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ACTIONS OF SYMPLECTIC GROUPS ON CERTAIN MANIFOLDS

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0. Introduction. In previous papers [12], [14] smooth actions of special unitary (resp. symplectic) groups on a product of complex (resp. quaternion) projective spaces have been studied. Here we shall study smooth actions of symplectic group $\boldsymbol{Sp}(n)$ on certain product manifolds and we shall prove the following.

THEOREM. *Let X be a closed orientable manifold on which $\boldsymbol{Sp}(n)$ acts smoothly and non-trivially. Suppose $n \geqq 7$.*

(i) *Suppose $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$, $1 \leqq b \leqq a < 2n$, and $a + b \leqq 4n - 3$. Then $a = 2n - 1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_0$, where Y_0 is a closed orientable manifold such that $Y_0 \sim \boldsymbol{P}_b(\boldsymbol{C})$, and $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$ and trivially on Y_0.*

(ii) *Suppose $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{C})$, $1 \leqq a \leqq n - 1$, $1 \leqq b \leqq 2n - 1$, and $2a + b \leqq 4n - 4$. Then there are three cases:*

(a) *$a = n - 1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{n-1}(\boldsymbol{H}) \times Y_1$, where Y_1 is a closed orientable manifold such that $Y_1 \sim \boldsymbol{P}_b(\boldsymbol{C})$, and $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and trivially on Y_1,*

(b) *$b = 2n - 1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_2$, where Y_2 is a closed orientable manifold such that $Y_2 \sim \boldsymbol{P}_a(\boldsymbol{H})$, and $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$ and trivially on Y_2,*

(c) *$b = 2n - 1$ and X is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/\boldsymbol{Sp}(1)$, where Y_3 is a closed orientable $\boldsymbol{Sp}(1)$ manifold such that $Y_3 \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$, $\boldsymbol{Sp}(1)$ acts as right scalar multiplication on S^{4n-1}, the unit sphere of $\boldsymbol{H}^n$, and $\boldsymbol{Sp}(n)$ acts naturally on S^{4n-1} and trivially on Y_3. In addition, $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$ and the induced homomorphism $i^*: H^2(Y_3) \to H^2(F)$ is trivial, where F denotes the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y_3. Conversely, if Y_3 satisfies the above conditions, then $(S^{4n-1} \times Y_3)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_a(\boldsymbol{H})$ for $1 \leqq a \leqq n - 2$.*

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients. By $X_1 \sim X_2$ we mean $H^*(X_1) \cong H^*(X_2)$ as graded algebras. Denote by $\boldsymbol{P}_n(\boldsymbol{C})$ and $\boldsymbol{P}_n(\boldsymbol{H})$ the complex (resp. quaternion) projective n-space.

1. Preliminary results. First we present the following two lemmas which are proved by a standard method (cf. [6], [7], [11]). We shall give an outline of the proof in the final section for completeness.

LEMMA 1.1. *Suppose* $n \geqq 7$. *Let* G *be a closed connected proper subgroup of* $\boldsymbol{Sp}(n)$ *such that* $\dim \boldsymbol{Sp}(n)/G < 8n$. *Then* G *coincides with* $\boldsymbol{Sp}(n-i) \times K$ $(i = 1, 2, 3)$ *up to an inner automorphism of* $\boldsymbol{Sp}(n)$, *where* K *is a closed connected subgroup of* $\boldsymbol{Sp}(i)$.

LEMMA 1.2. *Suppose* $r \geqq 5$ *and* $k < 8r$. *Then an orthogonal non-trivial representation of* $\boldsymbol{Sp}(r)$ *of degree* k *is equivalent to* $(\nu_r)_{\boldsymbol{R}} \oplus \theta^{k-4r}$. *Here* $(\nu_r)_{\boldsymbol{R}}: \boldsymbol{Sp}(r) \to \boldsymbol{O}(4r)$ *is the canonical inclusion, and* θ^t *is the trivial representation of degree* t.

In the following, let X be a closed connected orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action, and suppose $n \geqq 7$ and $\dim X < 8n$. Put

$$F_{(i)} = \{x \in X: \boldsymbol{Sp}(n-i) \subset \boldsymbol{Sp}(n)_x \subset \boldsymbol{Sp}(n-i) \times \boldsymbol{Sp}(i)\} ,$$
$$X_{(i)} = \boldsymbol{Sp}(n)F_{(i)} = \{gx: g \in \boldsymbol{Sp}(n),\ x \in F_{(i)}\} .$$

Here $\boldsymbol{Sp}(n)_x$ denotes the isotropy group at x. Then, by Lemma 1.1, we obtain $X = X_{(0)} \cup X_{(1)} \cup X_{(2)} \cup X_{(3)}$.

PROPOSITION 1.3. *If* $X_{(k)}$ *is non-empty, then* $X_{(i)}$ *is empty for each* $i \geqq k+2$.

PROOF. This is proved essentially in [13], [14], but we give a proof for completeness. Let us denote by $F(\boldsymbol{Sp}(n-j), X_{(i)})$ the fixed point set of the restricted $\boldsymbol{Sp}(n-j)$ action on $X_{(i)}$. It is easy to see that $F(\boldsymbol{Sp}(n-j), X_{(i)})$ is empty for each $j < i \leqq n-i$. Suppose that $X_{(k)}$ is non-empty and fix $x \in F_{(k)}$. Let σ be the slice representation at x. Then the restriction $\sigma | \boldsymbol{Sp}(n-k)$ is trivial or equivalent to $(\nu_{n-k})_{\boldsymbol{R}} \oplus \theta^t$ by Lemma 1.2. Anyhow, a principal isotropy group of the given action contains $\boldsymbol{Sp}(n-k-1)$, and hence $F(\boldsymbol{Sp}(n-k-1), X_{(i)})$ is non-empty if so is $X_{(i)}$. q.e.d.

PROPOSITION 1.4. *Suppose* $X = X_{(k)} \cup X_{(k+1)}$. *If* $X_{(k)}$ *and* $X_{(k+1)}$ *are non-empty, then the codimension of each connected component of* $F_{(k)}$ *in* X *is equal to* $4(k+1)(n-k)$.

PROOF. Fix $x \in F_{(k)}$. Let σ and ρ denote the slice representation at x and the isotropy representation of the orbit $\boldsymbol{Sp}(n)x$, respectively. The restriction $\sigma | \boldsymbol{Sp}(n-k)$ is equivalent to $(\nu_{n-k})_{\boldsymbol{R}} \oplus \theta^s$ by Lemma 1.2 and the assumption that $X_{(k+1)}$ is non-empty. On the other hand, $\rho | \boldsymbol{Sp}(n-k)$

is equivalent to $k(\nu_{n-k})_R \oplus \theta^t$ by considering adjoint representations. Hence $(\sigma \oplus \rho) | \boldsymbol{Sp}(n-k)$ is equivalent to $(k+1)(\nu_{n-k})_R \oplus \theta^{s+t}$. This shows that the codimension of $F_{(k)}$ at x is equal to $4(k+1)(n-k)$. q.e.d.

COROLLARY 1.5. *Suppose* $X = X_{(2)} \cup X_{(3)}$. *Then either* $X_{(2)}$ *or* $X_{(3)}$ *is empty.*

REMARK. $\dim \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k) \times \boldsymbol{Sp}(k) = 4k(n-k)$ and $\chi(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k) \times \boldsymbol{Sp}(k)) = {}_nC_k$, where $\chi(\ )$ denotes the Euler characteristic, and ${}_nC_k$ denotes the binomial coefficient.

REMARK. If $\dim X < 4n$, then we see $X = X_{(1)}$. In addition, if $H^{\text{odd}}(X) = 0$, then X is equivariantly diffeomorphic to $\boldsymbol{P}_{n-1}(\boldsymbol{H})$, $\boldsymbol{P}_{n-1}(\boldsymbol{H}) \times S^2$ or $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$, where $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{P}_{n-1}(\boldsymbol{H})$, $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$ and trivially on S^2. So we assume $\dim X \geqq 4n$, in the following sections.

