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THE PRINCIPLE OF SUBORDINATION AND COMPARISON THEOREMS FOR APPROXIMATION PROCESSES OF ABEL-BOUNDED ORTHOGONAL EXPANSIONS

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1. Introduction. In this paper we resume the discussion in [2] on the comparison of approximation processes on a fixed Banach space X, a problem going back to Favard [5]. We assume that any $f \in X$ is in some sense representable via an orthogonal expansion and that the approximation processes are multiplier operators. The idea of the following is to combine global multiplier methods employed in Butzer, Nessel and Trebels [2] with direct estimates as used by Zamansky [12] in the case of $(C, 1)$-bounded expansions (cf. [10]). The common root of both techniques is to be seen in the principle of subordination [1; Ch. 4]. A family of bounded linear operators $\{T(t)\}_{t>0}$ on X, i.e., $\{T(t)\} \subset [X]$, is called an approximation process if

$$\lim_{t\to 0} \| T(t)f - f\| = 0 \qquad (f \in X) .$$

If $\{A(t)\}$ is a further approximation process, Favard's comparison problem asks for estimates between the two quantities $\| T(t)f - f\|$ and $\| A(t)f - f\|$ with $f \in Y \subset X$. To describe the global comparison theorem in [2; I] let us repeat the notation of [2; III]. Let H be a Hilbert space, $\{P_k\}_{k=0}^{\infty}$ a sequence of mutually orthogonal projections which are continuous and complete on H, i.e., $P_kP_j = \delta_{jk}P_k$, $j, k \in N_0$; $P_k \subset [H]$, $k \in N_0$; $f = \sum_{k=0}^{\infty} P_kf$, $f \in H$. Further we assume that $P_k(H) \subset X$, $k \in N_0$, and that the set of polynomials $\Pi = \{\sum_{k=0}^{n} f_k : f_k \in P_k(H), n \in N_0\}$ is dense in X. Any (complex valued) sequence $\lambda = (\lambda_k)_{k=0}^{\infty}$ then generates an operator L of factor type on Π defined via

$$Lf = \sum_{k=0}^{\infty} \lambda_k P_k f \qquad (f \in \Pi) .$$

We call λ a (bounded) multiplier on X, notation $\lambda \in M(X)$, if $L \in [X]$ and define its multiplier norm via the operator norm of L

$$(1.1) \qquad \|\lambda\|_M = \inf \{C : \|Lf\| \leqq C, f \in \Pi, \|f\| \leqq 1\} .$$

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For approximation processes of multiplier type, i.e.,

$$T(t)f = \sum \tau_k(t)P_kf \ , \qquad A(t)f = \sum \alpha_k(t)P_kf \qquad (f \in \Pi) \ ,$$

the following result is proved in [2; I]

$$\| T(t)f - f\| \leqq \| ((\tau_k(t) - 1)/(\alpha_k(t) - 1))_k \|_M \| A(t)f - f\| \ . \tag{1.2}$$

Giving bounds on the multiplier norm (1.1) in the actual problem in the applications. Sufficient multiplier criteria can be derived from summability properties of the orthogonal expansion; it the case of Cesàro-boundedness see e.g. [2; I, II] or of Abel-boundedness [2; III]. For the following we assume Abel-boundedness, i.e., that

$$\| \sum e^{-tk}P_kf \| \leqq C_A \| f \| \qquad (f \in \Pi) \tag{1.3}$$

holds uniformly in $t > 0$, which is quite a weak property of the expansion. Examples of Abel-bounded orthogonal expansions are the Hermite series in weighted L^p-spaces

$$L^p_w(\boldsymbol{R}) = \left\{f\colon \| f \|_{p,w} = \left(\int_{-\infty}^{\infty} |f(x)|^p e^{-x^2}dx\right)^{1/p} < \infty\right\}$$

which are only Abel-bounded and not Cesàro-bounded of any order as well as Laguerre series in appropriate weighted L^p-spaces; for details see e.g. [2; III]. There it is also shown that, if (1.3) holds then

$$\| \lambda \|_M \leqq C \| m \|_{CBV} \ , \tag{1.4}$$

where λ is the restriction to the nonnegative integers of a function $m \in CBV$; here the set of functions of completely bounded variation is defined by

$$\begin{aligned} CBV = \Big\{ & m \in L^\infty(0, \infty)\colon \lim_{x\to\infty} m(x) = 0 \\ & \text{and for some } g \in BV[0, \infty),\ m(x) = \int_0^\infty e^{-xs}dg(s)\Big\} \ , \end{aligned} \tag{1.5}$$

$$\| m \|_{CBV} = \int_0^\infty |dg| \ .$$

REMARK 1. We mention two equivalent characterizations of CBV. The first one (see [11; p. 306]),

$$\frac{1}{\Gamma(k)} \int_0^\infty x^{k-1} |m^{(k)}(x)| dx \leqq M \tag{1.6}$$

for some $M > 0$ and all $k \in N$, shows nicely the connection with the BV_k-classes in [2; II], which play an important role in multiplier theory for Cesàro bounded expansions. The second is often more practicable;

