\|_\infty \prod_{k=1}^{r-1} \|a_k\|_* \prod_{r+1}^m \|a_k\|_\infty \Big(\int_0^\infty \int_{\boldsymbol{R}^n} |f * \phi_{0,t}|^2 t^{-1} dxdt\Big)^{1/2} \\ \times \Big(\int_{\boldsymbol{R}^n} \int_0^\infty |h * \phi_{\eta t}|^2 |a_r * \psi_{r,t}|^2 t^{-1} dtdx\Big)^{1/2} . \end{aligned} \tag{16}$$

The last term equals $\Big(\int_{\boldsymbol{R}^n} \int_0^\infty |h * \phi_t|^2 |a_r * \psi_{r,t/\eta}|^2 t^{-1} dtdx\Big)^{1/2}$. By (11) and

Lemma 3.4 this is bounded by $C\|h\|_2\|a_r\|_*$. By Lemma 6.6 the first integral on the right hand side of (16) is bounded by $C\|f\|_2$. Next the case $\delta_m = \min(\delta_r, \delta_m)$ can be treated in a quite similar way. Hence we have

$$\left|\int_{R^n} T_\delta(f, a_1, \cdots, a_m)(x)h(x)dx\right| \leqq C\|u\|_\infty \prod_{k=1}^{r} \|a_k\|_* \prod_{r+1}^{m} \|a_k\|_\infty \|f\|_2\|h\|_2$$

for all $h \in L^2$, which implies the desired result.

Proof of (ii). Let $f \in \mathrm{BMO} \cap L^2$, $a_k \in \mathrm{BMO}$ $(k = 1, \cdots, r)$ and $a_k \in L^\infty$ $(k = r + 1, \cdots, m)$. Let $h \in H^1_{00}$. We use the notation in (i). We have by (10), (11) and Lemma 3.7

$$|I_1| \leqq C\|u\|_\infty \prod_{j=1}^{r} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|h\|_{H^1}\|f\|_* .$$

In the other terms there are three typical ones

Type 1. $r_1+r_2 \geqq 2$ and $\delta_r < (1/2m)\min(\delta_{r_1+1}, \cdots, \delta_{r-1}, \delta_{r+r_2+1}, \cdots, \delta_m) = \delta'$ or $\delta_m < (1/2m)\min(\delta_{r_1+1}, \cdots, \delta_r, \delta_{r+r_2+1}, \cdots, \delta_{m-1}) = \delta''$.

Type 2. $r_1 + r_2 \geqq 2$ and $\delta_r \geqq \delta'$ or $\delta_m \geqq \delta''$.

Type 3. $r_1 + r_2 = 1$.

We treat first the case of type 1 and $\delta_r < \delta'$. Let radial $\psi \in \mathscr{S}$ be such that $\hat{\psi}(\xi) = 1$ on $\{1/4 < |\xi| < 4\}$, $=0$ on $\{|\xi| < 1/8\}$ and $\{|\xi| > 8\}$. Then we get

$$S(f) = \int_0^\infty \psi_{\delta_r t} * A(t, x)t^{-1}dt .$$

Thus we have for any $h \in H^1_{00}$

$$\begin{aligned} I_2 &= \int_{R^n} h(x)S(f)(x)dx = \int_0^\infty \int_{R^n} (\psi_{\delta_r t} * h)A(t, x)t^{-1}dxdt \\ &= \int_0^\infty \int_{R^n} (\psi_t * h)A(t/\delta_r, x)t^{-1}dxdt . \end{aligned}$$

Using (10) and (11) we obtain by Lemma 3.7

$$|I_2| \leqq C\|u\|_\infty \prod_{j=1}^{r-1} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|a_r\|_* \|h\|_{H^1}\|f\|_* .$$

The case $\delta_m < \delta''$ can be treated in a quite similar way.

For type 2, let $\phi \in \mathscr{S}$ be radial and $\hat{\phi} = 1$ $(|\xi| < 2m)$, $= 0$ $(|\xi| > 4m)$. We treat first the case $2m\delta' = \delta_{r-1}$ and $\delta_r \geqq \delta'$. The other cases can be treated in the same way. We have then

$$S(f) = \int_0^\infty \phi_{\delta_r t} * A(t, x)t^{-1}dt .$$

For any $h \in H^1_{00}$ we get

$$I_3 = \int_{R^n} h(x)S(f)(x)dx = \int_0^\infty \int_{R^n} (h*\phi_{\delta_r t})A(t, x)t^{-1}dxdt$$
$$= \int_0^\infty \int_{R^n} (h*\phi_t)A(t/\delta_r, x)t^{-1}dxdt .$$

Hence from (10) we have

$$|I_3| \leqq C\|u\|_\infty \|f\|_* \prod_{j=1}^{r-2} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty$$
$$\times \int_{R^n} \int_0^\infty |(h*\phi_t)(a_{r-1}*\psi_{r-1,t/\delta_r})(a_r*\psi_{r,t/\delta_r})|t^{-1}dtdx .$$

Since $\delta_{r-1} \geqq \delta_r \geqq \delta_{r-1}/2m$, we have from (8)

$$|\psi_{r-1,1/\delta_r}(x)| \leqq C(\delta_r/\delta_{r-1})^n(1+\delta_r|x|/\delta_{r-1})^{-n}w'_{r-1}((1+\delta_r|x|/\delta_{r-1})^{-1})$$
$$\leqq C_1(1+|x|)^{-n}w''_{r-1}(1/1+|x|)$$

for some $w''_{r-1} \in W_2$. Since clearly $\int_{R^n} \psi_{r-1,1/\delta_r}(x)dx = 0$, we have by Proposition 6.1

$$\gamma(|a_{r-1}*\psi_{r-1,t/\delta_r}|^2 t^{-1}dxdt) \leqq C_2\|a_{r-1}\|_*^2 .$$

Hence by (11) and Cauchy-Schwarz's inequality

$$\gamma(|a_{r-1}*\psi_{r-1,t/\delta_r}||a_r*\psi_{r,t/\delta_r}|t^{-1}dxdt) \leqq C\|a_{r-1}\|_*\|a_r\|_* .$$

Therefore by Lemma 3.6 we have

$$|I_3| \leqq C\|u\|_\infty \prod_{j=1}^{r} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|f\|_* \|h\|_{H^1} .$$

Next we treat the case of type 3. Let $\delta' = \delta_r$ or δ_m. If $\delta' < 1/2m$, one can proceed as in type 1. If $2m \geqq \delta' \geqq 1/2m$, one can proceed as in type 2. If $\delta' > 2m$, as for the first term in (12). Thus we have the desired result.