2. Cohomological aspects. Throughout this section, suppose that X is a closed orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action, $n \geqq 7$ and $X = X_{(0)} \cup X_{(1)}$.

PROPOSITION 2.1. *Suppose either* $X \sim P_a(\boldsymbol{C}) \times P_b(\boldsymbol{C})$, $1 \leqq b \leqq a < 2n \leqq a + b \leqq 4n - 3$, *or* $X \sim P_a(\boldsymbol{H}) \times P_b(\boldsymbol{C})$, $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a + b \leqq 4n - 4$. *Then* $X_{(0)}$ is empty.

PROOF. Suppose that $X_{(0)}$ is non-empty. Let U be an invariant closed tubular neighborhood of $X_{(0)}$ in X, and put $E = X - \operatorname{int} U$. Let $i\colon E \to X$ be the inclusion. Then $i^*\colon H^t(X) \to H^t(E)$ is an isomorphism for each $t \leqq 4n-2$, because the codimension of each connected component of $X_{(0)}$ is $4n$ by Lemma 1.2. Put $Y = E \cap F_{(1)}$. Then Y is a connected compact orientable manifold with non-empty boundary ∂Y, and $\boldsymbol{Sp}(1)$ acts naturally on Y. There is a natural diffeomorphism $E = (S^{4n-1} \times Y)/\boldsymbol{Sp}(1)$. By the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle $p\colon S^{4n-1} \times Y \to E$, we obtain an exact sequence:

$$0 \to H^{2k-1}(S^{4n-1} \times Y) \to H^{2k-4}(E) \to H^{2k}(E) \to H^{2k}(S^{4n-1} \times Y) \to 0\ ,$$

where $2k = \dim Y = \dim X - (4n-4)$. Hence we obtain $\operatorname{rank} H^{2k}(Y) - \operatorname{rank} H^{2k-1}(Y) \geqq 1$, by the cohomology ring structure of X. Considering the homology exact sequence of the pair $(Y, \partial Y)$ and the Poincaré-Lefschetz duality, we obtain

$$\operatorname{rank} H_0(\partial Y) \leqq \operatorname{rank} H_0(Y) + \operatorname{rank} H^{2k-1}(Y) - \operatorname{rank} H^{2k}(Y) \leqq 0\ .$$

Therefore ∂Y is empty; this is a contradiction. q.e.d.

In the remaining of this section, we assume $X = X_{(1)} = (S^{4n-1} \times F_{(1)})/\boldsymbol{Sp}(1)$, where $F_{(1)}$ is a closed connected orientable manifold with a natural $\boldsymbol{Sp}(1)$ action.

Here we describe certain situations which appear repeatedly in the following. Let Y be a closed orientable $\boldsymbol{Sp}(1)$ manifold such that $H^{\mathrm{odd}}(Y)=0$. Put $M=S^{4n-1}\times Y$, where $\boldsymbol{Sp}(1)$ acts as right scalar multiplication on S^{4n-1}. Let T be a closed toral subgroup of $\boldsymbol{Sp}(1)$. Consider the following commutative diagram:

$$(D-1)\qquad \begin{array}{ccc} M/T & \xrightarrow{p_1} & M/\boldsymbol{Sp}(1) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \boldsymbol{P}_{2n-1}(\boldsymbol{C}) & \xrightarrow{q} & \boldsymbol{P}_{n-1}(\boldsymbol{H}) \end{array},$$

where π_1, π_2 are projections of fiber bundles with Y as the fiber, and p_1, q are projections of 2-sphere bundles. Since $H^{\mathrm{odd}}(Y)=0$, we can apply the Leray-Hirsch theorem to the fibrations π_1, π_2. In particular, we see $H^{\mathrm{odd}}(M/\boldsymbol{Sp}(1))=0$. By the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle $p\colon M\to M/\boldsymbol{Sp}(1)$, we obtain an exact sequence:

$$(A_i)\quad 0\to H^{2i-1}(M)\to H^{2i-4}(M/\boldsymbol{Sp}(1))\xrightarrow{\mu} H^{2i}(M/\boldsymbol{Sp}(1))\xrightarrow{p^*} H^{2i}(M)\to 0$$

for each i, where μ is the multiplication by $e(p)$, the Euler class.

We regard S^∞ as the inductive limit of S^{4N-1} on which T acts naturally. Let F denote the fixed point set of the restricted T action on Y. Consider the following commutative diagram:

$$(D-2)\qquad \begin{array}{ccc} H^r((S^\infty\times Y)/T) & \xrightarrow{j^*} & H^r(M/T) \\ \downarrow{\scriptstyle i_\infty^*} & & \downarrow{\scriptstyle i_1^*} \\ H^r((S^\infty/T)\times F) & \xrightarrow{j_F^*} & H^r(\boldsymbol{P}_{2n-1}(\boldsymbol{C})\times F) \end{array},$$

where i_1, i_∞, j, j_F are natural inclusions. Since $H^{\mathrm{odd}}(Y)=0$, we see that (cf. [5])

(1) i_∞^* is injective, j^* is surjective and i_∞^* is surjective for $r>\dim Y$.

On the other hand, j_F^* is an isomorphism for $r\leqq 4n-2$, and hence

(2) i_1^* is injective for $r\leqq 4n-2$.

2-A. Now we consider the case $X\sim \boldsymbol{P}_a(\boldsymbol{C})\times\boldsymbol{P}_b(\boldsymbol{C})$.

PROPOSITION 2.2. *Suppose* $X\sim\boldsymbol{P}_a(\boldsymbol{C})\times\boldsymbol{P}_b(\boldsymbol{C})$, $1\leqq b\leqq a<2n\leqq a+b\leqq 4n-3$. *Then* $a=2n-1$ *and* $F_{(1)}\sim S^2\times\boldsymbol{P}_b(\boldsymbol{C})$.

PROOF. The cohomology ring is as follows.

$$H^*(X)=\boldsymbol{Q}[u,v]/(u^{a+1},v^{b+1});\ \deg u=\deg v=2\,.$$

We can express $e(p) = \alpha u^2 + \beta uv + \gamma v^2$; $\alpha, \beta, \gamma \in \boldsymbol{Q}$, where $p: S^{4n-1} \times F_{(1)} \to X$ is the principal $\boldsymbol{Sp}(1)$ bundle. By (A_1), we obtain $H^1(F_{(1)}) = 0$ and hence $H^{2k-1}(F_{(1)}) = 0$ by the Poincaré duality, where $2k = \dim F_{(1)} = 2(a + b - 2n + 2)$. Then by (A_k) we obtain an exact sequence:

$$0 \to H^{2k-4}(X) \xrightarrow{\mu} H^{2k}(X) \xrightarrow{p^*} H^{2k}(F_{(1)}) \to 0 .$$

By the ring structure of $H^*(X)$, we obtain

$$\operatorname{rank} H^{2k-4}(X) = k - 1 ,$$
$$\operatorname{rank} H^{2k}(X) = k + 1 \text{ (for } k \leqq b) \quad \text{and} \quad k \text{ (for } k = b + 1) .$$

Since $F_{(1)}$ is a closed connected orientable $2k$-manifold, we obtain $k = b + 1$ and hence $a = 2n - 1$. Next, we shall show $e(p) = \alpha u^2$, $\alpha \neq 0$. By definition, the $\boldsymbol{Sp}(1)$ bundle p is a pull-back of the canonical principal $\boldsymbol{Sp}(1)$ bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$, and hence $e(p)^n = 0$. Thus we obtain $\alpha\beta = 0$, by considering the term $u^{2n-1}v$ in the expression of $e(p)^n$. Suppose $\alpha = 0$. Then $p^*(u^{2n-1}) \neq 0$ by (A_{2n-1}), and hence $\dim F_{(1)} \geqq 4n - 2$. Thus we obtain $k = b + 1 = 2n - 1$. By considering the term $u^n v^n$ in the expression of $e(p)^n$, we obtain $\beta = 0$, and hence $e(p) = \gamma v^2$. Then $p^*(u^{2n-1}v) \neq 0$ by (A_{2n}). On the other hand $H^{4n}(S^{4n-1} \times F_{(1)}) = 0$, since $H^1(F_{(1)}) = 0$ and $\dim F_{(1)} = 4n - 2$. This is a contradiction. Thus we obtain $e(p) = \alpha u^2 + \gamma v^2$, $\alpha \neq 0$. By considering the term $u^{2n-2}v^2$ in the expression of $e(p)^n$, we obtain $\alpha\gamma = 0$. Therefore we obtain $e(p) = \alpha u^2$, $\alpha \neq 0$, and hence $F_{(1)} \sim S^2 \times \boldsymbol{P}_b(\boldsymbol{C})$, by (A_i). q.e.d.