$m \in CBV$ if and only if m is the difference of two completely monotone functions on $[0, \infty)$. Here $n(x)$ is completely monotone on $[0, \infty)$ if $n \in C[0, \infty)$ and $(-1)^k n^{(k)}(x) \geqq 0$ for all $0 < x < \infty$, $k \in N_0$.

2. By a result of Schoenberg $m(x) \in CBV$ implies $m(x^\kappa) \in CBV$, $0 < \kappa \leqq 1$ (see [2; III]).

3. If a family of sequences $\{\lambda(t)\}_{t>0}$ is of Fejèr's type, i.e., $\lambda_k(t) = m(tk)$ for all $t > 0$, $k \in N_0$ and if $m \in CBV$, then it is obvious that $\|\lambda(t)\|_M \leqq C\|m\|_{CBV}$ holds uniformly in $t > 0$.

There are two disadvantages in the approach via bounded multipliers. (i) The verification of (1.4) is mostly quite hard, (ii) the method gives only comparison theorems on the whole Banach space X. In some situations we can avoid these difficulties by making use of the principle of subordination, i.e., in those situations where the multiplier operator L is subordinated to the Abel means:

$$\lambda_k = \int_0^\infty e^{-ks} dm(s) , \qquad Lf = \int_0^\infty A_1(s) f dm(s) \qquad (f \in \Pi \backslash P_0(H)) ,$$

for a suitable function m of locally bounded variation on $(0, \infty)$.

2. **Comparison theorems.** To illustrate this idea let us consider for $0 < \kappa \leqq 1$ the Abel-Cartwright means

$$A_\kappa(t)f = \sum e^{-tk^\kappa} P_k f \qquad (f \in \Pi) \tag{2.1}$$

and the generalized Picard means

$$P_\kappa(t)f = \sum (1 + tk^\kappa)^{-1} P_k f \qquad (f \in \Pi) . \tag{2.2}$$

By Remarks 2 and 3 and the hypothesis (1.3) it is clear that $A_\kappa(t)$ is a family of uniformly bounded linear transformations on X; the same is true for $P_\kappa(t)$ since

$$\frac{1}{1+x} = \int_0^\infty e^{-xs} e^{-s} ds .$$

Motivated by results for one- and more- dimensional Fourier series one can expect that the approximation behavior of $A_\kappa(t)$ and $P_\kappa(t)$ is the same. With the following generalization of Shapiro's [8; p. 219] σ-modulus of continuity

$$\| T(t)f - f\|^* = \sup_{s \leqq t} \| T(s)f - f\| , \tag{2.3}$$

where $\{T(t)\}_{t>0}$ is an approximation process on X, we have:

THEOREM 1. *There exist constants C, K such that for all $f \in X$, $t > 0$*

(a) $C \leqq \| A_\kappa(t)f - f\|^*/\| P_\kappa(t)f - f\|^* \leqq K \quad (0 < \kappa \leqq 1)$;

(b) $\| A_1(t)f - f\|^* \leqq C \| A_\kappa(t^\kappa)f - f\|^* \leqq K \| A_\gamma(t^\gamma)f - f\|^*$ $(0 < \gamma < \kappa < 1)$.

PROOF. (a) To establish the right hand inequality, set $t = \tau^\kappa$. Then, by Remarks 2, 3 and (1.2) it is sufficient to show that $m(x) = (e^{-x} - 1)/(1/(1 + x) - 1) \in CBV$ which is obvious since

$$m(x) = (1 - e^{-x}) + \int_0^1 e^{-xs} ds .$$

Conversely, we have for all $f \in \Pi$

$$P_\kappa(t)f - f = \kappa \int_0^\infty \sum (e^{-\tau^\kappa k^\kappa s^\kappa} - 1) P_k f e^{-s^\kappa} s^{\kappa-1} ds$$
$$= \kappa \int_0^\infty (A_\kappa(ts^\kappa)f - f) e^{-s^\kappa} s^{\kappa-1} ds$$

and therefore, by a standard argument,

$$\| P_\kappa(t)f - f\|^* \leqq \kappa \int_0^\infty C_A(1 + s^\kappa) \| A_k(t)f - f\|^* e^{-s^\kappa} s^{\kappa-1} ds$$
$$\leqq C^{-1} \| A_\kappa(t)f - f\|^* .$$