We remark here that the assumption $f \in \mathrm{BMO} \cap L^2$ is used only to apply Fubini's theorem in I_1, I_2 and I_3. So the conclusion is valid if $f, a_1, \cdots, a_r \in \mathrm{BMO}$, $a_{r+1}, \cdots, a_m \in L^\infty$ and at least one of them is in L^2.

Proof of (iii). We shall use notations in the proof of (ii). However we take here $h \in L^\infty \cap L^2$, $f \in H^1_{00}$ in place of $h \in H^1_{00}$, $f \in \mathrm{BMO} \cap L^2$ (respectively). For I_1 we have by (10) and Lemma 3.7

$$|I_1| \leqq C\|u\|_\infty \prod_{j=1}^{r} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|f\|_{H^1}\|h\|_* .$$

For I_2 we have by (10)

$$|I_2| \leqq C\|u\|_\infty \prod_{j=1}^{r-1} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \int_{R^n}\int_0^\infty |(h*\psi_{\delta_r t})(f*\phi_{0,t})(a_r*\psi_{r,t})|t^{-1}dtdx .$$

As before we get

$$\gamma(|h*\psi_{\delta_r t}|^2 t^{-1}dtdx) \leqq C\max(1, \delta_r^n)\|h\|_* ,$$
$$\gamma(|a_r*\psi_{r,t}|^2 t^{-1}dtdx) \leqq C\max(1, \delta_r^n)\|a_r\|_* .$$

Furthermore we may assume $\delta_r \leqq 16m$. Since $\|h\|_* \leqq C\|h\|_\infty$, we have thus

$$\gamma(|(h*\psi_{\delta_r t})(a_r*\psi_{r,t})|t^{-1}dtdx) \leqq C\|h\|_\infty\|a_r\|_* .$$

Hence by Lemma 3.6 we have

$$|I_2| \leqq C\|u\|_\infty \prod_{j=1}^{r-1} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|a_r\|_* \|h\|_\infty \|f\|_{H^1} .$$

For I_3 we have, using $\delta_j \leqq 16m$,

$$\|h*\phi_{\delta_r t}\|_\infty \leqq C\|h\|_\infty , \qquad \gamma(|a_j*\psi_{j,t}|^2 t^{-1}dtdx) \leqq C\|a_j\|_* \quad (j = r-1, r) .$$

Hence also for I_3 we have the same inequality as for I_2.

For type 3 we proceed as for I_1, I_2 and I_3 if $\delta_r > 2m$, $2m \geqq \delta_r \geqq 1/2m$ and $\delta_r < 1/2m$, respectively. The other cases can be treated in the same way. Hence we have the desired inequality.

REMARKS. (1) If $\hat{\phi}_j(0) = 0$ for some $j = r+1, \cdots, m$, then in the conclusion $\|a_j\|_\infty$ can be replaced by $\|a_j\|_*$. (2) If for some $j = 1, \cdots, m$ one has $a_j \in L^2$, then in (ii) BMO $\cap L^2$ can be replaced by BMO.

Next we give one more proposition similar to the former.

PROPOSITION 4.2. *Let $w_j \in W_2$ $(j = 0, 1, \cdots, m)$ and $w_0 \in W_4$. Assume $|\phi_j(x)| \leqq (1+|x|)^{-n} w_j(1/1+|x|)$, $\operatorname{supp}\hat{\phi}_j \subset \{|\xi| < 2\}$ $(j = 0, r+1, \cdots, m)$ and $\operatorname{supp}\hat{\phi}_j \subset \{1/2 < |\xi| < 2\}$ $(j = 1, \cdots, r)$. Let $\eta_j > 0$ $(j = 1, 2, \cdots, r)$. Then for any $0 < \delta_j \leqq \eta_j$ $(j = 1, 2, \cdots, r)$ we have*

$$\|T_\delta(f, a_1, \cdots, a_m)\|_{\alpha(p)} \leqq C_p\|u\|_\infty \prod_{j=1}^{r} \|a_j\|_* \prod_{r+1}^{m} \|a_j\|_\infty \|f\|_{\beta(p)} \quad \textit{for} \quad f \in L^{\beta(p)}$$

where

(i) *if $p = 2$, $\alpha(p) = \beta(p) = 2$,*

(ii) *if $p = \infty$, $\alpha(p) = *$, $\beta(p) = \infty$ and $L^{\beta(p)}$ stands for $L^\infty \cap L^2$,*

(iii) *if $p = 1$, $\alpha(p) = 1$, $L^{\beta(p)}$ stands for H^1_{00} and $\beta(p)$ for H^1.*

Here $\delta = (1, \delta_1, \cdots, \delta_r, 1, \cdots, 1)$ and C_1, C_2, C_∞ do not depend on δ, a_j and f.

PROOF. We may assume $\eta_1 = \eta_2 = \cdots = \eta_r = \eta$ and $\delta_1 \leqq \delta_2 \leqq \cdots \leqq \delta_r$ without loss of generality. Let $\zeta(x)$ be a radial function in $\mathscr{S}$ such that $\hat{\zeta}(\xi) = 1$ on $\{|\xi| < 1/8\eta m\}$, $= 0$ on $\{|\xi| > 1/4\eta m\}$ and $\theta_j = \zeta*\phi_{j,\delta_j}$, $\psi_j = \phi_{j,\delta_j} - \theta_j$ $(j = 0, 2, 3, \cdots, m)$. Then by Lemmas 3.2 and 3.3 we have

$$(17) \qquad |\psi_j(x)|, |\theta_j(x)| \leqq C\delta_j^{-n}(1+|x|/\delta_j)^{-n} w_j'((1+|x|/\delta_j)^{-1})$$

for some $w'_j \in W_2$ and

$$\int \psi_j(x)dx = 0 \quad (j = 0, 2, \cdots, m)\,, \qquad \int \theta_j(x)dx = 0 \quad (j = 2, \cdots, r)\,.$$

