Tôhoku Math. Journ.
34 (1982), 553–558.

THE p-ADIC HURWITZ L-FUNCTIONS

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(Received January 29, 1982, revised March 27, 1982)

0. Introduction. Recently various p-adic analogues of arithmetic functions are constructed. Kubota and Leopoldt [4] constructed p-adic L-functions by interpolating the values at nonpositive integers of Dirichlet L-functions. Morita [6] constructed a p-adic analogue of the Hurwitz-Lerch L-function $L(s, a, b, \chi) = \sum_{n=1}^{\infty} \chi(n)b^n(a+n)^{-s}$ from the same point of view. In this paper, we construct a p-adic analogue of the Hurwitz L-function $L(s, a, \chi) = L(s, a, 1, \chi)$ as a power series at $s = 0$.

In §1, we calculate the higher derivatives at $s = 0$ of the complex Hurwitz L-function (Theorem 1). In §2, we obtain a lemma for p-adic interpolation. We construct in §3.1 p-adic analogues $\alpha_l^*(a, \chi)$ of the coefficients of the expansion at $s = 0$ of the complex Hurwitz L-function. We then define a p-adic function $\zeta_p(s, a, \chi)$ by

$$\zeta_p(s, a, \chi) = \sum_{l=0}^{\infty} \alpha_l^*(a, \chi)s^l$$

and show in §3.2 that this function coincides with Morita's p-adic analogue in the case of $b = 1$.

The author wishes to express his thanks to Professors S.-N. Kuroda and Y. Morita for their encouragement during the preparation of this paper. The author also wishes to express his thanks to the referee for the valuable advice.

1. Complex Hurwitz L-functions. 1.1. We denote by $\boldsymbol{Q}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ the fields of rational numbers, real numbers and complex numbers, respectively. We denote by $\operatorname{Re} s$ the real part of s.

Let χ be a Dirichlet character with conductor f, and let $L(s, \chi)$ be the Dirichlet L-function for the character χ. Put $\delta_\chi = 1$ if χ is trivial, and $\delta_\chi = 0$ otherwise. We define complex numbers $\beta_l(\chi)$ ($l \geqq 0$) by

$$\beta_l(\chi) = (-1)^l(l!)^{-1} \sum_{a=1}^{m} \chi(a) \lim_{n\to\infty} \left[\sum_{k=0}^{n} \frac{\{\log(mk+a)\}^l}{mk+a} - \frac{\{\log(mn+a)\}^{l+1}}{m(l+1)}\right].$$

Then we have the following:

PROPOSITION 1. *For any positive multiple m of f, we have*

$$L(s, \chi) = \delta_\chi (s-1)^{-1} + \sum_{l=0}^{\infty} \beta_l(\chi)(s-1)^l \,.$$

PROOF. Let

$$\zeta(s, m, a) = m^{-1}(s-1)^{-1} + \sum_{l=0}^{\infty} \beta_l(m, a)(s-1)^l$$

be the Laurent expansion of the partial zeta function $\zeta(s, m, a) = \sum_{k=0}^{\infty} (mk+a)^{-s}$. Then we can prove

$$\beta_l(m, a) = (-1)^l (l!)^{-1} \lim_{n\to\infty} \sum_{k=0}^{n} \left[\frac{\{\log (mk+a)\}^l}{mk+a} - \frac{\{\log (mn+a)\}^{l+1}}{m(l+1)} \right],$$

as in [8, Chapter I], where Siegel studied Riemann's zeta function. Since $L(s, \chi) = \sum_{a=1}^{m} \chi(a)\zeta(s, m, a)$, the proposition follows from this formula.

REMARK. The formula for $\beta_l(1, a)$ was obtained by Berndt [1], and the formula for $\beta_l(\chi)$ for the trivial character was obtained by Briggs and Chowla [2], Ferguson [3] and Verma [9].

