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THE DEGREES OF THE STANDARD IMBEDDINGS OF R-SPACES

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1. Introduction. R-spaces constitute an important class of homogeneous submanifolds in the Euclidean spheres: they are the orbits of the isotropy representations of symmetric spaces of noncompact type (cf. Takeuchi and Kobayashi [6]). This class includes many examples appearing in differential geometry of submanifolds. For example, all homogeneous hypersurfaces and all parallel submanifolds in spheres are realized as R-spaces.

Ferus [1] showed that the standard imbeddings of symmetric R-spaces have the parallel second fundamental forms and exhaust all submanifolds in spheres with the parallel second fundamental forms. So the following arises as a natural problem:

Problem. *Characterize the standard imbedding of each R-space in the sense of differential geometry.*

The first step in answering the Problem is to find many differential geometric properties of the standard imbeddings of R-spaces. In Kitagawa and Ohnita [3] we showed that the standard imbedding of every R-space has the parallel mean curvature vector.

Let M^n be a compact rank one symmetric space, that is, one of the following: S^n, $\boldsymbol{RP}^n$, $\boldsymbol{CP}^n$, $\boldsymbol{QP}^n$ and $\boldsymbol{CayP}^{16}$. Let f_k be the standard minimal isometric immersion of M^n into a sphere $S^{m(k)}$ induced by the k-th eigenfunctions of the Laplace-Beltrami operator of M^n (cf. Wallach [7]). If $k=1$, the immersion f_k is just the standard imbedding of a compact symmetric R-space of rank one. It is called a generalized Veronese submanifold except when M^n is a sphere. Wallach used the notion of its degree in studying the rigidity of a minimal isometric immersion. The degree of f_k coincides with k (cf. Wallach [7]) if M^n is a sphere, and with $2k$ (cf. Mashimo [4], [5]) otherwise. In particular, the degree of a generalized Veronese submanifold is 2.

In this note we show the following theorems:

Let Φ be the proper standard imbedding (cf. §2) of an R-space K/L.

THEOREM A. *The degree of Φ is equal to* 2.

THEOREM B. *If Φ is regular* (cf. §2), *then there exists a normal* *frame field* $\{\tilde{\xi}_1, \cdots, \tilde{\xi}_p\}$ *defined globally on* K/L *such that each* $\tilde{\xi}_\alpha$ $(\alpha = 1, \cdots, p)$ *is parallel with respect to the normal connection of* Φ. *In particular, the normal connection of* Φ *is flat.*

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2. **Preliminaries.** Let $(\mathfrak{g}, \theta)$ be an orthogonal symmetric Lie algebra of noncompact type (cf. Helgason [2]). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k} = \{X \in \mathfrak{g}; \theta(X) = X\}$ and $p = \{X \in \mathfrak{g}; \theta(X) = -X\}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Let K be the analytic subgroup of the inner automorphism group $\mathrm{Int}(\mathfrak{g})$ corresponding to the Lie algebra $\mathrm{ad}(\mathfrak{k})$. $\mathfrak{p}$ is invariant under K. Let ψ denote the Killing form of $\mathfrak{g}$ and we consider $(\mathfrak{p}, \langle\ ,\ \rangle)$ as a Euclidean space, where $\langle\ ,\ \rangle = \psi|_{\mathfrak{p}}$. Put $S = \{X \in \mathfrak{p}; \langle X, X\rangle = 1\}$ and also denote by the same $\langle\ ,\ \rangle$ the Riemannian metric on S induced by $\langle\ ,\ \rangle$. We denote by ∇^0 and $\bar{\nabla}$ the Riemannian connection of $\mathfrak{p}$ and S, respectively.

For an arbitrarily fixed element H_0 in $S \cap \mathfrak{a}$, put $L = \{k \in K; k(H_0) = H_0\}$ and an imbedding $\Phi: K/L \to S$ by $\Phi(kL) = k(H_0)$. Then $M = K/L$ is called an *R-space* and Φ its *standard imbedding*. If $\mathrm{rank}(\mathfrak{g}, \theta) = 1$, Φ is the identity map. Φ is said to be *proper* if $\mathrm{rank}(\mathfrak{g}, \theta) \geqq 2$ and $\Phi(M)$ is not a great sphere of S. We can show that if $\Phi(M)$ is a great sphere of S then there are two orthogonal symmetric Lie algebras $(\mathfrak{g}_1, \theta_1)$ and $(\mathfrak{g}_2, \theta_2)$ such that (1) $(\mathfrak{g}, \theta) = (\mathfrak{g}_1, \theta_1) \oplus (\mathfrak{g}_2, \theta_2)$, (2) $\mathrm{rank}(\mathfrak{g}_1, \theta_1) = 1$, (3) $H_0 \in \mathfrak{p}_1 = \{X \in \mathfrak{g}_1; \theta_1(X) = -X\}$.