Now we consider the $\boldsymbol{Sp}(1)$ action on $F_{(1)} \sim S^2 \times \boldsymbol{P}_b(\boldsymbol{C})$. Let T be a toral subgroup of $\boldsymbol{Sp}(1)$. Denote by F the fixed point set of the restricted T action on $F_{(1)}$. Since $\chi(F_{(1)}) \neq 0$, we see that F is non-empty.

PROPOSITION 2.3. $F \sim S^0 \times \boldsymbol{P}_b(\boldsymbol{C})$.

PROOF. Put $Y = F_{(1)}$ in the diagram $(D - 1)$. Let $t \in H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$ and $w \in H^4(\boldsymbol{P}_{n-1}(\boldsymbol{H}))$ be the canonical generators such that $q^*(w) = t^2$. By definition, $\pi_2^*(w) = e(p) = \alpha u^2$. Put $u_1 = p_1^*(u)$, $v_1 = p_1^*(v)$ and $t_1 = \pi_1^*(t)$. We can apply the Leray-Hirsch theorem to the bundles π_1, π_2 in the diagram $(D - 1)$, and we obtain

$$H^*(M/T) = \boldsymbol{Q}[t_1, u_1, v_1]/(u_1^{2n}, v_1^{b+1}, t_1^2 - \alpha u_1^2) , \quad \alpha \neq 0 .$$

Consider the diagram $(D - 2)$ for $Y = F_{(1)}$. Let u_2, v_2 be elements of $H^2((S^\infty \times F_{(1)})/T)$ such that $j^*(u_2) = u_1$ and $j^*(v_2) = v_1$. Let t be the canonical generator of $H^2(S^\infty/T) = H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$. Then we can express

$$i_\infty^*(u_2) = t \times f_0 + 1 \times f_1 , \quad i_\infty^*(v_2) = t \times g_0 + 1 \times g_1 ,$$

where f_k, g_k are elements of $H^{2k}(F)$ for $k = 0, 1$. Since

$$j_F^* i_\infty^*(\alpha u_2^2) = i_1^*(\alpha u_1^2) = i_1^*(t_1^2) = j_F^*(t^2 \times 1) ,$$

we obtain $f_0^2 = \alpha^{-1}$ and $f_1 = 0$. Moreover we see that f_0 is not constant, and hence F is not connected. Since $j_F^* i_\infty^*(v_2^{b+1}) = 0$, we obtain $g_0 = 0$ and hence $i_\infty^*(v_2) = 1 \times g_1$. Then $H^*(F)$ is generated by two elements f_0 and g_1, because i_∞^* is surjective for sufficiently large degree and $H^*((S^\infty \times F_{(1)})/T)$ is generated by two elements u_2, v_2 as a graded $H^*(S^\infty/T)$-algebra. Let F_1 (resp. F_2) be the union of connected components F_σ of F on which $f_0 | F_\sigma$ is positive (resp. negative). Then $H^*(F_s)$ is generated by only one element $g_1 | F_s$ for $s = 1, 2$. Since $i_1^*(v_1) = 1 \times g_1$, we obtain $(g_1 | F_s)^{b+1} = 0$, and hence $F_s \sim \boldsymbol{P}_b(\boldsymbol{C})$ for $s = 1, 2$, because $\chi(F_1) + \chi(F_2) = \chi(F_{(1)}) = 2b$.

q.e.d.

We need the following.

LEMMA 2.4. *Let S be a closed connected smooth $\boldsymbol{Sp}(1)$ manifold. Let F be the fixed point set of the restricted T action on S, where T is a closed toral subgroup of $\boldsymbol{Sp}(1)$. Suppose that* $\operatorname{codim} F = 2$ *and F is not connected. Then there is an equivariant diffeomorphism: $S = \boldsymbol{Sp}(1)/T \times F_1$, where F_1 is a connected component of F.*

PROOF. Since codim $F = 2$, T is the identity component of a principal isotropy group (cf. [9]), and hence there is an equivariant diffeomorphism:

$$S - F_0 = (\boldsymbol{Sp}(1)/T \times (F - F_0))/(NT/T) ,$$

where F_0 denotes the fixed point set of the $\boldsymbol{Sp}(1)$ action and NT denotes the normalizer of T. Since codim $F_0 > 2$, $S - F_0$ is connected and hence the orbit space of the NT/T action on $F - F_0$ is connected. Therefore F has just two components and NT/T acts freely on F. In particular, F_0 is empty and there is an equivariant diffeomorphism: $F = NT/T \times F_1$. Hence we obtain the desired result. q.e.d.

By Proposition 2.3 and Lemma 2.4, there is an equivariant diffeomorphism: $F_{(1)} = \boldsymbol{Sp}(1)/T \times Y_0$, where Y_0 is a connected component of F. Thus we obtain an equivariant diffeomorphism:

$$X = X_{(1)} = (S^{4n-1} \times F_{(1)})/\boldsymbol{Sp}(1) = \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_0 .$$

Consequently we obtain the following.

THEOREM 2.5. *Let X be a closed orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action. Suppose $n \geqq 7$, $X = X_{(0)} \cup X_{(1)}$ and $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$, $1 \leqq b \leqq a < 2n \leqq a + b \leqq 4n - 3$. Then $a = 2n - 1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_0$, where Y_0 is a closed orientable manifold such that $Y_0 \sim \boldsymbol{P}_b(\boldsymbol{C})$.*

2-B. Next we consider the case $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{C})$.

PROPOSITION 2.6. *Suppose* $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{C})$, $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a+b \leqq 4n-4$. *Then either* $a = n-1$ *and* $F_{(1)} \sim \boldsymbol{P}_b(\boldsymbol{C})$, *or* $b = 2n-1$ *and* $F_{(1)} \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$.

PROOF. The cohomology ring is as follows.

$$H^*(X) = \boldsymbol{Q}[u, v]/(u^{a+1}, v^{b+1}) ; \quad \deg u = 4 , \quad \deg v = 2 .$$

We can express $e(p) = \alpha u + \beta v^2$; $\alpha, \beta \in \boldsymbol{Q}$, where $p: S^{4n-1} \times F_{(1)} \to X$ is the principal $\boldsymbol{Sp}(1)$ bundle. By definition, the $\boldsymbol{Sp}(1)$ bundle p is a pull-back of the canonical principal $\boldsymbol{Sp}(1)$ bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$, and hence $e(p)^n = 0$. Thus we obtain $\alpha\beta = 0$, by considering the term $u^a v^{2n-2a}$ in the expression of $e(p)^n$. On the other hand, we can prove $e(p) \neq 0$ by making use of the exact sequence (A_i). Moreover we see, from (A_i), that if $\beta = 0$ then $a = n-1$ and $F_{(1)} \sim \boldsymbol{P}_b(\boldsymbol{C})$; if $\alpha = 0$ then $b = 2n-1$ and $F_{(1)} \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$. q.e.d.

Now we consider the $\boldsymbol{Sp}(1)$ action on $F_{(1)}$. Let T be a toral subgroup of $\boldsymbol{Sp}(1)$. Denote by F the fixed point set of the restricted T action on $F_{(1)}$. Since $\chi(F_{(1)}) \neq 0$, we see that F is non-empty. We shall show the following.