1.2. Let $L(s, a, \chi) = \sum_{n=1}^{\infty} \chi(n)(n+a)^{-s}$. Then this series converges absolutely for $\mathrm{Re}\, s > 1$ and $\mathrm{Re}\, a > -1$, and defines a holomorphic function in (s, a). Further, $L(s, a, \chi)$ can be continued to a meromorphic function in (s, a) on $\{(s, a); s \in \boldsymbol{C}, \mathrm{Re}\, a > 1\}$ (cf. Morita [6]). We call $L(s, a, \chi)$ the Hurwitz L-function for the character χ. We see that $L(s, a, \chi)$ is the restriction of the Hurwitz-Lerch L-function $L(s, a, b, \chi)$ and $L(s, 0, \chi)$ is the Dirichlet L-function $L(s, \chi)$. Now we calculate the Laurent expansion of $L(s, a, \chi)$ at $s = 1$.

PROPOSITION 2.

$$L(s, a, \chi) = \delta_\chi (s-1)^{-1} + \sum_{l=0}^{\infty} \beta_l(a, \chi)(s-1)^l \,,$$

where

$$\beta_l(a, \chi) = (-1)^l (l!)^{-1} \sum_{n=1}^{\infty} \chi(n) \left[\frac{\{\log (n+a)\}^l}{n+a} - \frac{(\log n)^l}{n} \right] + \beta_l(\chi) \,.$$

Moreover, this series converges uniformly on any compact subset of $\boldsymbol{C} \setminus (-\infty, -1]$.

PROOF. It is easy to see that the series

$$\sum_{n=1}^{\infty} \chi(n)(\partial/\partial s)^l \{(n+a)^{-s} - n^{-s}\} \qquad (l \geqq 0)$$

converges absolutely and uniformly on any compact subset of $\{(s, a) \in \boldsymbol{C} \times \boldsymbol{C}; \mathrm{Re}\, s > 0, \mathrm{Re}\, a > -1\}$. Thus Proposition 2 follows easily from the equality

$$L(s, a, \chi) - L(s, \chi) = \sum_{l=0}^{\infty} \{\beta_l(a, \chi) - \beta_l(\chi)\}(s-1)^l = \sum_{n=1}^{\infty} \chi(n)\{(n+a)^{-s} - n^{-s}\} .$$

COROLLARY. *Let $\Gamma_\chi(1+a)$ be the function introduced by Morita* [7]. *Then we have*

$$\beta_0(a, \chi) = -(d/da) \log \Gamma_\chi(1+a) .$$

1.3. We now study the expansion of $L(s, a, \chi)$ at $s = 0$.

THEOREM 1. *Let*

$$L(s, a, \chi) = \sum_{l=0}^{\infty} \alpha_l(a, \chi) s^l \quad \textit{and} \quad L(s, \chi) = \sum_{l=0}^{\infty} \alpha_l(\chi) s^l$$

be the Taylor expansions of $L(s, a, \chi)$ and $L(s, \chi)$ at $s = 0$. Then we have

$$\alpha_0(a, \chi) = -\delta_\chi a + \alpha_0(\chi) ,$$

$$\alpha_{l+1}(a, \chi) = (-1)^{l+1}\{(l+1)!\}^{-1} \sum_{n=1}^{\infty} \chi(n)[\{\log (n+a)\}^{l+1} - (\log n)^{l+1} - (l+1)(\log n)^l \cdot (a/n)] - \beta_l(\chi)a + \alpha_{l+1}(\chi)$$

for $l \geqq 0$.

COROLLARY. $\alpha_1(a, \chi) = \log \Gamma_\chi(1+a) + \alpha_1(\chi)$.

PROOF. Since $(\partial/\partial a)L(s, a, \chi) = -sL(s+1, a, \chi)$, we have $(d/da)\alpha_0(a, \chi) = -\delta_\chi$ and $(d/da)\alpha_{l+1}(a, \chi) = -\beta_l(a, \chi)$ for $l \geqq 0$. Hence Theorem 1 and its corollary follow from the formulas, Proposition 2 and its corollary.