Hence we get by Lemma 6.4 for any $t > 0$

$$\begin{aligned} \|a_j * \theta_{j,t}\|_\infty, \|a_j * \psi_{j,t}\|_\infty &\leqq C\|a_j\|_* \quad (j = 2, \cdots, r) \\ &\leqq C\|a_j\|_\infty \quad (j = 0, r+1, \cdots, m)\,, \end{aligned} \tag{18}$$

and by Proposition 6.1

$$\begin{aligned} &\gamma(|a_1 * \phi_{1,\delta_1 t}|^2 t^{-1}dtdx) \leqq C \max(1, \delta_1^n)\|a_1\|_*^2\,, \\ &\gamma(|a_j * \psi_{j,t}|^2 t^{-1}dtdx) \leqq C \max(1, \delta_j^n)\|a_j\|_*^2 \quad (j = 2, \cdots, r)\,, \\ &\gamma(|a_j * \psi_{j,t}|^2 t^{-1}dtdx) \leqq C\|a_j\|_\infty \quad (j = 0, r+1, \cdots, m)\,. \end{aligned} \tag{19}$$

Using θ_j and ψ_j, $T_\delta(f, a_1, \cdots, a_m)$ can be written in the form

$$\begin{aligned} &\int_0^\infty (f * \psi_{0,t}) \prod_{j=1}^m (a_j * \phi_{j,\delta_j t}) t^{-1}u(t)dt \\ &\quad + \int_0^\infty (f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{j=2}^m (a_j * \theta_{j,t})u(t)t^{-1}dt \\ &\quad + \sum \int_0^\infty (f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{k=2}^i (a_{j_k} * \theta_{j_k,t}) \prod_{i+1}^m (a_{j_k} * \psi_{j_k,t})u(t)t^{-1}dt\,. \end{aligned} \tag{20}$$

The first term can be treated by Proposition 4.1. The second term can be written in the following form

$$g(x) = \int_0^\infty \psi_{\delta_1 t} * \left[(f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{j=2}^m (a_j * \theta_{j,t})\right] u(t)t^{-1}dt$$

where $\psi \in \mathscr{S}$ is radial and $\operatorname{supp} \hat{\psi} \subset \{1/4 < |\xi| < 4\}$. For any $h \in L^2$ we have

$$I_2 = \int h(x)g(x)dx = \int_0^\infty \int_{R^n} (h * \psi_{\delta_1 t})(f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{j=2}^m (a_j * \theta_{j,t})u(t)t^{-1}dt\,.$$

Hence by (18) and Cauchy-Schwarz's inequality

$$\begin{aligned} |I_2| &\leqq C\|u\|_\infty \prod_{j=2}^r \|a_j\|_* \prod_{r+1}^m \|a_j\|_\infty \left(\int_{R^n} \int_0^\infty |h * \psi_{\delta_1 t}|^2 t^{-1}dtdx\right)^{1/2} \\ &\quad \times \left(\int_{R^n} \int_0^\infty |f * \theta_{0,t}|^2 |a_1 * \phi_{1,t}|^2 t^{-1}dtdx\right)^{1/2}\,. \end{aligned}$$

By Lemmas 6.6 and 3.4 the last two terms are smaller than $C\|h\|_2$ and $C\|f\|_2 \max(1, \eta^n)\|a_1\|_*$ (respectively), since $\delta_1 \leqq \eta$. This implies

$$\|g\|_2 \leqq C\|u\|_\infty \prod_{j=1}^r \|a_j\|_* \prod_{r+1}^m \|a_j\|_\infty \|f\|_2\,.$$

Now, if $\phi \in \mathscr{S}$ is radial and $\hat{\phi}(\xi) = 1$ on $\{|\xi| < 2(m+1)\max(1, \eta)\}$, $=0$ on

$\{|\xi| > 4m \max(1, \eta)\}$, then any term in the last terms in (20) has the form

$$S(x) = \int_0^\infty \phi_{\delta_1 t} * \left[(f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{k=2}^{i} (a_{j_k} * \theta_{j_k,t}) \prod_{i+1}^{m} (a_{j_k} * \psi_{j_k,t}) \right] u(t) t^{-1} dt .$$

Hence for any $h \in L^2$ we have

$$\begin{aligned} I_3 &= \int h(x) S(x) dx \\ &= \int_0^\infty \int_{R^n} (h * \phi_{\delta_1 t})(f * \theta_{0,t})(a_1 * \phi_{1,\delta_1 t}) \prod_{k=2}^{i} (a_{j_k} * \theta_{j_k,t}) \prod_{i+1}^{m} (a_{j_k} * \psi_{j_k,t}) u(t) t^{-1} dt dx . \end{aligned}$$

Thus we get by (18) and Cauchy-Schwarz's inequality

$$\begin{aligned} |I_3| \leqq C \| u \|_\infty \prod_{\substack{k=2 \\ k \neq j_m}}^{r} \| a_k \|_* \prod_{\substack{k=r+1 \\ k \neq j_m}}^{m} \| a_k \|_\infty \left(\int_{R^n} \int_0^\infty | h * \phi_{\delta_1 t} |^2 | a_1 * \phi_{1,\delta_1 t} |^2 t^{-1} dt dx \right)^{1/2} \\ \times \left(\int_{R^n} \int_0^\infty | f * \theta_{0,t} |^2 | a_{j_m} * \psi_{j_m,t} |^2 t^{-1} dt dx \right)^{1/2} . \end{aligned}$$

Hence by Lemma 3.4 and (19) the last two terms in the above are smaller than

$C \max(1, \eta^n) \| a_1 \|_* \| h \|_2$ and $C \max(1, \eta^n) \| a_{j_m} \|_* \| f \|_2$ (respectively),

which implies

$$\| S \|_2 \leqq C \| u \|_\infty \prod_{j=1}^{r} \| a_j \|_* \prod_{r+1}^{m} \| a_j \|_\infty \| f \|_2 .$$

We thus obtain the inequality (i).

Proof of (ii). We may assume $\delta = (1/\delta_1, 1, \delta_2/\delta_1, \cdots, \delta_r/\delta_1, 1/\delta_1, \cdots, 1/\delta_1)$ without loss of generality. Since $1/\delta_1 \geqq 1/\eta_1$, by Proposition 4.1 (ii) and its remark we get the inequality (ii).