PROPOSITION 2.7. *If* $a = n-1$ *and* $F_{(1)} \sim \boldsymbol{P}_b(\boldsymbol{C})$, *then the* $\boldsymbol{Sp}(1)$ *action on* $F_{(1)}$ *is trivial. If* $b = 2n-1$ *and* $F_{(1)} \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$, *then* $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{H})$ *or* $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$. *Moreover the induced homomorphism* $i^*: H^2(F_{(1)}) \to H^2(F)$ *is trivial.*

PROOF. Put $Y = F_{(1)}$ in the diagram $(D-1)$. Let $t \in H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$ and $w \in H^4(\boldsymbol{P}_{n-1}(\boldsymbol{H}))$ be the canonical generators as before. Then $\pi_2^*(w) = e(p)$ by definition. We see that $e(p) = \alpha u, \alpha \neq 0$ or $e(p) = \beta v^2, \beta \neq 0$ in Proposition 2.6.

Suppose first $e(p) = \alpha u$. Then $a = n-1$ and $F_{(1)} \sim \boldsymbol{P}_b(\boldsymbol{C})$. We can prove $M/T \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$, $b \leqq 2n-2$ by the Leray-Hirsch theorem, and hence the T action on $F_{(1)} \sim \boldsymbol{P}_b(\boldsymbol{C})$ is trivial (cf. [12, Proposition 3.3]). Therefore the $\boldsymbol{Sp}(1)$ action on $F_{(1)}$ is trivial.

Suppose next $e(p) = \beta v^2$. Then $b = 2n-1$ and $F_{(1)} \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$. Put $u_1 = p_1^*(u)$, $v_1 = p_1^*(v)$ and $t_1 = \pi_1^*(t)$. We can apply the Leray-Hirsch theorem to the bundles π_1, π_2 in the diagram $(D-1)$, and we obtain

$$H^*(M/T) = \boldsymbol{Q}[t_1, u_1, v_1]/(u_1^{a+1}, v_1^{2n}, t_1^2 - \beta v_1^2) , \quad \beta \neq 0 .$$

Consider the diagram $(D-2)$ for $Y = F_{(1)}$. Let u_2, v_2 be homogeneous elements of $H^*((S^\infty \times F_{(1)})/T)$ such that $j^*(u_2) = u_1$ and $j^*(v_2) = v_1$. Let t

be the canonical generator of $H^2(S^\infty/T)=H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$. Then we can express

$$i_\infty^*(u_2) = t^2 \times f_0 + t \times f_1 + 1 \times f_2\ , \quad i_\infty^*(v_2) = t \times g_0 + 1 \times g_1\ ,$$

where f_k, g_k are elements of $H^{2k}(F)$. Since

$$j_F^* i_\infty^*(\beta v_2^2) = i_1^*(\beta v_1^2) = i_1^*(t_1^2) = j_F^*(t^2 \times 1)\ ,$$

we obtain $g_0^2 = \beta^{-1}$ and $g_1 = 0$. Moreover we see that g_0 is not constant, and hence F is not connected. Since $j_F^* i_\infty^*(u_2^{a+1}) = 0$ and $a+1 \leqq n-1$, we obtain $f_0 = 0$ and hence $i_\infty^*(u_2) = t \times f_1 + 1 \times f_2$. Let F_1 (resp. F_2) be the union of connected components F_σ of F on which $g_0 \mid F_\sigma$ is positive (resp. negative). Then each element of $H^k((S^\infty \times F_s)/T)$ for $k > 4a+2$ is expressed as a polynomial of $t \times 1$ and $t \times (f_1 \mid F_s) + 1 \times (f_2 \mid F_s)$ with rational coefficients for $s = 1, 2$, because $H^*((S^\infty \times F_{(1)})/T)$ is generated by two elements u_2, v_2 as a graded $H^*(S^\infty/T)$-algebra and i_∞^* is surjective for $k > 4a+2$. In particular, if $f_1 \mid F_s \neq 0$, then we can express

$$t^{4a-1} \times (f_1 \mid F_s) = \sum_j c_j (t \times (f_1 \mid F_s) + 1 \times (f_2 \mid F_s))^j (t \times 1)^{4a-2j}\ ,$$

for $c_j \in \boldsymbol{Q}$. Then we obtain $c_0 = 0$, $c_1 = 1$ and $f_2 \mid F_s = -c_2(f_1 \mid F_s)^2$. Therefore

$$H^*(F_s) = \boldsymbol{Q}[x_s]/(x_s^{a+1})\ ; \quad \deg x_s = 2 \quad \text{or} \quad 4\ ,$$

because $f_k^{a+1} = 0$ $(k = 1, 2)$ and $\chi(F_1) + \chi(F_2) = \chi(F_{(1)}) = 2a$. If $F_s \sim \boldsymbol{P}_a(\boldsymbol{H})$ for some s, then $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{H})$ by Lemma 2.4. Thus we obtain $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{H})$ or $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$. Finally we shall show that $i^*\colon H^2(F_{(1)}) \to H^?(F)$ is trivial for the case $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$. Consider the following commutative diagram:

$$\begin{array}{ccc} H^2(M/T) & \xrightarrow{k_1^*} & H^2(F_{(1)}) \\ \big\downarrow i_1^* & & \big\downarrow i^* \\ H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times F) & \xrightarrow{k_0^*} & H^2(F)\ , \end{array}$$

where i, i_1 are natural inclusions and k_0, k_1 are inclusions of typical fiber of bundles over $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$. We see that $k_1^*(v_1)$ generates $H^2(F_{(1)})$ and $i_1^*(v_1) = t \times g_0$, and hence $i^* k_1^*(v_1) = k_0^*(t \times g_0) = 0$. Thus $i^*\colon H^2(F_{(1)}) \to H^2(F)$ is trivial. q.e.d.

Suppose $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{H})$. Then by Lemma 2.4, there is an equivariant diffeomorphism: $F_{(1)} = \boldsymbol{Sp}(1)/T \times Y_2$, where Y_2 is a connected component of F. Thus we obtain an equivariant diffeomorphism:

$$X = X_{(1)} = (S^{4n-1} \times F_{(1)})/\boldsymbol{Sp}(1) = \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_2\ .$$

Consequently we obtain the following.

THEOREM 2.8. *Let X be a closed orientable manifold with a nontrivial smooth $\boldsymbol{Sp}(n)$ action. Suppose $n \geqq 7$, $X = X_{(0)} \cup X_{(1)}$ and $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{C})$; $1 \leqq a \leqq n-1$, $1 \leqq b \leqq 2n-1$, $2n \leqq 2a+b \leqq 4n-4$. Then there are three cases*:

(a) *$a = n-1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{n-1}(\boldsymbol{H}) \times Y_1$, where Y_1 is a closed orientable manifold such that $Y_1 \sim \boldsymbol{P}_b(\boldsymbol{C})$,*

(b) *$b = 2n-1$ and X is equivariantly diffeomorphic to $\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times Y_2$, where Y_2 is a closed orientable manifold such that $Y_2 \sim \boldsymbol{P}_a(\boldsymbol{H})$,*

(c) *$b = 2n-1$ and X is equivariantly diffeomorphic to $(S^{4n-1} \times Y_3)/\boldsymbol{Sp}(1)$, where Y_3 is a closed orientable $\boldsymbol{Sp}(1)$ manifold such that $Y_3 \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$, $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$ and $i^*\colon H^2(Y_3) \to H^2(F)$ is trivial, where F denotes the fixed point set of the restricted T action on Y_3. Conversely, if Y_3 satisfies the above conditions, then $(S^{4n-1} \times Y_3)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_a(\boldsymbol{H})$ for $a \leqq n-2$.*