(b) By an application of (1.2) and Remark 2 it follows that

$$\| P_\alpha(t^\alpha)f - f\| \leqq C \| P_\beta(t^\beta)f - f\| \qquad (0 < \beta < \alpha \leqq 1) , \tag{2.4}$$

since with $t^\alpha = \tau$

$$m(x) - 1 = \left(\frac{1}{1 + x} - 1\right) \Big/ \left(\frac{1}{1 + x^{\beta/\alpha}} - 1\right) - 1 = \frac{x^{1-\beta/\alpha}}{1 + x} - \frac{1}{1 + x} \in CBV$$

(cf. [2; III]). A combination of (2.4) with (a) gives the assertion. Observe that showing (b) and the left inequality of (a) directly via (1.2) would have led us to quite complicated verifications, e.g. that $((e^{-tk}-1)/(e^{-t^\kappa k^\kappa}-1))$ is a uniformly bounded multiplier family. Theorem 1, (b) gives us in particular:

COROLLARY 2. *If $f \in X$ is such that*

$$\| A_\kappa(t)f - f\|^* = 0(t^\gamma) , \qquad t \to 0+ , \quad 0 < \gamma \leqq 1 , \quad 0 < \kappa < 1 ,$$

then

$$\| A_1(t)f - f\|^* = 0(t^{\kappa\gamma}) , \qquad t \to 0+ .$$

Obviously, the converse of Corollary 2 cannot be deduced via bounded multiplier techniques. Nevertheless we can show:

THEOREM 3. *If $f \in X$ is such that*

$$\| A_1(t)f - f \|^* = 0(t^\gamma) , \qquad t \to 0+ , \quad 0 < \gamma \leqq 1$$

then, for $0 < \kappa < 1$, *we have*

$$\| A_\kappa(t)f - f \|^* = 0(\varphi_{\gamma,\kappa}(t)) , \qquad t \to 0+ ,$$

where $\varphi_{\gamma,\kappa}(t) = t^{\gamma/\kappa}$ *or* $t \ln 1/t$ *or* t *if* $\gamma < \kappa$ *or* $\gamma = \kappa$ *or* $\gamma > \kappa$, *resp.*

The proof easily follows in the case $\kappa = 1/2$, since (cf. [9; p. 61])

$$e^{-\sqrt{x}} = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-s}}{\sqrt{s}} e^{-x/4s} ds , \qquad x \geqq 0$$

implies

$$\| A_{1/2}(t)f - f \| = \left\| \frac{1}{\sqrt{\pi}} \int_0^\infty \sum (e^{-kt/4s} - 1) P_k f \frac{e^{-s}}{\sqrt{s}} ds \right\|$$
$$\leqq C \int_0^\infty \frac{e^{-s}}{\sqrt{s}} \left\| A_1\Big(\frac{t}{4s}\Big) f - f \right\| ds$$

whose evaluation readily gives the assertion in the case $\kappa = 1/2$. The general case is based upon the following representation (see [6])

$$e^{-tx^{\alpha/2}} = \int_0^\infty e^{-x/4u} \frac{1}{\sqrt{u}} d\sigma(u, t; \alpha) , \qquad 0 < \alpha < 2 , \tag{2.5}$$

where $\sigma(u, t; \alpha)$ is a bounded nondecreasing function of u, $0 \leqq u < \infty$. Hence

$$A_\kappa(t)f - f = \int_0^\infty \sum (e^{-k/4u} - 1) P_k f \frac{1}{\sqrt{u}} d\sigma(u, t; 2\kappa)$$

and therefore, by the hypothesis and (1.3),

$$\| A_\kappa(t)f - f \| \leqq \int_0^\infty \left\| A_1\Big(\frac{1}{4u}\Big) f - f \right\| \frac{1}{\sqrt{u}} d\sigma(u, t; 2\kappa)$$
$$\leqq C \int_0^\infty \min\{1, u^{-\gamma}\} \frac{1}{\sqrt{u}} d\sigma(u, t; 2\kappa)$$

from which the assertion follows by:

LEMMA 4. *With* σ *as in* (2.5) *and* $\varphi_{\gamma,\kappa}$ *as in Theorem* 3 *there holds*

$$\int_0^\infty u^{-1/2} \min\{1, u^{-\gamma}\} d\sigma(u, t; 2\kappa) \leqq C \varphi_{\gamma,\kappa}(t) .$$

