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# ON YAMABE'S PROBLEM —BY A MODIFIED DIRECT METHOD—

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**1. Introduction.** In his paper [12], Yamabe asked whether any compact $C^\infty$-Riemannian manifold of dimension $\geqq 3$ can be deformed conformally to a $C^\infty$-Riemannian manifold of constant scalar curvature, and proposed to solve the question by reducing it to a non-linear eigenvalue problem which consists of finding a strict positive $C^\infty$ function $u$ on a compact Riemannian manifold $(M, g)$ together with a constant $\lambda$ so that one has

$$-(4(n-1)/(n-2))\Delta u + Ru = \lambda u^{(n+2)/(n-2)} \tag{1.1}$$

where $\Delta$ is the usual Laplace-Beltrami operator and $R = R(x)$ denotes the scalar curvature defined by the metric $g$.

As pointed out by Trudinger [11] there was a gap in Yamabe's proof, and the problem is still unsolved as it stands. An almost complete solution, however, has recently been achieved by Aubin [1], [2]. In fact, introducing the functional

$$J(u) = \int_M (\kappa|\nabla u|^2 + Ru^2)d\omega \Big/ \left(\int_M |u|^N d\omega\right)^{2/N}, \tag{1.2}$$
$$\kappa = 4(n-1)/(n-2), \qquad N = 2n/(n-2)$$

where $\nabla$ denotes the covariant derivation, $d\omega$ the volume element relative to the metric $g$, and the number

$$\mu = \inf\{J(u); u \in H^1(M), u \neq 0\} \tag{1.3}$$

here $H^1(M)$ implying as usual the Sobolev space of degree one, he proved that for any compact Riemannian manifold of dimension $\geqq 3$ there holds an inequality $\mu \leqq n(n-1)\omega_n^{2/n}$ with $\omega_n$ denoting the surface area of $n$-sphere $S^n$, and, among others, the following:

THEOREM. *If $\mu < n(n-1)\omega_n^{2/n}$, there exists a strictly positive $C^\infty$ function on $(M, g)$ satisfying (1.1) with $\lambda = \mu$. (This gives a partial answer to Yamabe's problem.)*

The purpose of this paper is to give a simplified proof of the above result, using a different approach from that of Aubin, but similar in some way to that of Eliasson whose argument seems to be incomplete [6, p. 325].

Actually, we apply directly to the functional (1.2) the so-called variational method, combined with the steepest decent method to obtain a modified minimizing sequence which converges in $H^1(M)$ to an extremum $u$ attaining the minimum $\mu$. In this context, it becomes clearer that the sharp Sobolev inequality due to Aubin plays a crucial role in the proof of the non-triviality of the extremum.

In the final section, we consider Yamabe's problem on non-compact manifolds. Rather restrictive conditions herein imposed are in order to guarantee the applicability of Trudinger's regularity theorem for weak solutions of (1.1), and the validity of a sharp Sobolev inequality.

The results of this paper were presented in the Symposium on "Nonlinear problem in geometry", held at Katata, Japan, in 1978.

**2. A general theory of the isoperimetric problem by the steepest descent method.** Let $\mathscr{A}$, $\mathscr{B}$ be two real valued $C^2$-functionals defined on a Hilbert space $H$ with $\| \ \|$ and $( \ , \ )$ denoting the norm and the scalar product, respectively. Define $\mathscr{M}$ as $\mathscr{M} = \{u \in H; \mathscr{B}(u) = 1\}$ and

$$(\mathscr{A}'(u), v) = (d/d\varepsilon)\mathscr{A}(u + \varepsilon v)|_{\varepsilon=0} , \qquad (\mathscr{B}'(u), v) = (d/d\varepsilon)\mathscr{B}(u + \varepsilon v)|_{\varepsilon=0} .$$

Here we identify the operators $\mathscr{A}'(u)$, $\mathscr{B}'(u)$ with the elements in $H$ by Riesz's theorem.