PROOF. It remains to prove the final statement in the case (c). Let Y be a closed orientable $\boldsymbol{Sp}(1)$ manifold such that $Y \sim S^2 \times \boldsymbol{P}_a(\boldsymbol{H})$, $F \sim S^0 \times \boldsymbol{P}_a(\boldsymbol{C})$ and $i^*\colon H^2(Y) \to H^2(F)$ is trivial, where F denotes the fixed point set of the restricted T action. We shall show $(S^{4n-1} \times Y)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_a(\boldsymbol{H})$ for $a \leqq n-2$. Put $M = S^{4n-1} \times Y$. Consider the following commutative diagrams as before:

$$\begin{array}{ccc} M/T & \xrightarrow{p_1} & M/\boldsymbol{Sp}(1) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \boldsymbol{P}_{2n-1}(\boldsymbol{C}) & \xrightarrow{q} & \boldsymbol{P}_{n-1}(\boldsymbol{H}) \end{array}, \qquad \begin{array}{ccc} F & \xrightarrow{i} & Y \\ \downarrow{\scriptstyle k_0} & & \downarrow{\scriptstyle k_1} \\ \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times F & \xrightarrow{i_1} & M/T \end{array}.$$

Let $t \in H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$ and $w \in H^4(\boldsymbol{P}_{n-1}(\boldsymbol{H}))$ be the canonical generators such that $q^*(w) = t^2$. Because π_1, π_2 are projections of bundles with Y as the fiber and $H^{\mathrm{odd}}(Y) = 0$, we can apply the Leray-Hirsch theorem and we see that there is an element $u_k \in H^{2k}(M/\boldsymbol{Sp}(1))$ for $k = 1, 2$ such that $H^*(M/\boldsymbol{Sp}(1))$ is freely generated by $1, u_2, u_2^2, \cdots, u_2^a, u_1, u_1u_2, u_1u_2^2, \cdots, u_1u_2^a$ as an $H^*(\boldsymbol{P}_{n-1}(\boldsymbol{H}))$-module, and $u_1^2 = c\pi_2^*(w)$ for some $c \in \boldsymbol{Q}$. Put $v_k = p_1^*(u_k)$. Express $i_1^*(v_1) = t \times g_0 + 1 \times g_1$ for some $g_j \in H^{2j}(F)$. Then $g_1 = k_0^*(t \times g_0 + 1 \times g_1) = i^*k_1^*(v_1) = 0$, because $i^*\colon H^2(Y) \to H^2(F)$ is trivial. Hence $i_1{}^*(v_1) = t \times g_0$ and $c = g_0^2$. We see that g_0 is not constant and $c \neq 0$, because i_1^* is injective for each degree $\leqq 4n-2$ and $v_1, \pi_1^*(t)$ are linearly independent in $H^2(M/T)$. Hence $H^*(M/\boldsymbol{Sp}(1))$ is generated by u_1, u_2 as a graded algebra. Express $i_1^*(v_2) = t^2 \times f_0 + t \times f_1 + 1 \times f_2$ for some $f_j \in H^{2j}(F)$. Let F_1 be a connected component of F and put $w_2 = u_2 - d\pi_2^*(w)$, where $d = f_0 | F_1$. Then $H^*(M/\boldsymbol{Sp}(1))$ is freely generated by

$u_1^i w_2^j$; $0 \leqq i \leqq 2n-1$, $0 \leqq j \leqq a$ as a graded module, and

$$w_2^{a+1} = \sum_{j=0}^{a} c_j w_2^j \pi_2^*(w^{a+1-j}) , \quad c_j \in \boldsymbol{Q} .$$

By definition, $i_1^* p_1^*(w_2) = t \times f_1 + 1 \times f_2$ on $H^*(\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{F}_1)$, a direct summand of $H^*(\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times F)$. Since $F_1 \sim \boldsymbol{P}_a(\boldsymbol{C})$, we obtain $i_1^* p_1^*(w_2^{a+1}) = 0$ on $H^*(\boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times F_1)$ and hence

$$0 = \sum_{j=0}^{a} c_j (t \times (f_1 | F_1) + 1 \times (f_2 | F_1))^j (t^2 \times 1)^{a+1-j} .$$

Moreover we obtain $f_1 | F_1 \neq 0$, because $f_1 | F_1$ and $f_2 | F_1$ generate the graded algebra $H^*(F_1)$ and $F_1 \sim \boldsymbol{P}_a(\boldsymbol{C})$. Then we obtain $c_j = 0$ for $j = 0, 1, \cdots, a$ inductively, and hence $w_2^{a+1} = 0$. On the other hand, $u_1^{2n} = c^n \pi_2^*(w^n) = 0$. Hence we obtain

$$H^*(M/\boldsymbol{Sp}(1)) = \boldsymbol{Q}[u_1, w_2]/(u_1^{2n}, w_2^{a+1}); \deg u_1 = 2, \deg w_2 = 4 .$$

Therefore $M/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_a(\boldsymbol{H})$. q.e.d.

3. Cohomology of certain homogeneous spaces. Let ζ be a quaternion k-plane bundle and ζ_C its complexification under the restriction of the field. Its i-th symplectic Pontrjagin class $e_i(\zeta)$ is by definition [3, §9.6]

$$e_i(\zeta) = (-1)^i c_{2i}(\zeta_C) ,$$

where $c_{2i}(\zeta_C)$ is the $2i$-th Chern class. Denote by $\boldsymbol{HP}(\zeta)$ the total space of the associated projective space bundle. Let $\hat{\zeta}$ be the canonical quaternion line bundle over $\boldsymbol{HP}(\zeta)$ and $t = e_1(\hat{\zeta})$. It is known that there is an isomorphism:

$$H^*(\boldsymbol{HP}(\zeta)) = H^*(B)[t] \Big/ \Big(\sum_i e_{k-i}(\zeta) t^i \Big) , \tag{3.1}$$

where B is the base space of the bundle ζ (cf. [4, §3]).

We now consider the cohomology of $V_{n,2}/G = \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-2) \times G$ for certain closed connected subgroups G of $\boldsymbol{Sp}(2)$. Let ξ be the canonical quaternion line bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and ζ its orthogonal complement, that is, ζ is a quaternion $(n-1)$-plane bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ such that its total space is

$$E(\zeta) = \{(u, [v]) \in \boldsymbol{H}^n \times \boldsymbol{P}_{n-1}(\boldsymbol{H}) : u \perp v\} .$$

It is easy to see that $\boldsymbol{HP}(\zeta)$ is naturally diffeomorphic to $V_{n,2}/\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$. Since $\xi \oplus \zeta$ is a trivial bundle, we obtain $e_k(\zeta) = (-1)^k e_1(\xi)^k$. By definition, $\boldsymbol{HP}(\zeta)$ is naturally identified with a subspace of $\boldsymbol{P}_{n-1}(\boldsymbol{H}) \times \boldsymbol{P}_{n-1}(\boldsymbol{H})$. Let $i: \boldsymbol{HP}(\zeta) \to \boldsymbol{P}_{n-1}(\boldsymbol{H}) \times \boldsymbol{P}_{n-1}(\boldsymbol{H})$ be the inclusion. Then $\hat{\zeta} = i^*(\xi \times 1)$. Hence by (3.1) there is an isomorphism:

$$(3.2)\qquad H^*(V_{n,2}/Sp(1)\times Sp(1)) = Q[u, v]\Big/\Big(u^n, \sum_i u^i v^{n-1-i}\Big)\,,$$

$\deg u = \deg v = 4$, by the identification $u = i^*(1\times e_1(\xi))$, $v = i^*(e_1(\xi)\times 1)$.