For an $\boldsymbol{R}$-linear form λ on $\mathfrak{a}$, we put $\mathfrak{g}_\lambda = \{X \in \mathfrak{g}; (\mathrm{ad}\, H)X = \lambda(H)X$ for all $H \in \mathfrak{a}\}$. If $\mathfrak{g}_\lambda \neq \{0\}$, then λ is called a root of $(\mathfrak{g}, \theta)$ with respect to $\mathfrak{a}$. Let Δ be the set of all nonzero roots on $\mathfrak{a}$. We put $\psi_\theta(X, Y) = \psi(X, \theta(Y))$ for $X, Y \in \mathfrak{g}$. ψ_θ is a negative definite symmetric bilinear form on $\mathfrak{g}$. Then $\mathfrak{g}$ has the following orthogonal direct sum with respect to $\psi_\theta: \mathfrak{g} = \mathfrak{g}_0 + \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$. For an element $H \in \mathfrak{a}$, we put $\Delta_H = \{\lambda \in \Delta; \lambda(H) \neq 0\}$. If $\Delta = \Delta_H$, then H is called a regular element in $\mathfrak{a}$.

The standard imbedding Φ is called *regular* if H_0 is a regular element.

We now explain the notion of the degree of an isometric imbedding Φ of a Riemannian manifold $(M, \Phi^*\langle\ ,\ \rangle)$ into a Euclidean sphere S. We denote by B the second fundamental form of Φ and let A be the shape operator of Φ defined by $\langle A_\xi X, Y\rangle = \langle B(X, Y), \xi\rangle$ for $X, Y \in T_x(M)$ and $\xi \in T_x(M)^\perp$. For $x \in M$, put $O_x^2(M) = \mathrm{span}_{\boldsymbol{R}}\{B(X, Y): X, Y \in T_x(M)\}$ and let N_2 be the orthogonal projection of $T_x(M)^\perp = O_x^2(M) \oplus (O_x^2(M))^\perp$ onto $(O_x^2(M^\perp))$, where $(O_x^2(M))^\perp$ is the orthogonal complement of $O_x^2(M)$ in $T_x(M)^\perp$. Let

$\mathscr{R}_1 = M$ and $\mathscr{R}_2 = \{x \in M; \dim O_x^2(M)$ is maximal in $\mathscr{R}_1\}$. We define a symmetric 3-tensor field B_3 on $\mathscr{R}_2$ by $(B_3)(X, Y, Z) = N_2((\nabla_X^* B)(Y, Z))$ for $X, Y, Z \in T_x(M)$, where $(\nabla_X^* B)(Y, Z) = \nabla_X^{\perp}(B(\tilde{Y}, \tilde{Z})) - B(\nabla_X \tilde{Y}, Z) - B(Y, \nabla_X \tilde{Z})$. Here $\nabla^{\perp}$ is the normal connection of Φ and $\tilde{Y}, \tilde{Z}$ are vector fields defined locally around x with $(\tilde{Y})_x = Y$, $(\tilde{Z})_x = Z$. B_3 is called the third fundamental form of Φ. We can define $O_x^j(M)$, $\mathscr{R}_j$, B_j for $j = 2, 3, \cdots$, recursively. We call B_j *the j-th fundamental form* of Φ. There exists a natural number d such that $B_d \not\equiv 0$ on $\mathscr{R}_d$ and $B_{d+1} \equiv 0$ on $\mathscr{R}_d$. We call d the *degree* of Φ.

For example, $d = 1$ means that Φ is totally geodesic. The degree of Φ is 2 if Φ has the parallel second fundamental form.

3. Proof of Theorems. Let $\mathfrak{l}$ be the Lie algebra of L and $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{k}$ with respect to ψ_θ. We identify the tangent space $T_o(M)$ at the origin $o = \{L\} \in M = K/L$ with $\mathfrak{m}$. Then the differential $\Phi_*: T_o(M) \to T_{\Phi(o)}(\mathfrak{p})$ is identified with the mapping $-\mathrm{ad}(H_0): \mathfrak{m} \to \mathfrak{p}$. Put $\mathfrak{p}_0 = \Phi_*(T_o(M)) = (-\mathrm{ad}(H_0))\mathfrak{m}$ and let $\mathfrak{n}$ be the orthogonal complement of $\mathrm{span}_R\{H_0\} + \mathfrak{p}_0$ in $\mathfrak{p}$ with respect to ψ_θ. Then the normal space $T_o(M)^{\perp}$ at the origin of M in S is identified with $\mathfrak{n}$. We have an orthogonal decomposition of $\mathfrak{p}$:

$$\mathfrak{p} = \mathrm{span}_R\{H_0\} + \mathfrak{p}_0 + \mathfrak{n} .$$

Now we put

$$\mathfrak{h} = \sum_{\lambda \in \Delta_{H_0}} \mathfrak{g}_\lambda , \qquad \mathfrak{b} = \mathfrak{g}_0 + \sum_{\lambda \in \Delta - \Delta_{H_0}} \mathfrak{g}_\lambda .$$