PROOF. First introduce the Fourier transformation F on $L^1(\boldsymbol{R})$ by

$$f^\wedge(\xi) = F[f](\xi) = \int_{-\infty}^\infty f(x) e^{-i\xi x} dx , \qquad \xi \in \boldsymbol{R} , \quad f \in L^1(\boldsymbol{R})$$

and denote by F^{-1} the inverse Fourier transformation. We note that

$$F^{-1}[f^\wedge(t\xi)](x) = t^{-1}f(x/t) \quad \text{and} \quad (\text{cf. [6]})$$

$$0 \leqq F^{-1}[e^{-|\xi|^\kappa}](x) = 0((1+|x|^{1+\kappa})^{-1}), \qquad x \in \boldsymbol{R}. \tag{2.6}$$

Then it follows from (2.6) that for $0 < t \leqq 1$ there holds

$$\begin{aligned}
&\int_0^\infty \min\{x^\gamma, 1\} F^{-1}[e^{-t|\xi|^\kappa}](x)dx \\
&\leqq t^{\gamma/\kappa}\int_0^1 (t^{-1/\kappa}x)^\gamma F^{-1}[e^{-|\xi|^\kappa}](t^{-1/\kappa}x)t^{-1/\kappa}dx \\
&\quad + C\int_1^\infty (1+(t^{-1/\kappa}x)^{1+\kappa})^{-1}t^{-1/\kappa}dx \\
&\leqq t^{\gamma/\kappa}\int_0^1 y^\gamma F^{-1}[e^{-|\xi|^\kappa}](y)dy + Ct^{\gamma/\kappa}\int_1^{t^{-1/\kappa}} y^{\gamma-1-\kappa}dy + 0(t) \\
&= 0(t^{\gamma/\kappa}) + 0(\varphi_{\gamma,\kappa}(t)) + 0(t).
\end{aligned}$$

Hence an interchange of integration and a substitution lead to

$$\begin{aligned}
C\varphi_{\gamma,\kappa}(t) &\geqq \int_0^\infty \min\{x^\gamma, 1\}\int_0^\infty \frac{4u}{1+16u^2x^2}\frac{1}{\sqrt{u}}d\sigma(u, t; 2\kappa)dx \\
&= K\int_0^\infty u^{-\gamma-1/2}\int_0^{4u}\frac{y^\gamma}{1+y^2}dyd\sigma(u, t; 2\kappa) + \int_0^\infty u^{-1/2}\int_{4u}^\infty \frac{dy}{1+y^2}d\sigma(u, t; 2\kappa) \\
&\geqq K'\left\{\int_0^{1/4} u^{-1/2}d\sigma(u, t; 2\kappa) + \int_{1/4}^\infty u^{-\gamma-1/2}d\sigma(u, t; 2\kappa)\right\} \\
&\geqq K''\int_0^\infty u^{-1/2}\min\{1, u^{-\gamma}\}d\sigma(u, t; 2\kappa)
\end{aligned}$$

for some constant $K'' > 0$, i.e., the assertion.

REMARK 4. If one replaces s by s^β, $0 < \beta < 1$, in (2.5) and proceeds as above then one obviously obtains analogous results for the comparison of $\|A_\kappa(t)f - f\|^*$ and $\|A_\beta(t)f - f\|^*$. Furthermore, by the same procedure, more general orders of approximation can be treated, e.g. $t^\gamma|\log t|^\beta$.

5. In [7] approximation processes in a Banach space are constructed via approximate identities in $L^1_{2\pi}$ and related results (direct theorems) are derived.

3. Bernstein- and Zamansky-type inequalities. Here we indicate how the same method works in the case of Bernstein- and Zamansky-type inequalities. To this end we need the analog of a derivative. In [2; III] it is already shown that the factor type operator

$$D^\alpha f = \sum k^\alpha P_k f \qquad (f \in \Pi, \alpha > 0)$$

can be extended to a closed linear operator with domain $D(D^\alpha) \subset X$; e.g. in the case of Hermite series in weighted L^p-spaces D^1 has the representation $D^1 = (1/2)\ (d/dx)^2 - x(d/dx)$.