Let $\pi\colon P_{2n-1}(C)\to P_{n-1}(H)$ be the natural projection defined by

$$(u_1\colon \cdots \colon u_n\colon v_1\colon \cdots \colon v_n) \to (u_1 + jv_1\colon \cdots \colon u_n + jv_n)\,,$$

where j is a quaternion such that $j^2 = -1$ and $jz = \bar{z}j$ for each complex number z. Then $\pi^*(\xi_C) = \eta \oplus \eta^*$, where η is the canonical complex line bundle over $P_{2n-1}(C)$ and η^* its dual line bundle. Moreover $\pi^* e_1(\xi) = c_1(\eta)^2$. We see that the total space $CP(\pi^*(\zeta_C))$ of the complex projective space bundle is naturally diffeomorphic to $V_{n,2}/T^2$ and there is a natural inclusion $i'\colon CP(\pi^*(\zeta_C))\to P_{2n-1}(C)\times P_{2n-1}(C)$, where T^2 is the standard maximal torus of $Sp(2)$. Then we obtain an isomorphism (cf. [4, §3]):

$$(3.3)\qquad H^*(V_{n,2}/T^2) = Q[x, y]\Big/\Big(x^{2n}, \sum_i x^{2i}y^{2n-2-2i}\Big)\,,$$

$\deg x = \deg y = 2$, by the identification $x = i'^*(1\times c_1(\eta))$, $y = i'^*(c_1(\eta)\times 1)$.

Let $p\colon V_{n,2}/Sp(1)\times Sp(1)\to V_{n,2}/Sp(2)$ be the natural projection and ξ_2 be the canonical quaternion 2-plane bundle over $V_{n,2}/Sp(2)$.

LEMMA 3.4. *The graded algebra $H^*(V_{n,2}/Sp(2))$ is generated by $e_1(\xi_2)$, $e_2(\xi_2)$. The algebra is isomorphic to the subalgebra of $Q[u, v]/(u^n, \sum_i u^i v^{n-1-i})$, consisting of symmetric polynomials.*

PROOF. Since the fibration p is a 4-sphere bundle and $H^{\mathrm{odd}}(V_{n,2}/Sp(2)) = 0$ (cf. [2, § 26]), the homomorphism $p^*\colon H^*(V_{n,2}/Sp(2))\to H^*(V_{n,2}/Sp(1)\times Sp(1))$ is injective. Since $p^*(\xi_2) = i^*(\xi\times\xi)$, we obtain

$$p^* e_1(\xi_2) = i^* e_1(\xi\times\xi) = u + v\,,$$
$$p^* e_2(\xi_2) = i^* e_2(\xi\times\xi) = uv\,.$$

Then the desired result is obtained by the Leray-Hirsch theorem.

q.e.d.

Let $p_1\colon V_{n,2}/T^2\to V_{n,2}/U(2)$ be the natural projection and η_2 be the canonical complex 2-plane bundle over $V_{n,2}/U(2)$. Then we obtain the following by the same argument as above.

LEMMA 3.5. *The graded algebra $H^*(V_{n,2}/U(2))$ is generated by $c_1(\eta_2)$, $c_2(\eta_2)$. The algebra is isomorphic to the subalgebra of $Q[x, y]/(x^{2n}, \sum_i x^{2i}y^{2n-2-2i})$, consisting of symmetric polynomials.*

LEMMA 3.6. *The graded algebra $H^*(V_{n,2}/U(1)\times Sp(1))$ is isomorphic to the subalgebra of $Q[x, y]/(x^{2n}, \sum_i x^{2i}y^{2n-2-2i})$, generated by x^2, y.*

PROOF. Consider the natural mappings

$$V_{n,2}/T^2 \xrightarrow{p_2} V_{n,2}/\boldsymbol{U}(1) \times \boldsymbol{Sp}(1) \xrightarrow{i_2} \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_{n-1}(\boldsymbol{H}) .$$

We see that i_2^* is surjective and p_2^* is injective. On the other hand, there are the following equations

$$p_2^* i_2^*(1 \times e_1(\xi)) = x^2 , \quad p_2^* i_2^*(c_1(\eta) \times 1) = y .$$

Thus we obtain the desired result. q.e.d.

Here we state the following for later use.

PROPOSITION 3.7. *Let G be one of T^2, $\boldsymbol{U}(2)$ and $\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$. Let w_1, w_2 be any non-zero homogeneous elements of $H^*(V_{n,2}/G)$ such that* $\deg w_k = 2k$. *Then w_1^{2n-1} and w_2^{n-1} are non-zero elements.*

PROOF. For $G = T^2$, we obtain the result from (3.3). For $G = \boldsymbol{U}(2)$ or $\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$, we obtain the result from Lemmas 3.5, 3.6 and the result for $G = T^2$. q.e.d.

4. Finish of the proof. Throughout this section, suppose that $n \geqq 7$ and X is a closed orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action, and $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$ for some a, b such that

$$1 \leqq b \leqq a < 2n \leqq a + b \leqq 4n - 3 ,$$

or $X \sim \boldsymbol{P}_c(\boldsymbol{H}) \times \boldsymbol{P}_d(\boldsymbol{C})$ for some c, d such that

$$1 \leqq c \leqq n - 1, 1 \leqq d \leqq 2n - 1 \quad \text{and} \quad 2n \leqq 2c + d \leqq 4n - 4 .$$

We shall show that $X_{(2)}$ and $X_{(3)}$ are empty sets.

PROPOSITION 4.1. $X \neq X_{(k)}$; $k = 2, 3$.

PROOF. Suppose $X = X_{(k)}$. Then there is an equivariant diffeomorphism: $X = (\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k) \times F_{(k)})/\boldsymbol{Sp}(k)$. In particular, we obtain $\chi(X) = {}_nC_k\chi(F_{(k)})$. Looking at the Euler characteristic of X, we see that $k \neq 3$. Thus only the following possibilities remain:

(a) $\dim F_{(2)} = 8$, $\chi(F_{(2)}) = 8$; $(a, b) = (2n-1, 2n-3)$,

(b) $\dim F_{(2)} = 6$, $\chi(F_{(2)}) = 4$; $(c, d) = (n-1, 2n-3), (n-2, 2n-1)$,

(c) $\dim F_{(2)} \leqq 4$.

If $\dim F_{(2)} \leqq 4$, then $X = V_{n,2}/\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$ or $X = V_{n,2}/\boldsymbol{Sp}(2) \times F_{(2)}$, and hence the case (c) does not happen by (3.2) and Lemma 3.4. In the cases (a), (b) if the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ is transitive, then $X = V_{n,2}/T^2$, $V_{n,2}/\boldsymbol{U}(2)$ or $V_{n,2}/\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$, and hence such cases do not happen by Proposition 3.7.

Consider the case (a). Since $\chi(F_{(2)}) \neq 0$ and the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ is non-transitive, the restricted G action on $F_{(2)}$ has a fixed point, and hence the natural projection $\pi_1\colon (V_{n,2} \times F_{(2)})/G \to V_{n,2}/G$ has a cross-section s, where $G = \boldsymbol{U}(2)$ or $\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$. Consider the following commutative diagram:

$$\begin{array}{ccc} (V_{n,2} \times F_{(2)})/G & \xrightarrow{\pi_1} & V_{n,2}/G \\ \downarrow q & & \downarrow p \\ X = (V_{n,2} \times F_{(2)})/\boldsymbol{Sp}(2) & \xrightarrow{\pi_2} & V_{n,2}/\boldsymbol{Sp}(2) , \end{array}$$

where π_1, π_2, p, q are natural projections. We can express

$$\pi_2^* e_1(\xi_2) = \alpha x^2 + \beta xy + \gamma y^2;\ x, y \in H^2(X),\ \alpha, \beta, \gamma \in \boldsymbol{Q} .$$

Moreover we can express $s^*q^*x = \mu t$, $s^*q^*y = \nu t$ ($\mu, \nu \in \boldsymbol{Q}$) for some non-zero element $t \in H^2(V_{n,2}/G)$ because $\operatorname{rank} H^2(V_{n,2}/G) = 1$ by Lemmas 3.5, 3.6. Hence we obtain

$$p^* e_1(\xi_2) = s^*\pi_1^*p^*e_1(\xi_2) = s^*q^*\pi_2^*e_1(\xi_2) = \delta t^2 ,$$

where $\delta = \alpha\mu^2 + \beta\mu\nu + \gamma\nu^2$. Let $i\colon \boldsymbol{Sp}(2)/G \to V_{n,2}/G$ be the natural inclusion. Then $i^*t \neq 0$ and $i^*p^*e_1(\xi_2) = 0$, and hence $\delta = 0$, because $\boldsymbol{Sp}(2)/G \sim \boldsymbol{P}_3(\boldsymbol{C})$. Thus we obtain $p^*e_1(\xi_2) = 0$; this is a contradiction to the fact that p^* is injective. Therefore, the case (a) does not happen.