THEOREM 2.1 (Berger [5, p. 124]). *Consider the critical points of the functional $\mathscr{A}$ restricted to $\mathscr{M}$. If $\mathscr{B}'(u) \neq 0$ on $\mathscr{M}$, then a critical point $u_0$ of $\mathscr{A}$ on $\mathscr{M}$ satisfies the equation*

$$(2.1) \qquad \mathscr{A}'(u_0) - \lambda(u_0)\mathscr{B}'(u_0) = 0 \quad \textit{where} \quad \lambda(u_0) = (\mathscr{A}'(u_0), \mathscr{B}'(u_0))\, \|\mathscr{B}'(u_0)\|^{-2} .$$

The following theorem is a slight generalization of Theorem (3.2.11) of Berger [5].

THEOREM 2.2. *Assume that $\mathscr{A}$ is bounded from below on $\mathscr{M}$ and $\mathscr{B}'(u) \neq 0$ for $u \in \mathscr{M}$. We consider the following initial value problem:*

$$(2.2) \qquad (d/dt)u(t) = \mathscr{X}(u(t)) , \qquad u(0) = u_0$$

*where $\mathscr{X}(u) = -\mathscr{A}'(u) + \lambda(u)\mathscr{B}'(u)$ with $\lambda(u) = (\mathscr{A}'(u), \mathscr{B}'(u))\, \|\mathscr{B}'(u)\|^{-2}$.*

*Then, there exists a solution $u(t) \in C^1([0, \infty); H)$ of (2.2) satisfying*

$$(2.3) \qquad \int_0^\infty \|\mathscr{X}(u(s))\|^2 ds < \infty .$$

PROOF. As $\mathscr{A} \in C^2(H, \boldsymbol{R})$ and $\mathscr{B}'(u) \neq 0$ on $\mathscr{M}$, $\mathscr{M}$ is a $C^2$-Hilbert manifold. Moreover, as $(\mathscr{X}(u), \mathscr{B}'(u)) = 0$ on $\mathscr{M}$, $\mathscr{X}$ is a $C^1$-tangent vector field on $\mathscr{M}$. We thus have a solution $u(t)$ of (2.2) at least locally in time. (See, Lang [8, p. 83].) Along $u(t)$, we have

$$(2.4)\qquad \begin{aligned}(d/dt)\mathscr{A}(u(t)) &= (\mathscr{A}'(u(t)), (d/dt)u(t)) \\ &= -\|\mathscr{A}'(u(t))\|^2 + (\mathscr{A}'(u(t)), \mathscr{B}'(u(t)))^2\|\mathscr{B}'(u(t))\|^{-2} \leqq 0 .\end{aligned}$$

$$(2.5)\qquad (d/d)\mathscr{B}(u(t)) = (\mathscr{B}'(u(t)), (d/dt)u(t)) = 0 , \quad \text{i.e.,} \quad u(t) \in \mathscr{M} .$$

Suppose now that $u(t)$ exists only locally in time, that is, the maximal interval of existence is given by $[0, t^*)$ with $t^* < \infty$. Then for $0 \leqq t_1, t_2 < t^*$, we have

$$(2.6)\qquad \begin{aligned}\|u(t_2) - u(t_1)\| &= \left\|\int_{t_1}^{t_2} (d/ds)u(s)ds\right\| \leqq \int_{t_1}^{t_2} \|\mathscr{X}(u(s))\| ds \\ &\leqq \left(\int_{t_1}^{t_2} \|\mathscr{X}(u(s))\|^2 ds\right)^{1/2} |t_2 - t_1|^{1/2} .\end{aligned}$$

On the other hand,

$$(2.7)\qquad \mathscr{A}(u(t_2)) - \mathscr{A}(u(t_1)) = \int_{t_1}^{t_2} (d/ds)\mathscr{A}(u(s))ds = -\int_{t_1}^{t_2} \|\mathscr{X}(u(s))\|^2 ds .$$

As $\mathscr{A}$ is bounded from below on $\mathscr{M}$, we find from (2.6), (2.7) that $\{u(t)\}$ forms a Cauchy sequence in $H$ when $t$ converges to $t^*$. Thus, $\lim_{t\to t^*} u(t)$ exists in $H$. Consequently, applying the local existence theorem at $t = t^*$, we can extend $u(t)$ up to $t = t^*$, contradicting thus the maximality of $t^*$. Therefore $t^* = \infty$. Finally by our assumption, (2.3) follows from (2.7). q.e.d.