Proof of (iii). One can prove (iii) in a way similar to the proof of (i), by using Lemmas 3.6 and 3.7 instead of Lemmas 3.4 and 6.6. Thus the proof of our proposition is complete.

To complete the proof of Theorems 1 and 2 we need one more step. We introduce the following decomposition of functions and operators as in Coifman and Meyer [4, p. 152]. Let $p(\xi)$ be a radial function in $\mathscr{S}$ such that $\operatorname{supp} p \subset \{2/3 < |\xi| < 2\}$, $\sum_{-\infty}^{\infty} p(2^j \xi) = 1$ $(\xi \neq 0)$ and $\sum_0^\infty p(2^j \xi) = 1$ $(0 < |\xi| < 1)$. For a function $\phi \in L^1(\boldsymbol{R}^n)$ we introduce ψ_j^1, ψ_j^2, ψ_j^3, R_j and ϕ_0 as follows

$$\hat{\psi}_j^1(\xi) = 2^j \hat{\phi}(2^{-j}\xi) p(\xi) , \quad \hat{\psi}_j^2(\xi) = \hat{\phi}(2^{-j}\xi) p(\xi) , \quad \hat{\psi}_j^3(\xi) = 2^j \hat{\phi}(2^j \xi) p(\xi) ,$$

$$\hat{R}_j(\xi) = \left(\sum_{k=j+1}^{\infty} p(2^{k-j}\xi) \right) \hat{\phi}(2^{-j}\xi) , \quad \hat{\phi}_0(\xi) = \left(1 - \sum_{k=0}^{\infty} p(2^j \xi) \right) \hat{\phi}(\xi) .$$

Then we have by Lemma 3.1 and easy calculation:

LEMMA 4.3. *Let* $|\phi(x)| \leqq (1+|x|)^{-n}w(1/1+|x|)$ *for a* $w \in W_2^\varepsilon$ *(or* W_3^ε*)* $(\varepsilon > 0)$. *Then we have the following.*

(i) *If* $|\partial^\alpha\hat{\phi}(\xi)||\xi|^{|\alpha|} \leqq C_\alpha$, $|\xi| < A$, $|\alpha| \leqq n+1$, *then there exist* $w_1 \in W_2^\varepsilon$ *(or* W_3^ε*) and* $C = C_A > 0$ *such that*

$$|\psi_j^2(x)|,\ |R_j(x)| \leqq C(1+|x|)^{-n}w_1(1/1+|x|) \quad x \in \boldsymbol{R}^n\,, \quad j \in \boldsymbol{N}\,.$$

(ii) *If* $|\partial^\alpha\hat{\phi}(\xi)||\xi|^{|\alpha|} \leqq C_\alpha|\xi|$, $|\xi| < A$, $|\alpha| \leqq n+1$, *then there exist* $w_2 \in W_2^\varepsilon$ *(or* W_3^ε*) and* $C = C_A > 0$ *such that*

$$|\psi_j^1(x)| \leqq C(1+|x|)^{-n}w_2(1/1+|x|) \quad x \in \boldsymbol{R}^n\,, \quad j \in \boldsymbol{N}\,.$$

(iii) *If* $|\partial^\alpha\hat{\phi}(\xi)||\xi|^{|\alpha|+1} \leqq C_\alpha$, $|\xi| > B$, $|\alpha| \leqq n+1$, *then there exist* $w_3 \in W_2^\varepsilon$ *(or* W_3^ε*) and* $C = C_B > 0$ *such that*

$$|\psi_j^3(x)| \leqq C(1+|x|)^{-n}w_3(1/1+|x|) \quad x \in \boldsymbol{R}^n\,, \quad j \in \boldsymbol{N}\,.$$

(iv) *There exist* $w_4 \in W_2^\varepsilon$ *(or* W_3^ε*) and* $C > 0$ *such that*

$$|\phi_0(x)| \leqq C(1+|x|)^{-n}w_4(1/1+|x|) \quad x \in \boldsymbol{R}^n\,.$$

Now for our operator $T(a_0, a_1, \cdots, a_k; \phi_0, \phi_1, \cdots, \phi_k)$ we get the following formulas. If $\operatorname{supp}\hat{\phi}_j \subset \{|\xi| < 1\}$ $(j = 0, 1, \cdots, k)$, then

$$\begin{aligned}(21)\qquad & T(a_0, a_1, \cdots, a_k; \phi_0, \phi_1, \cdots, \phi_k)\\ &= \sum_{N=0}^{\infty} 2^{-N} \sum_{r=0}^{k} \sum_{j_1<j_2<\cdots<j_k} \sum_{i_1=0}^{N} \cdots \sum_{i_r=0}^{N} T(a_0, a_{j_1}, \cdots, a_{j_k};\\ &\qquad \psi_{0,N}^1, \psi_{j_1,i_1}^2, \cdots, \psi_{j_r,i_r}^2, R_{j_{r+1},N}, \cdots, R_{j_k,N})\,.\end{aligned}$$

Here $\psi_{j,r}^1$ is ψ_r^1 for $\phi = \phi_j$, and so on. In general we have

$$\begin{aligned}(22)\qquad & T(a_0, \cdots, a_k; \phi_0, \cdots, \phi_k)\\ &= \sum_{0\leqq j_0<\cdots<j_r\leqq k} \sum_{i_0=0}^{\infty} \cdots \sum_{i_r=0}^{\infty} 2^{-(i_0+\cdots+i_r)} T(a_{j_0}, \cdots, a_{j_k};\\ &\qquad \psi_{j_0,i_0}^3, \cdots, \psi_{j_r,i_r}^3, \phi_{j_{r+1},0}, \cdots, \phi_{j_k,0})\,.\end{aligned}$$

Now we can prove our main theorems.