Consider the case (b). Since the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ is non-transitive, the identity component of an isotropy group is conjugate to $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$ or $\boldsymbol{Sp}(2)$. If the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ is trivial, then $X = V_{n,2}/\boldsymbol{Sp}(2) \times F_{(2)}$ and hence such a case does not happen by Lemma 3.4.

Suppose first that the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ has no fixed point. Denote by F the fixed point set of the restricted $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$ action on $F_{(2)}$. Then we see that F is a closed orientable surface with $\chi(F) = 4$ and F has at most two components. Therefore, $X = (V_{n,2}/\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)) \times S^2$, and hence such a case does not happen by (3.2).

Suppose next that the $\boldsymbol{Sp}(2)$ action on $F_{(2)}$ has a fixed point. Then we see that the fixed point set of the $\boldsymbol{Sp}(2)$ action is one-dimensional by considering the isotropy representations. Let U be its closed invariant tubular neighborhood and denote by F' the fixed point set of the restricted $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$ action on $F_{(2)} - \operatorname{int} U$. Then we see that F' is a compact orientable surface with $\chi(F') = 4$, F' has at most two components and each component of F' has a non-empty boundary. Such a case does not happen, because $\chi \leqq 1$ for each compact connected orientable surface with non-empty boundary. q.e.d.

PROPOSITION 4.2. *If $X_{(1)}$ is non-empty, then $X_{(2)}$ is empty.*

PROOF. Suppose that both $X_{(1)}$ and $X_{(2)}$ are non-empty. Then $X =$

$X_{(1)} \cup X_{(2)}$ and codim $F_{(1)} = 8n - 8$, by Propositions 1.3, 1.4. Since $\dim X \leqq 8n - 6$, we obtain $\dim F_{(1)} = 0$ or 2.

Suppose first that the $\boldsymbol{Sp}(1)$ action on $F_{(1)}$ is non-trivial. Then $\dim F_{(1)} = 2$ and $X \sim \boldsymbol{P}_{2n-1}(\boldsymbol{C}) \times \boldsymbol{P}_{2n-2}(\boldsymbol{C})$. Considering the slice representation at a point of $F_{(1)}$, we see that the $\boldsymbol{Sp}(n)$ action on X has a codimension one orbit, and hence X is a union of closed invariant tubular neighborhoods of just two non-principal orbits (cf. [10]). Calculating the Euler characteristics, we see that two non-principal orbits are $\boldsymbol{P}_{2n-1}(\boldsymbol{C})$ and $V_{n,2}/T^2$. Since codim $\boldsymbol{P}_{2n-1}(\boldsymbol{C}) = 4n - 4$ in X, the inclusion $i: V_{n,2}/T^2 \to X$ induces an isomorphism $i^*: H^2(X) \to H^2(V_{n,2}/T^2)$, and hence $x^{2n-1} \neq 0$ for each non-zero element $x \in H^2(X)$ by Proposition 3.7. This is a contradiction.

Suppose next that the $\boldsymbol{Sp}(1)$ action on $F_{(1)}$ is trivial. Considering the slice representation at a point of $F_{(1)}$, we see that the codimension of the principal orbit is equal to $1 + \dim F_{(1)}$, for the $\boldsymbol{Sp}(n)$ action on X. There are just two cases:

(d) $\dim F_{(1)} = 0$; $(a, b) = (2n - 1, 2n - 3)$ or $(2n - 2, 2n - 2)$,
$(c, d) = (n - 1, 2n - 2)$,

(e) $\dim F_{(1)} = 2$; $(a, b) = (2n - 1, 2n - 2)$.

Consider the case (d). The $\boldsymbol{Sp}(n)$ action has a codimension one orbit. Calculating the Euler characteristics, we see that two non-principal orbits are $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and $V_{n,2}/G$, where $G = \boldsymbol{U}(2)$ or $\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$, and the possibility remains only when $X \sim \boldsymbol{P}_{n-1}(\boldsymbol{H}) \times \boldsymbol{P}_{2n-2}(\boldsymbol{C})$. Since codim $\boldsymbol{P}_{n-1}(\boldsymbol{H}) = 4n - 4$ in X, the inclusion $i: V_{n,2}/G \to X$ induces an isomorphism $i^*: H^2(X) \to H^2(V_{n,2}/G)$, and hence $x^{2n-1} \neq 0$ for each non-zero element $x \in H^2(X)$ by Proposition 3.7. This is a contradiction.

Consider the case (e). The isotropy group is $\boldsymbol{Sp}(n - 1) \times \boldsymbol{Sp}(1)$ at each point of $F_{(1)}$. Considering the slice representation at a point of $F_{(1)}$, we see that the principal isotropy group is $\boldsymbol{Sp}(n - 2) \times K$, where K is a closed connected 3-dimensional subgroup of $\boldsymbol{Sp}(2)$. Denote by G the identity component of the normalizer of K in $\boldsymbol{Sp}(2)$. Then G is conjugate to $\boldsymbol{U}(2)$ or $\boldsymbol{U}(1) \times \boldsymbol{Sp}(1)$. Suppose that the restricted G action on $F_{(2)}$ has a fixed point. Then the natural projection of $(V_{n,2} \times F_{(2)})/G$ to $V_{n,2}/G$ has a cross-section. Since the inclusion $i: X_{(2)} \to X$ induces an isomorphism $i^*: H^k(X) \to H^k(X_{(2)})$ for $k \leqq 4n - 6$, we obtain a contradiction by the same way as in the proof of Proposition 4.1. Therefore the $\boldsymbol{Sp}(n)$ action on $X_{(2)}$ has no singular orbit. Denote by T^n the standard maximal torus of $\boldsymbol{Sp}(n)$. Since $X_{(1)} = \boldsymbol{P}_{n-1}(\boldsymbol{H}) \times F_{(1)}$ and the restricted T^n action on $X_{(2)}$ has no fixed point, we see that the fixed point set of the restricted T^n action on X is diffeomorphic to n copies of $F_{(1)}$, and hence $\chi(F_{(1)}) = \chi(X)/n = 4n - 2$. Let U be a closed invariant tubular neighborhood of $X_{(1)}$ in X. Put $E = X - \operatorname{int} U$, and $E_{(2)} = E \cap F_{(2)}$. Then E is an equivariant deformation retract of $X_{(2)}$, and $E_{(2)}$ is a compact connected orientable 10-manifold. Moreover the $\boldsymbol{Sp}(n)$ action on $\partial E = \partial U$ has only one isotropy type $\boldsymbol{Sp}(n-2) \times K$, and its orbit space is diffeomorphic to $F_{(1)}$. We shall evaluate the number of connected components of ∂E. Let $\boldsymbol{Sp}(1)$ be standardly embedded in $\boldsymbol{Sp}(2)$. Considering the Gysin sequences for sphere bundles

$$\boldsymbol{Sp}(2)/\boldsymbol{Sp}(1) \to (V_{n,2} \times E_{(2)})/\boldsymbol{Sp}(1) \to E\ ,$$
$$\boldsymbol{Sp}(1) \to V_{n,2} \times E_{(2)} \to (V_{n,2} \times E_{(2)})/\boldsymbol{Sp}(1)\ ,$$

we obtain $\operatorname{rank} H^9((V_{n,2} \times E_{(2)})/\boldsymbol{Sp}(1)) \leqq 2$, $\operatorname{rank} H^6((V_{n,2} \times E_{(2)})/\boldsymbol{Sp}(1)) \leqq 4$, and hence $\operatorname{rank} H^9(V_{n,2} \times E_{(2)}) \leqq 6$. Thus we obtain $\operatorname{rank} H^0(\partial E_{(2)}) \leqq 7$, by the cohomology exact sequence of the pair $(E_{(2)}, \partial E_{(2)})$ and the Poincaré-Lefschetz duality for $E_{(2)}$. Therefore the number of connected components of ∂E is at most seven, and hence the number of components of the closed surface $F_{(1)}$ is at most seven. This is a contradiction to $\chi(F_{(1)}) = 4n-2$. q.e.d.