REMARK. For the trivial character χ^0, formulas

$$\alpha_0(a, \chi^0) = -a - 1/2 ,$$

$$\alpha_1(\alpha, \chi^0) = \log (\Gamma(1+a)/\sqrt{2\pi}) \quad \text{and} \quad \alpha_1(\chi^0) = -\log \sqrt{2\pi}$$

are well-known (cf. Whittacker and Watson [10, Chap. XIII]).

2. A preparatory lemma. 2.1. We denote by $\boldsymbol{Z}$, $\boldsymbol{Z}_+$, $\boldsymbol{Z}_p$ and $\boldsymbol{Q}_p$ the ring of rational integers, the set of positive integers, the ring of p-adic integers and the field of p-adic numbers, respectively. We denote by $\boldsymbol{Z}_p^\times$ the group of units in $\boldsymbol{Z}_p$, and by $\boldsymbol{C}_p$ the p-adic completion of the algebraic closure of $\boldsymbol{Q}_p$. We assume that the values of any Dirichlet character are in $\boldsymbol{C}_p$. Put $q = 4$ if $p = 2$, and $q = p$ otherwise. Let ω be the Dirichlet character satisfying $\omega(x) \equiv x \pmod q$. We define $\langle x \rangle$ by $x = \omega(x)\langle x \rangle$ for $x \in \boldsymbol{Z}_p^\times$. We denote by $|x|$ the valuation of $x \in \boldsymbol{C}_p$, which is normalized by $|p| = 1/p$. We denote by $\sum_{x \in I}^*$ the sum over all $x \in I$ prime to p.

2.2. We now prove:

LEMMA 1. *Let $g(x)=\sum_{n=0}^{\infty} a_n(x-1)^n$ ($a_n \in \boldsymbol{C}_p$) be a power series which converges on $U_\varepsilon=\{x\in \boldsymbol{C}_p; |x-1|\leqq |q|^{1-\varepsilon}\}$ ($\varepsilon>0$). Let $\bar{f}$ be the least common multiple of f and q. We define a function on $f\boldsymbol{Z}_+$ by*

$$G(m, \chi)=\sum_{k=1}^{m}{}^{*}\, \chi(k) g(\langle k\rangle) .$$

Then $G(m, \chi)$ can be uniquely extended to a continuous function on $f\boldsymbol{Z}_p$, and the restriction of $G(z, \chi)$ to $\bar{f}\boldsymbol{Z}_p$ is analytic. Moreover its Taylor expansion at $z=0$ converges for $|z|\leqq|q|$.

PROOF. Let $\mathfrak{M}_{\chi,z}(A)$ and $F_{A,\chi}(z)=\mathfrak{M}_{\chi,z}(A)-\mathfrak{M}_{\chi}(A)$ be as in Morita [5]. Let $h(x)$ be a power series which satisfies $(d/dx)h(x)=g(x)$. Then $h(x)$ converges on $U_{\varepsilon'}$ for any ε' with $0<\varepsilon'<\varepsilon$. By [5, Theorem 2], $F_{h,\chi\omega}(z)$ is an analytic function on $\{z\in \boldsymbol{C}_p; |z|\leqq|\bar{f}|\}$ and coincides with $G(z, \chi)$ on $f\boldsymbol{Z}_+$. We put

$$G(m, \chi)=\sum_{k=1}^{m_1}{}^{*}\, \chi(k) g(\langle k\rangle)+\sum_{k=m_1+1}^{m}{}^{*}\, \chi(k) g(\langle k\rangle) ,$$

where m_1 denotes the maximal integer which is a multiple of $\bar{f}$ and does not exceed m. This formula shows continuity, because the first term is an analytic function and the second is a finite sum of continuous functions. Therefore the lemma is proved.

REMARK. The author had originally proved Lemma 1 for $g(x)=(\log x)^l$. It is due to a suggestion of Y. Morita that the author generalized the result to the present case.