REMARK. As $\mathscr{A}$ is a bounded from below on $\mathscr{M}$, there exists $\mu = \inf_{\mathscr{M}} \mathscr{A}(u)$. Whether there exists $u_0 \in \mathscr{M}$ such that $\mu = \mathscr{A}(u_0)$, is the so-called isoperimetric problem in some generalized sense.

THEOREM 2.3. *Assume the following.*

(1) *$\mathscr{A}$ is bounded from below on $\mathscr{M}$ and $\mathscr{B}'(u) \neq 0$ on $\mathscr{M}$.*

(2) *$\mathscr{A}^{-1}(I) \cap \mathscr{M}$ is bounded in $H$ for a bounded set $I \subset \boldsymbol{R}$.*

(3) *If a sequence $u_k$ converges to $u$ weakly and $\mathscr{X}(u_k)$ converges to $v$ strongly in $H$, then $\mathscr{X}(u) = v$.*

*Then, the solution $u(t)$ of (2.2) has a weak limit $\bar{u}$, which satisfies $\mathscr{X}(\bar{u}) = 0$. Moreover, if $\bar{u}$ belongs to $\mathscr{M}$, $\bar{u}$ is the desired critical point of $\mathscr{A}$ on $\mathscr{M}$.*

PROOF. By the preceding theorem, there exists an increasing sequence $t_k$ such that $\mathscr{X}(u(t_k))$ converges strongly to 0. On the other hand, the set $\{u(t)\}$ is bounded since $\mathscr{A}(u(t)) \subset [\mu, \mathscr{A}(u_0)]$ where $\mu = \inf_{\mathscr{M}} \mathscr{A}(u)$. Therefore, $\{u(t_k)\}$ has a weakly convergent subsequence with limit $\bar{u}$. So by (3), we have $\mathscr{X}(\bar{u}) = 0$. The last statement is obvious. q.e.d.

**3. Proof of Aubin's result.** Let us define two functionals on $H^1(M)$ by

$$\mathscr{A}(u) = \int_M (\kappa |\nabla u|^2 + R(x)u^2)d\omega\,, \qquad \mathscr{B}(u) = \int_M |u|^N d\omega \tag{3.1}$$

$$\text{where} \quad \kappa = 4(n-1)/(n-2) \quad \text{and} \quad N = 2n/(n-1)\,.$$

By the Sobolev imbedding theorem, we have $H^1(M) \subset L^N(M)$. Therefore $\mathscr{B}(u)$ is well-defined on $H^1(M)$. We denote the scalar product and the norm of $H^1(M)$ by $(u, v)_1 = (u, v) + (\nabla u, \nabla v)$ and $|||u|||_1^2 = (u, u)_1$ where $(u, v) = \int_M uv d\omega$. Moreover, we put $\|u\|_p^p = \int_M \|u\|^p d\omega$, and $\|u\|_2^2$ will be denoted simply by $\|u\|^2$.

PROPOSITION 3.1. (1) *$\mathscr{A}, \mathscr{B} \in C^2(H^1(M), \boldsymbol{R})$.*

(2) *$\mathscr{A}$ is bounded from below on $\mathscr{M} = \{u \in H^1(M) : \mathscr{B}(u) = 1\}$.*

(3) *$\mathscr{A}^{-1}(I) \cap \mathscr{M}$ is a bounded set in $H^1(M)$ for any bounded set $I \subset \boldsymbol{R}$.*

PROOF. (1) is clear as $N > 2$. By Hölder's inequality, we have

$$\mathscr{A}(u) = \kappa \|\nabla u\|^2 + \int_M Ru^2 d\omega \geqq \kappa |||u|||_1^2 - \left(\int_M |R - \kappa|^{n/2} d\omega\right)^{2/n} \quad \text{for} \quad u \in \mathscr{M}\,. \tag{3.2}$$

(2) and (3) follow at once from (3.2). q.e.d.