PROOF OF THEOREM 1. (i) The case when $\operatorname{supp}\hat{\phi}_j \subset \{|\xi| < 1\}$ $(j = 0, 1, \cdots, k)$. We use the formula (21). Then we get

$$\begin{aligned}&\|T(a_0, a_{j_1}, \cdots, a_{j_k}; \psi_{0,N}^1, \psi_{j_1,i_1}^2, \cdots, \psi_{j_r,i_r}^2, R_{j_{r+1},N}, \cdots, R_{j_k,N})\|_{\alpha(p)}\\ &\qquad \leqq C\|u\|_\infty \|a_0\|_* \prod_{j=2}^{k} \|a_j\|_\infty \|a_1\|_{\beta(p)}\end{aligned}$$

for $p = 2, \infty, 1$ in the notation of Proposition 4.2, by Lemma 4.3 and Proposition 4.1 if $1 \in \{j_1, \cdots j_r\}$, and by Lemma 4.3 and Proposition 4.2 if $1 \in \{j_{r+1}, \cdots, j_k\}$. Hence we get

$$\begin{aligned}\|T(a_0, \cdots, a_k; \phi_0, \cdots, \phi_k)\|_{\alpha(p)} \\ &\leqq C\|u\|_\infty \|a_0\|_* \prod_{j=2}^{k} \|a_j\|_\infty \|a_1\|_{\beta(p)} \sum_{N=0}^{\infty}\left(\sum_{r=0}^{k}\binom{k}{r}N^r\right)2^{-N} \\ &\leqq C'\|u\|_\infty \|a_0\|_* \prod_{j=2}^{k} \|a_j\|_\infty \|a_1\|_{\beta(p)} .\end{aligned}$$

(ii) The general case. We use the formula (22). Then we get

$$\begin{aligned}&\|T(a_{j_0}, \cdots, a_{j_r}, a_{j_{r+1}}, \cdots, a_{j_k}; \psi^3_{j_0,i_0}, \cdots, \psi^3_{j_r,i_r}, \phi_{j_{r+1},0}, \cdots, \phi_{j_k,0})\|_{\alpha(p)} \\ &\qquad \leqq C\|u\|_\infty \|a_0\|_* \prod_{j=2}^{k} \|a_j\|_\infty \|a_1\|_{\beta(p)}\end{aligned}$$

by the case (i) if $\{j_0, \cdots, j_r\} = \emptyset$, by Lemma 4.3 and Proposition 4.1 if $1 \in \{j_0, \cdots, j_r\}$, and by Lemma 4.3 and Proposition 4.2 if $1 \in \{j_{r+1}, \cdots, j_k\}$. This implies the desired inequality as above.

PROOF OF THEOREM 2. Similar to the above proof.

5. Applications and Examples for Theorems 1 and 2. As an application of our main theorems we can give another proof of a result in Coifman-Meyer [3, Théorème 1] and improve it somewhat.

THEOREM 5.1. *Let* $\sigma(x, \xi) \in C^\infty(\boldsymbol{R}^n \times \boldsymbol{R}^{nm})$ *satisfy*

$$\begin{aligned}&|\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi)| \leqq C_{\alpha,\beta}(1 + |\xi|)^{-|\alpha|} , \\ &\qquad \xi \in \boldsymbol{R}^{nm} , \quad x \in \boldsymbol{R}^n , \quad |\alpha| \leqq 2nm + 1 , \quad |\beta| \leqq n + 1 ,\end{aligned}$$

and define the operator T *as follows*

$$T(f_1, \cdots, f_m) = \int_{\boldsymbol{R}^{nm}} e^{ix\cdot(\xi_1+\cdots+\xi_m)}\sigma(x, \xi) f_1(\xi_1) \cdots f_m(\xi_m) d\xi_1 \cdots d\xi_m .$$

Then, for any $p_j \in [1, \infty]$ $(1 \leqq j \leqq m)$, $0 < 1/p = 1/p_1 + \cdots + 1/p_m \leqq 1$ *there exists* $C = C(n, m, p_j, C_{\alpha,\beta}) > 0$ *such that*

$$\|T(f_1, \cdots, f_m)\|_p \leqq C\|f_1\|_{p_1} \cdots \|f_m\|_{p_m} \quad (f_j \in \mathscr{S},\ j = 1, \cdots, m) ,$$

where we use temporarily the notation $\|\ \|_{p_j} = \|\ \|_{H^1}$ *and assume* $f_j \in H^1_{00}$ *if* $p_j = 1$.

Coifman and Meyer have given the above in the case $1 < p_j < \infty$ and $p \geqq 1$. However they have given the proof only for $p > 1$. We sketch our proof briefly. First, we prove the case $p_1 = p$, $p_2 = \cdots = p_m = \infty$. This case can be proved in a way quite similar to the proof of Théorème 34 in Coifman-Meyer [4, pp. 154–157], by using our Theorems 1 and 2 (or Propositions 4.1 and 4.2) and Sobolev's imbedding theorem for $1 \leqq p < \infty$. In the case $p = 1$ one must be careful. We have used the atomic decomposition of H^1 functions. When the support of an atom is small, we treat it as in the case $p > 1$. Otherwise we merely use the case $p = 2$. Next, by the multilinear interpolation theorem of Calderón we can establish Theorem 5.1.

REMARK. If $\sigma(x, \xi)$ does not depend on the variables x, the condition imposed on σ can be weakened as follows:

$$\sigma(\xi) \in C^\infty(\boldsymbol{R}^{nm}\setminus\{0\}) \quad \text{and} \quad |\partial_\xi^\alpha \sigma(\xi)| \leqq C_\alpha |\xi|^{-|\alpha|} , \quad \xi \neq 0 , \quad |\alpha| \leqq 2nm + 1 .$$

In this case p may be ∞, where $\| \ \|_p$ shall be replaced by the BMO norm. We do not know whether this is true in the general case.

Another application is concerned with Littlewood-Paley's g-function. The following may be known, but we could not find any explicit proof.

PROPOSITION 5.2. *It holds for some* $C > 0$ *that*

$$\|g(f)\|_* \leqq C\|f\|_* , \quad f \in \mathrm{BMO} \cap L^2 .$$

PROOF. Since

$$t\frac{\partial P_t}{\partial t}\Big|_{t=1} \quad \text{and} \quad t\frac{\partial P_t}{\partial x_j}\Big|_{t=1} , \quad j = 1, \cdots, n$$

satisfy the condition in Theorem 1, we apply Theorem 1 and get

$$\|g(f)^2\|_* \leqq C\|f\|_*^2 , \quad f \in \mathrm{BMO} \cap L^2 .$$

It is easily seen that for any non-negative valued $f \in \mathrm{BMO}$

$$\|f\|_*^s \leqq \|f^s\|_* , \quad (0 < s < 1) .$$

Thus we get the desired inequality. Here, $P_t(x)$ denotes the Poisson kernel for $\boldsymbol{R}_+^{n+1}$.