3. A p-adic analogue of the Hurwitz L-function. 3.1. Let $\bar{\boldsymbol{Q}}$ be the algebraic closure of $\bar{\boldsymbol{Q}}$, and fix embeddings of this field into $\boldsymbol{C}$ and $\boldsymbol{C}_p$. Let χ be a $\boldsymbol{Q}$-valued Dirichlet character with conductor f. Let m be a positive multiple of f. Then by using Proposition 1 and Theorem 1, we have

$$\alpha_l(m, \chi)=(-1)^{l+1}(l\,!)^{-1} \sum_{k=1}^{m} \chi(k)(\log k)^l+\alpha_l(\chi) .$$

Let $L_p(s, \chi\omega)$ be the p-adic L-function associated with $\chi\omega$ and let $\log x$ be the p-adic logarithmic function $\log x=\sum_{n=1}^{\infty}((-1)^{n+1}/n)(x-1)^n$. We define a $\boldsymbol{C}_p$-valued function $\alpha_l^*(m, \chi)$ by

$$\alpha_l^*(m, \chi)=(-1)^{l+1}(l\,!)^{-1} \sum_{k=1}^{m}{}^{*}\, \chi(k)(\log\langle k\rangle)^l+\alpha_l^*(\chi\omega) ,$$

where $\alpha_l^*(\chi\omega)$ is defined by $L_p(s, \chi\omega)=\sum_{l=0}^{\infty} \alpha_l^*(\chi\omega)\, s^l$. Then, by Lemma 1, we have the following:

PROPOSITION 3. *The C_p-valued function $\alpha_l^*(m, \chi)$ can be uniquely extended to a continuous function on fZ_p. Moreover the restriction of $\alpha_l^*(a, \chi)$ to $\bar{f}Z_p$ is given by a convergent power series*

$$\alpha_l^*(a, \chi) = \alpha_l^*(\chi\omega) + (-1)^{l+1}(l!)^{-1} \sum_{m=0}^{\infty} \mathfrak{M}_{\chi_\omega - m}\{(d/du)^m(\log u)^l\}a^{m+1}/(m+1)!\ .$$

This series converges for $|a| \leqq |q|$.

3.2. Since $\alpha_l^*(a, \chi)$ is a p-adic analogue of the higher derivative at $s = 0$ of the Hurwitz L-function,

$$(1)\qquad \zeta_p(s, a, \chi) = \sum_{l=0}^{\infty} \alpha_l^*(a, \chi)s^l$$

can be regarded as a p-adic analogue of $L(s, a, \chi)$. This power series converges for $|a|<|q|$ and $|s|<|q|^{-1}|p|^{1/(p-1)}$. We denote by $L_p(s, a, b, \chi)$ the p-adic analogue of the Hurwitz-Lerch L-function constructed by Morita [6]. Then we have:

THEOREM 2. $\zeta_p(s, a, \chi) = L_p(s, a, 1, \chi\omega)$.

PROOF. By Morita [6], we see

$$(2)\qquad L_p(s, a, 1, \chi\omega) = L_p(s, \chi\omega) - \sum_{0 \leq l, h < \infty} \Big[\sum_{n=0}^{\infty} (-1)^n/n!\, \mathfrak{M}_{\chi_\omega - l} \times \{(d/du)^l(\log u)^{n+h}\}a^{l+1}/(l+1)! \cdot (1-s)^h/h! \Big]\ .$$

Comparing the h-th derivatives at $s = 1$ of (1) and (2), we get Theorem 2.

COROLLARY.

$$(\partial/\partial s)L_p(s, a, 1, \chi)|_{s=0} = \log \Gamma_{p,\chi_\omega^{-1}}(1+a) + \alpha_1^*(\chi)\ ,$$

where $\log \Gamma_{p,\chi}(1+a) = F_{u(\log u - 1), \chi_\omega}(a)$ (*cf. Morita* [7]).

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