Now observe that, by [3; p. 246 (9)],

$$x^{\nu-1/2}e^{-\sqrt{x}} = \int_0^\infty e^{-xs}2^{-\nu}\pi^{-1/2}s^{-\nu-1/2}e^{-1/8s}D_{2\nu}(1/\sqrt{2s})ds \tag{3.1}$$

holds for $\nu > 0$. Here the parabolic cylinder function $D_{2\nu}(z)$ is an entire function (see [4; p. 117]) which for $|z|\to\infty$, $-3\pi/4 < \arg z < 3\pi/4$ behaves like [4; p. 122(1)]

$$D_{2\nu}(z) = z^{2\nu}e^{-z^2/4}\{1 + 0(|z|^{-2})\} .$$

Hence, for $\nu > 1/2$, we have for all $s > 0$

$$D_{2\nu}(1/\sqrt{2s}) = 0(1) \tag{3.2}$$

and therefore, for $0 < \kappa \leqq 1$ and $\alpha > 0$,

$$\begin{aligned}&\sum (tk)^{\alpha\kappa}e^{-(tk)^{\kappa/2}}P_kf\\ &\quad = C\int_0^\infty (\sum e^{-(tk)^\kappa s}P_kf)s^{-\alpha-1}e^{-1/8s}D_{2\alpha+1}(1/\sqrt{2s})ds ,\end{aligned}$$

which implies, by (1.3), Remarks 2, 3 and (3.2), that

$$\|D^{\alpha\kappa}A_{\kappa/2}(t^{\kappa/2})f\| \leqq Ct^{-\alpha\kappa}\|f\| .$$

We have thus proved part (a) of:

THEOREM 5. *Let $\alpha > 0$ and $0 < \kappa \leqq 1/2$.*

(a) *$A_\kappa(t)(X)$ is in the domain of D^α for each $t > 0$ and*

$$\|D^\alpha A_\kappa(t)f\| \leqq Ct^{-\alpha/\kappa}\|f\| \qquad (f\in X) .$$

(b) *For $\alpha > 1/2$ we even have*

$$\|D^\alpha A_\kappa(t)f\| \leqq Ct^{-\alpha/\kappa}\|A_\kappa(t)f - f\|^* .$$

PROOF. (b) From (3.1) we see

$$0 = C\int_0^\infty s^{-\nu-1/2}e^{-1/8s}D_{2\nu}(1/\sqrt{2s})ds \qquad (\nu > 1/2) ;$$

hence,

$$\sum t^{\alpha/\kappa}k^\alpha e^{-tk^\kappa}P_kf = C\int_0^\infty \sum (e^{-k^{2\kappa}t^2s} - 1)P_kfs^{-\alpha-1}e^{-1/8s}D_{2\alpha+1}(1/\sqrt{2s})ds$$

and therefore, by Theorem 1, (b), the assertion

$$\begin{aligned}\|D^\alpha A_\kappa(t)f\| &\leqq Ct^{-\alpha/\kappa}\int_0^\infty \|A_\kappa(t\sqrt{s})f - f\|s^{-\alpha-1}e^{-1/8s}ds\\ &\leqq Ct^{-\alpha/\kappa}\|A_\kappa(t)f - f\|^*\int_0^\infty (1 + \sqrt{s})s^{-\alpha-1}e^{-1/8s}ds ,\end{aligned} \tag{3.3}$$

since the latter integral converges for $\alpha > 1/2$.

In the case that $A_\kappa(t)f$ converges to f with a certain order, say

$$\|A_\kappa(t)f - f\| = 0(t^\gamma) \qquad (t \to 0+, 0 < \gamma \leqq 1)\,,$$

the restriction $\alpha > 1/2$ in Theorem 5, (b) can be dropped. For, by (3.3),

$$\|D^\alpha A_\kappa(t)f\| \leqq Ct^{-\alpha/\kappa}\int_0^\infty \min\{(t\sqrt{s})^\gamma, 1\}s^{-\alpha-1}e^{-1/8s}ds$$

$$\leqq Ct^{-\alpha/\kappa}\varphi_{\gamma,2\alpha}(t^{2\alpha})\,, \tag{3.4}$$

where $\varphi_{\gamma,2\alpha}$ has the same meaning as in Theorem 3.

In the case $\alpha = \kappa$, a certain converse of (3.4) can easily be derived; for, by Theorem 1, (a), there holds

$$\begin{aligned}\|A_\kappa(t)f - f\|^* &\leqq C\|P_\kappa(t)f - f\|^* \\ &= Ct\left\|\int_0^\infty \sum k^\kappa e^{-stk^\kappa}P_kfe^{-s}ds\right\|^* \\ &\leqq Ct\int_0^\infty \left\|D^\kappa A_\kappa(st)f\right\|^* e^{-s}ds \leqq C\varphi_{\gamma,2\alpha}(t^{2\alpha})\,.\end{aligned}$$

The above results give an impression how one can use the principle of subordination to derive, with simple means, a series of approximation theorems. We emphasize that no use is made of the Plancherel-L^2-theory (as a main source of multiplier criteria), that in concrete situations it may be more convenient to avoid multiplier techniques, and that they alone do not seem to be appropriate for the type of results obtained.

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