For $u, v \in H^1(M)$, we define

$$\begin{aligned} (\mathscr{A}'(u), v)_1 &= (d/d\varepsilon)\mathscr{A}(u + \varepsilon v)|_{\varepsilon=0} = 2\int_M (\kappa \nabla u \nabla v + Ruv)d\omega\,, \\ (\mathscr{B}'(u), v)_1 &= (d/d\varepsilon)\mathscr{B}(u + \varepsilon v)|_{\varepsilon=0} = N\int_M u|u|^{N-2} v d\omega\,. \end{aligned} \tag{3.3}$$

Or, using $(u, v)_1 = (u, (1 - \Delta)v)$, we define more explicitly

$$\mathscr{A}'(u) = 2(1 - \Delta)^{-1}(-\kappa \Delta u + Ru)\,, \quad \mathscr{B}'(u) = N(1 - \Delta)^{-1}(u|u|^{N-2})\,. \tag{3.4}$$

Here, we consider the operator $\mathscr{A}'(u)$ as an element of $H^1(M)$.

PROPOSITION 3.2. (1) *$\mathscr{B}'(u) \neq 0$ on $\mathscr{M}$.*

(2) *$\mathscr{A}', \mathscr{B}'$ are weakly continuous on $H^1(M)$.*

PROOF. (1) is obvious from the second equality of (3.3). Let $u_k$ converge weakly to $u$ in $H^1(M)$. From the first equality of (3.3), we have

$$\lim_{k \to \infty} (\mathscr{A}'(u_k) - \mathscr{A}'(u), v)_1 = 0 \quad \text{for each} \quad v \in H^1(M)\,. \tag{3.5}$$

By Rellich's theorem, we may assume that $u_k$ converges to $u$ a.e. Then, by the well-known procedure (see, for example, Lions [9, Lemma 1.3, p. 12]), we have that $u_k|u_k|^{N-2}$ converges weakly to $u|u|^{N-2}$ in $L^{N/(N-1)}(M)$. We thus have

$$(3.6)\qquad \lim_{k\to\infty} (\mathscr{B}'(u_k) - \mathscr{B}'(u), v)_1 = 0 \quad \text{for each} \quad v \in H^1(M)\,. \qquad \text{q.e.d.}$$

As before, we define

$$(3.7)\qquad \mathscr{X}(u) = -\mathscr{A}'(u) + \lambda(u)\mathscr{B}'(u)\,, \qquad \lambda(u) = (\mathscr{A}'(u), \mathscr{B}'(u))_1 |||\mathscr{B}'(u)|||_1^{-2}\,.$$

The following estimates are due to Eliasson [6].

PROPOSITION 3.3. *For $u, v \in \mathscr{M}$,*

$$(3.8)\qquad (u|u|^{N-2} - v|v|^{N-2}, u - v) = (N-1)\Big(\int_0^1 |v + s(u-v)|^{N-2}ds, (u-v)^2\Big) \leqq (N-1)\|u - v\|_N^2$$

$$(3.9)\qquad (\mathscr{X}(u) - \mathscr{X}(v), u - v)_1 \leqq -2\kappa\|\nabla(u-v)\|^2 - 2\int_M R(u-v)^2 d\omega + N|\lambda(u) - \lambda(v)|\,\|u - v\|_N + N\lambda(v)(u|u|^{N-2} - v|v|^{N-2}, u - v)\,.$$

PROPOSITION 3.4 (Aubin [4, Théorème 9]). *For any compact Riemannian manifold and for any $0 < \varepsilon \leqq 1$, there exists a constant $C_0$ such that*

$$(3.10)\qquad |u|_N^2 \leqq (\varepsilon + 4/(n(n-2)\omega_n^{2/n}))|\nabla u|^2 + C_0|u|^2\,, \quad \textit{where} \quad u \in H^1(M)\,.$$