$\mathfrak{h}$ and $\mathfrak{b}$ are invariant by θ since $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$. Hence we have $\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}) + (\mathfrak{b} \cap \mathfrak{k}) + (\mathfrak{b} \cap \mathfrak{p})$. Then it is easy to show the following:

$$\mathfrak{l} = \mathfrak{b} \cap \mathfrak{k} , \quad \mathfrak{m} = \mathfrak{h} \cap \mathfrak{k} , \quad \mathfrak{p}_0 = \mathfrak{h} \cap \mathfrak{p} , \qquad (3.1)$$
$$\mathrm{span}_R\{H_0\} + \mathfrak{n} = \mathfrak{b} \cap \mathfrak{p} .$$

By $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, $[\mathfrak{h}, \mathfrak{b}] \subset \mathfrak{h}$ and (3.1), we have

$$(\mathrm{ad}(\mathfrak{m}))\mathfrak{n} \subset \mathfrak{p}_0 . \qquad (3.2)$$

LEMMA. *Let X be an element of $\mathfrak{m}$ and ξ an element of $\mathfrak{n}$. We put $x_t = (\exp(tX)) \cdot o \in M$. If a normal vector field ξ_t along x_t is defined by $\xi_t = (\exp(tX)) \cdot \xi \in T_{x_t}(M)^{\perp}$ then we have $\nabla_t^{\perp} \xi_t = 0$.*

PROOF. $\nabla_t^0 \xi_t = (d/dt)\xi_t = (\exp(tX)) \cdot [X, \xi]$. By (3.2) and $\bar{\nabla}_t \xi_t = \nabla_t^0 \xi_t + \langle \dot{x}_t, \xi_t \rangle x_t = \nabla_t^0 \xi_t$, we have $\bar{\nabla}_t \xi_t \in (\exp(tX)) \cdot \mathfrak{p}_0 = \Phi_*(T_{x_t}(M))$. By Weingarten's fomula we obtain $\nabla_t^{\perp} \xi_t = 0$. q.e.d.

PROOF OF THEOREM A. If Φ is proper, then $B \not\equiv 0$. Since Φ is K-

equivariant, we have $\mathscr{R}_2 = M$. Let X, Y and Z be elements of $\mathfrak{m} = T_o(M)$. Put $x_t = (\exp(tX))\cdot o$, $Y_t = (\exp(tX))\cdot Y \in T_{x_t}(M)$ and $Z_t = (\exp(tX))\cdot Z \in T_{x_t}(M)$. Then by the K-equivariance of Φ we have $B_{x_t}(Y_t, Z_t) = (\exp(tX))\cdot B(Y, Z)$. Applying the Lemma to $\xi = B(X, Y) \in \mathfrak{n}$, we have $\nabla_t^{\perp}(B(Y_t, Z_t)) = 0$. Hence $(\nabla_X^* B)(Y, Z) = -B(\nabla_X Y_t, Z) - B(Y, \nabla_X Z_t) \in O_o^2(M)$. From the definition of the third fundamental form B_3 we have $(B_3)_o = 0$. B_3 vanishes everywhere by the homogeneity of M and the equivariance of Φ. q.e.d.

REMARK. If $(\mathfrak{k}, \mathfrak{l})$ is a symmetric Lie algebra, then Y_t and Z_t are parallel along x_t with respect to the Riemannian connection of M. From the above proof we have $\nabla^* B = 0$.

PROOF OF THEOREM B. It is known that the centralizer in K of a regular element of $\mathfrak{a}$ coincides with the centralizer in K of $\mathfrak{a}$ (for example, see Helgason [2, p. 289]). Thus L is the centralizer in K of $\mathfrak{a}$. Since $\Delta = \Delta_{H_0}$, we have $\mathfrak{b} = \mathfrak{g}_0$. By (3.1) and the maximality of $\mathfrak{a}$, we have $\mathrm{span}_R\{H_0\} + \mathfrak{n} = \mathfrak{b} \cap \mathfrak{p} = \mathfrak{a}$. Hence the action of L on $\mathfrak{n} = T_o(M)^{\perp}$ is trivial. Select an orthonormal basis $\{\xi_1, \cdots, \xi_p\}$ of $\mathfrak{n}$. We can extend ξ_α to a K-invariant normal vector field $\tilde{\xi}_\alpha$ defined globally on M. By Lemma we have $\nabla^{\perp}\tilde{\xi}_\alpha = 0$. Thus $\{\tilde{\xi}_1, \cdots, \tilde{\xi}_p\}$ is the desired normal frame field. q.e.d.

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