We shall next give some examples of ϕ in Theorem 1 such that $|\phi(x)| \leqq C(1 + |x|)^{-n} w(1/1 + |x|)$ just for some $w \in W_2^a$ in the one dimensional case.

EXAMPLE. Let $a > 0$ and h be as follows. h is infinitely differentiable on $(-3/4, 3/4)$ and

$$h(\xi) = \begin{cases} i(\operatorname{sgn} \xi) \log^{-a} (1 - |\xi|)^{-1} , & 1/2 < |\xi| < 1 \\ 0 , & |\xi| < 1/4 , \quad |\xi| \geqq 1 . \end{cases}$$

Let

$$\phi(x) = -\hat{h}(x) = 2\int_0^\infty h(\xi) \sin x\xi d\xi .$$

Then, integrating by parts we get

$$\phi(x) = \frac{2}{x}\int_0^{1/2} h'(\xi)\cos x\xi d\xi - \frac{2a}{x}\int_{1/2}^1 \frac{\cos x\xi}{1-\xi}\log^{-(1+a)}\frac{1}{1-\xi}d\xi .$$

The first term is clearly of order $O(1/x^2)$. The second integral is smaller than

$$\int_{1-2\pi/x}^1 \frac{1}{1-\xi}\log^{-(1+a)}\frac{1}{1-\xi}d\xi = \frac{1}{a}\log^{-a}\frac{x}{2\pi}, \quad \text{for} \quad x > 2\pi .$$

Since h is integrable, ϕ is bounded. Thus, summing up, we see that there exists $C_1 > 0$ such that

$$|\phi(x)| \leqq C_1(1+|x|)^{-1}\log^{-a}(1/1+|x|) .$$

In a similar way we see that there exists $C_2 > 0$ such that

$$\phi(x) \geqq C_2(x\log^a 4x/\pi)^{-1}, \quad x = 2j\pi, \quad j = 1, 2, \cdots .$$

Clearly, if $a > 5/2$, then $\log^{-a} t^{-1} \in W_2^b$ $(0 < b < a)$. And if $a > 3/2$, it belongs to W_3^b $(0 < b < a)$.

6. BMO and Carleson measures. In this section we investigate relations between functions of bounded mean oscillation and Carleson measures. The most important result in this direction is Theorem 3 in Fefferman-Stein [7, p. 145] and further extensions can be found in Fabes, Neri and Johnson [6], Ortiz and Torchinsky [8], Strömberg [12] and Strichartz [11]. Our aim is to extend some results in the last two papers. Our first result is the following.

PROPOSITION 6.1. *Let* $w \in W_2$ *and* $|\psi(x)| \leqq (1+|x|)^{-n}w(1/1+|x|)$, $\int \psi(x)dx = 0$. *Then* $|\psi_{at} * f|^2 t^{-1}dxdt$ *is a Carleson measure for any* $a > 0$ *and* $f \in \mathrm{BMO}$, *and there exists* $C > 0$ *such that*

$$\gamma(|\psi_{at} * f|^2 t^{-1}dxdt) \leqq C\max(1, a^n)\|f\|_*^2 .$$

Another extension will be given later (Proposition 6.7). When one only deals with bounded functions, one gets

PROPOSITION 6.2. *Let* $w \in W_3$. *Let* $|\psi(x)| \leqq (1+|x|)^{-n}w(1+|x|)$ *and* $\int \psi(x)dx = 0$. *Then* $|\psi_{at} * f|^2 t^{-1}dxdt$ *is a Carleson measure for any* $a > 0$ *and* $f \in L^\infty$, *and there exists* $C > 0$ *such that*

$$\gamma(|\psi_{at} * f|^2 t^{-1}dxdt) \leqq C\max(1, a^n)\|f\|_\infty^2 .$$

To prove these propositions we follow the proof-method in [7]. We begin with the following lemma.

LEMMA 6.3. *Let $w \in W_1$. Then there exists $C > 0$ such that*

$$\int_{R^n} |f(x) - f_{Q(y,s)}|(s^n + |x-y|^n)^{-1} w(s/(s+|x-y|))dx \leqq C\|f\|_* \quad (f \in \mathrm{BMO})$$

for any cube $Q(y, s)$ of side length s and center y.

PROOF. Since the BMO norm is invariant under dilation and translation, we may assume $s = 1$ and $y = 0$. As $w(t)$ is nondecreasing, by the arguments in Fefferman-Stein [7, p. 142] we see that

$$\begin{aligned}\int_{R^n} &|f(x) - f_{Q(0,s)}|(1 + |x|^n)^{-1} w(1/1 + |x|)dx \\ &\leqq \sum_{k=0}^{\infty} (1 + 2^n(k+1)) w(1/1 + 2^k)\|f\|_* + w(1)2^{-1}\|f\|_* \\ &\leqq C\left(\int_0^1 w(t)t^{-1} \log(e + 1/t)dt + w(1)\right)\|f\|_* .\end{aligned}$$

Using this one gets easily the first part of the following lemma. The second part is easy.

LEMMA 6.4. (i) *Let $w \in W_1$. Then there exists $C > 0$ such that*

$$\|g_t * f\|_\infty \leqq C\|f\|_*$$

for any $t > 0$, $f \in \mathrm{BMO}$ and g with $|g(x)| \leqq (1 + |x|)^{-n} w(1/1 + |x|)$.

(ii) *Let $w \in W_0$. Then there exists $C > 0$ such that*

$$\|g_t * f\|_\infty \leqq C\|f\|_\infty$$

for any $t > 0$, $f \in L^\infty$ and g with $|g(x)| \leqq (1 + |x|)^{-n} w(1/1 + |x|)$.

Our next lemma is as follows.

LEMMA 6.5. *Let $w \in W_3$. Then there exists $C > 0$ such that*

$$\int_0^\infty |\hat{g}(t\xi)|^2 t^{-1} dt \leqq C$$

for any $\xi \in R^n$ and g with $|g(x)| \leqq (1 + |x|)^{-n} w(1/1 + |x|)$ and $\int g(x)dx = 0$.