By (2) of Proposition 3.1, there exists $\mu = \inf_{\mathscr{M}} \mathscr{A}(u) > -\infty$, and a minimizing sequence $u_k \in \mathscr{M}$, i.e.,

$$(3.11)\qquad \lim_{k\to\infty} \mathscr{A}(u_k) = \mu\,.$$

Consider an auxiliary problem

$$(3.12)\qquad (d/dt)u_k(t) = \mathscr{X}(u_k(t))\,, \qquad u_k(0) = u_k\,.$$

By Theorem 2.2, there exists a solution $u_k(t) \in C^1([0, \infty); H^1(M))$ such that

$$(3.13)\qquad \mathscr{A}(u_k) \geqq \mathscr{A}(u_k(t)) \geqq \mu\,, \qquad \int_0^\infty |||\mathscr{X}(u_k(s))|||_1^2 ds < \infty\,.$$

Choosing an arbitrary positive sequence $\varepsilon_k$ tending to 0 as $k$ tends to $\infty$, we may find $t_k$ for each $k$ in such a way that

$$(3.14)\qquad |||\mathscr{X}(u_k(t_k))|||_1 \leqq \varepsilon_k\,.$$

PROPOSITION 3.5. $\lim_{k\to\infty} \lambda(u_k(t_k)) = (2/N)\mu$.

PROOF. As $u_k(t_k) \in \mathscr{M}$, we have

$$(\mathscr{X}(u_k(t_k)), u_k(t_k))_1 = -2\mathscr{A}(u_k(t_k)) + N\lambda(u_k(t_k)) . \tag{3.15}$$

As $\{u_k(t_k)\}$ is bounded in $H^1(M)$, combining (3.11), (3.13) and (3.14), we have the desired result. q.e.d.

Taking a subsequence if necessary, we may assume that $u_k(t_k)$ converges weakly to $u$ in $H^1(M)$. By (2) of Proposition 3.2, and Proposition 3.5, we have

$$(\mathscr{X}(u), v)_1 = -(\mathscr{A}'(u), v)_1 + (2/N)\mu(\mathscr{B}'(u), v)_1 = 0 \quad \text{for any} \quad v \in H^1(M) . \tag{3.16}$$

Rewriting this, we have

$$\kappa(\nabla u, \nabla v) + (Ru, v) = \mu \int_M u|u|^{N-2} v d\omega \quad \text{for any} \quad v \in H^1(M) . \tag{3.16'}$$

This implies

$$\mathscr{A}(u) = \mu\mathscr{B}(u) . \tag{3.17}$$

The last thing we must prove is:

PROPOSITION 3.6. *$\mathscr{B}(u) = 1$ if $\mu < n(n-1)\omega_n^{2/n}$.*

PROOF. As $M$ is compact, taking a subsequence if necessary, we may suppose that $v_k$ converges strongly to $u$ in $L^2(M)$. Here we denote $u_k(t_k)$ by $v_k$ for the sake of notational simplicity.

(I) The case where $\mu < (N-1)^{-1}n(n-1)\omega_n^{2/n}$. From (3.8) and (3.9), we have

$$\begin{aligned}2\kappa|||v_k - v_m|||_1^2 - N(N-1)\lambda(v_m)\Big(\int_0^1 |v_m + s(v_k - v_m)|^{N-2} ds, (v_k - v_m)^2\Big) \\ \leqq 2\int_M (R-\kappa)(v_k - v_m)^2 d\omega + N|\lambda(v_k) - \lambda(v_m)|\,||v_k - v_m||_N \\ - (\mathscr{X}(v_k) - \mathscr{X}(v_m), v_k - v_m)_1 .\end{aligned} \tag{3.18}$$

(a) If $\mu < 0$, taking $m$ sufficiently large, we may suppose that $\lambda(v_m) < 0$. In this case, it is clear that $v_k$ forms a Cauchy sequence in $H^1(M)$. So, $v_k \in \mathscr{M}$ converges strongly to $u$ in $L^N(M)$. This means that $\mathscr{B}(u) = 1$.

(b) If $0 \leqq \mu < (N-1)^{-1}n(n-1)\omega_n^{2/n}$, taking $\varepsilon$ sufficiently small and $m$ sufficiently large, we may estimate the left hand side of (3.18) from below by

$$\{2\kappa - 2(N-1)[(N-1)^{-1}n(n-1)\omega_n^{2/n} - \varepsilon](\varepsilon + 4/(n(n-2)\omega_n^{2/n}))\}|||v_k - v_m|||_1^2 \tag{3.9}$$

so, $\{v_k\}$ forms a Cauchy sequence in $H^1(M)$, implying $\mathscr{B}(u) = 1$ as above.

(II) The case where $0 < \mu < n(n-1)\omega_n^{2/n}$. Put $\rho = \|u\|_N$. If $\rho \neq 0$, then

$$\mathscr{A}(u/\rho) = \mu\rho^{N-2} . \tag{3.20}$$

As $u/\rho \in \mathscr{M}$, by the definition of $\mu$, we have $\rho \geqq 1$. On the other hand, $\rho \leqq 1$ follows easily, therefore we have $\rho = 1$. So it suffices to prove that $u$ is not identically zero.

Putting $u = v_k$ in (3.10), we have

$$1 \leqq (\varepsilon + 4/(n(n-1)\omega_n^{2/n}))\|\nabla v_k\|^2 + C_0\|v_k\|^2 . \tag{3.21}$$

By (3.15),

$$\kappa\|\nabla v_k\|^2 = (N/2)\lambda(v_k) - \int_M Rv_k^2 d\omega - (\mathscr{K}(v_k), v_k)_1 . \tag{3.22}$$

Inserting (3.22) into (3.21) and applying the assumption $\mu < n(n-1)\omega_n^{2/n}$, we find $\delta > 0$ such that

$$\liminf_{k\to\infty} \|v_k\| \geqq \delta > 0 . \tag{3.23}$$

As $v_k$ converges strongly to $u$ in $L^2(M)$, we thus obtain $\|u\| \neq 0$. q.e.d.

Summing up the above arguments, we have:

THEOREM 3.7. *If $\mu < n(n-1)\omega_n^{2/n}$, there exists a function $u \in H^1(M)$ satisfying*

$$\kappa(\nabla u, \nabla v) + \int_M Ruv d\omega = \mu \int_M u|u|^{N-2} v d\omega \qquad \textit{for} \quad v \in H^1(M) . \tag{3.16}$$

$$\mathscr{A}(u) = \mu \quad \textit{and} \quad \mathscr{B}(u) = 1 . \tag{3.17}$$

As $u \in H^1(M)$, it follows that $|u| \in H^1(M)$, $\mathscr{A}(|u|) = \mu$ and $\mathscr{B}(|u|) = 1$. So $|u|$ is also a critical point of $\mathscr{A}$ on $\mathscr{M}$. By Theorem 2.1,

$$-\mathscr{A}'(|u|) + \lambda(|u|)\mathscr{B}'(|u|) = 0 . \tag{3.24}$$

Taking the inner product in $H^1(M)$ with $|u|$, we get easily $\lambda(|u|) = (2/N)\mu$. We may assume therefore that $|u|$ itself is a weak solution of (1.1).

The regularity of the weak solution follows from:

LEMMA 3.8 (Trudinger [11, Theorem 3]). *Let $u \in H^1(M)$ be a weak solution of* (1.1). *Then $u \in C^\infty(M)$.*

Further, strict positivity follows from the maximum principle.