In particular, if $w \in W_2$, the conclusion follows.

PROOF. Note first, essentially $W_0 \supset W_3 \supset W_2$, as noted in Section 1. Hence $g \in L^1$. Now, since $\int g(x)dx = 0$, we have $\hat{g}(\xi) = \int (e^{-ix\cdot\xi} - 1)g(x)dx$. Hence we get

$$|\hat{g}(\xi)| \leqq \int_{|x| \leqq 1/|\xi|} |e^{-ix\cdot\xi} - 1||g(x)|dx + 2\int_{|x| > 1/|\xi|} |g(x)|dx .$$

Thus by easy calculation we get

$$|\hat{g}(\xi)| \leqq |\xi| \int_{|\xi|/2}^{1} w(t)t^{-2}dt + 2\int_0^{|\xi|} w(t)t^{-1}dt . \tag{23}$$

(a) Case $\xi = (b, 0, \cdots, 0)$. In this case we have

$$I = \int_0^\infty |\hat{g}(t\xi)|^2 \frac{dt}{t} = \int_0^\infty |\hat{g}((t, 0, \cdots, 0))|^2 \frac{dt}{t} = \int_0^1 \cdots + \int_1^\infty \cdots$$
$$= I_1 + I_2 , \quad \text{say} .$$

Using (23) we have

$$I_1 \leqq 2\int_0^1 t^2\left(\int_{t/2}^1 w(s)s^{-2}ds\right)^2 \frac{dt}{t} + 4\int_0^1 \left(\int_0^t w(s)\frac{ds}{s}\right)^2 \frac{dt}{t} = I_3 + I_4 , \quad \text{say} .$$

By easy calculation we get

$$I_3 \leqq 4\left(\int_0^1 w(s)s^{-1}ds\right)\int_0^1\int_{t/2}^1 w(s)s^{-2}dsdt .$$

Interchanging the order of integrations in the last integral we have

$$I_3 \leqq 16\left(\int_0^1 w(s)s^{-1}ds\right)^2 < +\infty .$$

Next since $w \in W_3$, $\int_0^1 w^2(t)\log^{2+a}(e + 1/t)t^{-1}dt < +\infty$ for some $a > 0$. Now by Cauchy-Schwarz's inequality

$$\left(\int_0^t w(s)\frac{ds}{s}\right)^2 \leqq \left(\int_0^t w^2(s)\log^{1+a}\left(e + \frac{1}{s}\right)\frac{ds}{s}\right)\left(\int_0^t \log^{-(1+a)}\left(e + \frac{1}{s}\right)\frac{ds}{s}\right) .$$

Hence we have

$$I_4 \leqq C\int_0^1\int_0^t w^2(s)\log^{1+a}\left(e + \frac{1}{s}\right)\frac{ds}{s}\frac{dt}{t}$$
$$= C\int_0^1 w^2(s)\log^{1+a}\left(e + \frac{1}{s}\right)\log\frac{1}{s}\frac{ds}{s} \leqq C .$$

For I_2 we get, using Parseval's identity and the monotonicity of w,

$$I_2 \leqq \int_{-\infty}^\infty |\hat{g}(\xi_1, 0, \cdots, 0)|^2 d\xi_1 = \int_{-\infty}^\infty \left|\int_{-\infty}^\infty \left(\int_{R^{n-1}} g(x_1, x')dx'\right)e^{-ix_1\xi_1}dx_1\right|^2 d\xi_1$$
$$= \int_{-\infty}^\infty \left|\int_{R^{n-1}} g(x_1, x')dx'\right|^2 dx_1$$
$$\leqq \int_{-\infty}^\infty \left(\int_{R^{n-1}} (1 + |x'|)^{n-1}(1 + |x_1|)^{-1}w(1/1 + |x'|)dx'\right)^2 dx_1$$
$$\leqq C\left(\int_{R^{n-1}} (1 + |x'|)^{n-1}w(1/1 + |x'|)dx'\right)^2 \leqq C\left(\int_0^1 w(t)t^{-1}dt\right)^2 .$$

These show the desired inequality in the case (a).

(b) General case. Let $\xi \in \boldsymbol{R}^n$ and $U \in SO(n)$ satisfy $U^{-1}\xi = (|\xi|, 0, \cdots, 0)$. Then we get easily

$$\hat{g}(\xi t) = \int_{R^n} g(Ux)e^{-ix(t|\xi|,0,\cdots,0)}dx .$$

Since the assumption for g is invariant under rotation, we can reduce this case to the case (a). This completes the proof.

LEMMA 6.6. *Let $w \in W_3$. Then there exists $C > 0$ such that*

$$\int_0^\infty \int_{R^n} |g_t * f|^2 t^{-1} dx dt \leqq C\|f\|_2^2$$

for any $f \in L^2$ and g with $|g(x)| \leqq (1 + |x|)^{-n} w(1/1 + |x|)$ and $\int g(x)dx = 0$.

PROOF. Since $\hat{g}_t(\xi) = \hat{g}(t\xi)$, we obtain the conclusion by Lemma 6.5 and Parseval's identity.

PROOF OF PROPOSITION 6.1. It is easily seen that it suffices to prove the proposition only in the case $a = 1$. Let Q be the cube whose sides have length $4s$, with center $\{0\}$. Put

$$f = f_Q + (f - f_Q)\chi_Q + (f - f_Q)\chi_{Q^c} = f_1 + f_2 + f_3 , \quad \text{say} ,$$

where χ_Q is the characteristic function of the set Q. Then, since $\int \psi(x)dx = 0$, we have $\psi_t * f_1 = 0$. Now by Lemma 6.6 we get

$$\int_0^s \int_{|x|<s} |\psi_t * f_2|^2 t^{-1} dx dt \leqq \int_0^\infty \int_{R^n} |\psi_t * f_2|^2 t^{-1} dx dt \leqq C\|f_2\|_2^2 \leqq C_1 s^n \|f\|_*^2 .$$