LEMMA 3.9 (See, for example, Aubin [2, Lemma 6]). *If a non-negative function $v$ of class $C^2$ satisfies a differential equation $\Delta v =$*

*$vf(x, v)$, where $f(x, v) \in C^0(M \times \boldsymbol{R}, \boldsymbol{R})$, then either $v$ is strictly positive or $v$ is identically zero.*

Combining the above, we now obtain the result (a) of Aubin. q.e.d.

**4. Non-compact case.** Let $M$ be a non-compact manifold with a Riemannian metric $g$. We seek sufficient conditions for the existence of a strictly positive smooth function $u$ and a real number $\lambda$ satisfying (1.1).

We assume the following.

(i) The injective radius $\delta_0$ of $M$ is bounded away from zero.

(ii) The sectional curvature is $C^\infty$-bounded on $TM$.

(iii) For any $0 < \delta < \delta_0$, there exists a locally finite uniform covering by $B_{P_i}(\delta)$, $i \in I$. $B_{P_i}(\delta)$ is a geodesic ball of radius $\delta$ with center $P_i$. That is, there exists a constant $k = k(\delta)$ such that each point $Q \in M$ has a neighborhood which intersects at most $k$ balls of $B_{P_i}(\delta)$.

We define two functionals $\mathscr{A}$, $\mathscr{B}$ on $H^1(M)$ and the set $\mathscr{M}$ as before. Then we have:

PROPOSITION 4.1. *In addition to* (i), (ii) *and* (iii) *above, we assume that there exists a constant $\kappa'$, $0 < \kappa' < \kappa$ such that $R - \kappa' \in L^{n/2}(M)$.*

*Then we have the following*:

(1) $\mathscr{A}, \mathscr{B} \in C^2(H^1(M), \boldsymbol{R})$.

(2) *$\mathscr{A}(u)$ is bounded from below on $\mathscr{M}$. So we define $\mu = \inf_{\mathscr{M}} \mathscr{A}(u) > -\infty$.*

(3) *$\mathscr{A}^{-1}(\tilde{I}) \cap \mathscr{M}$ is bounded in $H^1(M)$ for any bounded set $\tilde{I} \subset R$.*

(4) *$\mathscr{B}'(u) \neq 0$ on $\mathscr{M}$.*

(5) *$\mathscr{A}'$ and $\mathscr{B}'$ are weakly continuous on $H^1(M)$.*

PROOF. Since $M$ is not compact, we cannot use Rellich's theorem directly. So we must check the weak continuity of $\mathscr{B}'$. Apart from it, the other statements can be proved similarly.

Let $u_k$ converge to $u$ weakly in $H^1(M)$. We want to show that $u_k|u_k|^{N-2}$ converges to $u|u|^{N-2}$ weakly in $L^{N/(N-1)}(M)$. Let $M_j$ be an increasing family of bounded sets in $M$ tending to $M$. If we restrict $u_k$ to $M_j$, there exists a subsequence $\{u_{k',j}\}$ such that it converges to $u$ a.e. in $M_j$. By the diagonal argument, for any subsequence $\{u_{k'}\}$ of $\{u_k\}$, there exists a subsequence $\{u_{k''}\}$ such that $u_{k''}|u_{k''}|^{N-2}$ converges to $u|u|^{N-2}$ weakly in $L^{N/(N-1)}(M)$. This implies that $u_k|u_k|^{N-2}$ itself converges to $u|u|^{N-2}$ weakly in $L^{N/(N-1)}(M)$. q.e.d.

THEOREM 4.2. *In addition to the above assumptions, we assume that $\mu < (N-1)^{-1}\kappa'\kappa^{-1}n(n-1)\omega_n^{2/n}$. Then, there exists a strictly positive* *function* $u \in C^\infty(M) \cap H^1(M)$, *satisfying* (1.1) *with* $\lambda = \mu$.

PROOF. The existence of solution of (1.1) follows similarly as in the previous arguments. Lemmas 3.8 and 3.9 need no change under the assumptions in (i), (ii) and (iii). Moreover, Proposition 3.4 holds without any change. (See, Aubin [4, Corollaire 5, p. 595].)

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