For f_3 we get

$$\begin{aligned} |\psi_t * f_3(x)| &= \left|\int_{Q^c} \psi_t(x - y) f_3(y) dy\right| \\ &\leqq \int_{Q^c} |f(y) - f_Q|(t + |x - y|)^{-n} w(t/(t + |x - y|)) dy . \end{aligned}$$

Now if $y \in Q^c$, $|x| < s$ and $0 < t < s$, then we have

$$1 + \frac{|x - y|}{t} \geqq 1 + \frac{|y| + s}{4t} \geqq \left(1 + \frac{|y|}{5s}\right)\left(1 + \frac{s}{4t}\right) .$$

Let $b = 4/5$. Then, since w is nondecreasing, we have easily

$$w(t/(t + |x - y|)) \leqq w^b(5s/(5s + |y|)) w^{1-b}(4t/(4t + s)) .$$

Hence it follows that

$$\begin{aligned} |\psi_t * f_3(x)| \leqq Cw^{1-b}(4t/(4t + s)) s^{-n} \int |f(y) - f_Q| \left(1 + \frac{|y|}{s}\right)^{-n} \\ \times w^b(5s/(5s + |y|)) dy . \end{aligned}$$

Therefore by Lemma 6.3 we get

$$|\psi_t * f_3(x)| \leqq C_1 \| f \|_* w^{1-b}(4t/(4t+s)) .$$

Thus we have

$$\int_0^s \int_{|x|<s} |\psi_t * f_3(x)|^2 t^{-1} dx dt \leqq C_2 s^n \| f \|_* \int_0^s w^{2(1-b)}(4t/(4t+s)) t^{-1} dt . \tag{24}$$

Since $w \in W_2$, $w \in W_2^a$ for some $a > 0$. Hence by Cauchy-Schwarz's inequality we get

$$\left(\int_0^1 w^{2/5}(t)\frac{dt}{t}\right)^2 \leqq \int_0^1 w^{4/5}(t) \log^{1+a}\left(e+\frac{1}{t}\right)\frac{dt}{t} \int_0^1 \log^{-(1+a)}\left(e+\frac{1}{t}\right)\frac{dt}{t} . \tag{25}$$

Therefore the last integral in (24) is finite. Thus we obtain the desired inequality.

PROOF OF PROPOSITION 6.2. One can prove Proposition 6.2 in a way similar to the above proof. In this case we take $b = 2/3$. Then as in (25) we have $w^{2/3} \in W_0$. We use this and Lemma 6.4 instead of (25) and Lemma 6.3, respectively.

Modifying our arguments above, we can extend a recent result in Strichartz [11, Theorem 2.1] as follows. We leave the detailed proof to the reader.

PROPOSITION 6.7. *Let $w \in W_2$ and w_1 be a nonincreasing function on $[1, \infty)$, satisfying $\int_1^\infty w_1^2(t) t^{-1} dt < +\infty$. Let $a > 0$. Then there exists $C = C(a) > 0$ such that*

$$\gamma(|\psi_t * f|^2 t^{-1} dx dt) \leqq C \| f \|_*^2$$

for any $f \in \mathrm{BMO}$ and $\psi \in L^1$ satisfying $\|\psi\|_1 \leqq 1$, $\int \psi(x) dx = 0$ and

(i) $|\psi(x)| \leqq (1+|x|)^{-n} w(1/1+|x|)$ *for* $|x| > a$

(ii) $|\hat{\psi}(\xi)| \leqq w_1(|\xi|)$ *for* $|\xi| > 1$.

A similar result corresponding to Proposition 6.2 can also be formulated. And we can modify Proposition 6.7 so that it contains Proposition 6.1. However, we think that Proposition 6.7 itself has its meaning, because it contains no assumption on the Fourier transform.

As for the results converse to Propositions 6.1 and 6.7, we obtain a result similar to Theorem 2.5 in Strichartz [11], by modifying his arguments.

PROPOSITION 6.8. *Let $w \in W_2$. Let ψ_j satisfy the assumptions in*

Propositions 6.1 *or* 6.7 $(j = 1, 2, \cdots, k)$. *Suppose furthermore* $\hat{\psi}_j(\xi) \in C^\infty(\boldsymbol{R}^n \setminus \{0\})$ *and for each* $\xi \neq 0$ *there exists* j *such that* $\hat{\psi}_j(t\xi) \neq 0$ *for some* $t > 0$. *Then, if* $\int |f(x)|(1 + |x|)^{-n} w(1/1 + |x|)dx < +\infty$ *and* $|\psi_{j,t} * f|^2 t^{-1} dx dt$ *is a Carleson measure for each* $j = 1, 2, \cdots, k$, *it follows that* $f \in \mathrm{BMO}$.

Finally in this section we give two examples which show that the numbers 4/5 and 2 are best possible in Propositions 6.1 and 6.2, respectively.

EXAMPLES. Let $g_j \in \mathscr{S}$ with $\operatorname{supp} g_j \subset \{|x| < 1/2\}$ and $\int g_j(x)dx = \int_{|x| \geqq 1} (1 + |x|)^{-n} \log^{-a_j}(2 + |x|)dx$ $(j = 1, 2)$, where $a_1 = 5/2$ and $a_2 = 3/2$. Let $\psi_j(x) = -g_j(x)$ for $|x| < 1$, $= (1 + |x|)^{-n} \log^{-a_j}(2 + |x|)$ for $|x| \geqq 1$. Let $f_1(x) = \log^+ |x|$ and $f_2(x) = \chi_{\{|x| \geqq 1\}}$. Then $|\psi_{j,t} * f_j(x)|^2 t^{-1} dx dt$ is not a Carleson measure $(j = 1, 2)$. In fact, for $|x| < 1/2$ and $0 < t < 1$ we have by elementary calculations and estimates

$$\psi_{1,t} * f_1(x) \geqq C \log \frac{1}{t} \log^{-3/2}(2 + 2/t^2) \geqq C_1 \log^{1/2} \frac{1}{t} ,$$

where C_1 is independent of t, x; $0 < t < 1$ and $|x| < 1/2$. Hence, near $x = 0$, we have $\int_0^1 |\psi_{1,t} * f_1(x)|^2 t^{-1} dt = +\infty$. Similarly we get $\int_0^1 |\psi_{2,t} * f_2(x)|^2 t^{-1} dt = +\infty$, near $x = 0$. These imply our assertion.

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