Here we complete the proof of the main theorem stated in Introduction, by combining Theorems 2.5, 2.8 and Propositions 4.1, 4.2, in view of Section 1.

5. Proof of Lemmas. We shall give an outline of the proof of Lemmas 1.1, 1.2. The method used here is essentially due to Dynkin [6] (cf. [11, § 7]).

PROOF OF LEMMA 1.1. Let G be a closed connected subgroup of $\boldsymbol{Sp}(n)$, and suppose $\dim \boldsymbol{Sp}(n)/G < 8n$. Notice that the inclusion $i: G \to \boldsymbol{Sp}(n)$ gives a symplectic representation of G.

Suppose first that the representation i is reducible, that is, there is a positive integer k such that $k \leqq n/2$ and G is contained in $\boldsymbol{Sp}(n-k) \times \boldsymbol{Sp}(k)$ up to an inner automorphism of $\boldsymbol{Sp}(n)$. Then

$$2kn \leqq 4k(n-k) \leqq \dim \boldsymbol{Sp}(n)/G < 8n\ .$$

Hence we obtain $k \leqq 3$. Let p_1 (resp. p_2) be the natural projection of $\boldsymbol{Sp}(n-k) \times \boldsymbol{Sp}(k)$ onto $\boldsymbol{Sp}(n-k)$ (resp. $\boldsymbol{Sp}(k)$). We obtain $\dim \boldsymbol{Sp}(n-k)/p_1(G) < 8n - 4k(n-k)$, because

$$\dim \boldsymbol{Sp}(n-k)/p_1(G) \leqq \dim(\boldsymbol{Sp}(n-k) \times \boldsymbol{Sp}(k))/G < 8n - 4k(n-k)\ .$$

SUBLEMMA. *Suppose $p_1(G) = \boldsymbol{Sp}(n-k)$ and $2k < n$. Then $G = \boldsymbol{Sp}(n-k) \times K$ for some closed subgroup K of $\boldsymbol{Sp}(k)$.*

PROOF. Let G' be the kernel of the homomorphism $p_2|G$. Then

$p_1(G')$ is a positive dimensional normal subgroup of $\boldsymbol{Sp}(n-k) = p_1(G)$, and hence $p_1(G') = \boldsymbol{Sp}(n-k)$, because $\boldsymbol{Sp}(n-k)$ is simple. Therefore $G = \boldsymbol{Sp}(n-k) \times K$ for some closed subgroup K of $\boldsymbol{Sp}(k)$. q.e.d.

We can assume that the inclusion $i_1: p_1(G) \to \boldsymbol{Sp}(n-k)$ is irreducible. Here we assume that the representation $i: G \to \boldsymbol{Sp}(n)$ is irreducible and $\dim \boldsymbol{Sp}(n)/G < 8n$ (i.e. $\dim G > 2n^2 - 7n$) for $n \geqq 4$. In addition, suppose $\dim \boldsymbol{Sp}(n)/G < 32$ for $n = 6$, $\dim \boldsymbol{Sp}(n)/G < 16$ for $n = 5$ and $\dim \boldsymbol{Sp}(n)/G < 8$ for $n = 4$. We shall show that $G = \boldsymbol{Sp}(n)$ under the above condition. This is the final step of the proof of Lemma 1.1.

Denote by $i_C: G \to U(2n)$ the complexification of the quaternion representation i. If i_C is reducible, then

$$2n^2 - 7n < \dim G \leqq \dim U(n) = n^2 ,$$

and hence $n \leqq 6$. But $\dim \boldsymbol{Sp}(6)/\boldsymbol{U}(6) = 42 > 32$, $\dim \boldsymbol{Sp}(5)/\boldsymbol{U}(5) = 30 > 16$ and $\dim \boldsymbol{Sp}(4)/\boldsymbol{U}(4) = 20 > 8$. Therefore i_C is irreducible. Since $i_C(G)$ is contained in $\boldsymbol{SU}(2n)$, we see that G is semi-simple.

Suppose that G is not simple. There are closed normal subgroups H_1, H_2 of G and irreducible representations $r_j: H_j \to \boldsymbol{U}(n_j)$ such that the tensor product $r_1 \otimes r_2$ is equivalent to $i_C p$, where $n = n_1 n_2$, $n_j \geqq 2$ and $p: H_1 \times H_2 \to G$ is a covering projection. Since i_C has a quaternion structure, we can assume that (cf. [1, Proposition 3.56]) r_1 has a real form and r_2 has a quaternion structure. In particular,

$$\dim G = \dim H_1 + \dim H_2 \leqq \dim \boldsymbol{O}(n_1) + \dim \boldsymbol{Sp}(n_2/2) < n_1^2/2 + n_2^2 .$$

Then we obtain $n \leqq 3$. This is a contradiction. Therefore G is simple.

Put $r = \operatorname{rank} G$, and denote by G^* the universal covering group of G. Denote by $L_1, \cdots, L_r$ the fundamental weights of G^*. Then there is a one-to-one correspondence between complex irreducible representation of G^* and sequences $(a_1, \cdots, a_r)$ of non-negative integers such that $a_1L_1 + \cdots + a_rL_r$ is the highest weight of a corresponding representation (cf. [6, Theorems 0.8, 0.9]; [8, § 21.2]). Denote by $d(a_1L_1 + \cdots + a_rL_r)$ the degree of the complex irreducible representation of G^* with the highest weight $a_1L_1 + \cdots + a_rL_r$. The degree can be computed by Weyl's dimension formula (cf. [6, Theorem 0.24, (0.148)-(0.155)]; [8, § 24.3]). Notice that if $a_i \geqq a_i'$ for $i = 1, 2, \cdots, r$, then $d(a_1L_1 + \cdots + a_rL_r) \geqq d(a_1'L_1 + \cdots + a_r'L_r)$ and the equality holds only if $a_i = a_i'$ for $i = 1, 2, \cdots, r$.

If G is an exceptional Lie group, then G^* has no complex irreducible representation of degree $2n$ for each n such that $\dim G > 2n^2 - 7n$. Therefore G is a classical Lie group.

Suppose $G^* = \boldsymbol{SU}(r+1)$, $r \geqq 1$. Then $\dim G = r^2 + 2r$, and $r = \operatorname{rank} G \leqq \operatorname{rank} \boldsymbol{Sp}(n) = n$. Hence we obtain $n \leqq 8$ by the inequality

$$2n^2 - 7n < r^2 + 2r \leqq n^2 + 2n .$$

The possibilities remain only when $(n, r) = (8, 8), (7, 7), (6, 6), (6, 5), (5, 5), (5, 4), (4, 4), (4, 3)$ or $(4, 2)$. We see that there is no possibility, by the value $\dim \boldsymbol{Sp}(n)/\boldsymbol{SU}(n)$ for $n \leqq 6$ and the fact that $\boldsymbol{SU}(r+1)$ has no complex irreducible representation of degree $2r$ for each $r \geqq 4$.

Suppose $G^* = \boldsymbol{Spin}(r)$, $r \geqq 5$. Since $\dim G < \dim \boldsymbol{Sp}(n)$, we obtain $(2n-3)(2n-4) - 12 < r(r-1) < 2n(2n+1)$. Thus we obtain $r = 2n-3$, $2n-2$, $2n-1$ or $2n$. By Weyl's formula, we see that $\boldsymbol{Spin}(2n-1)$ for $n \geqq 5$, $\boldsymbol{Spin}(2n-3)$ and $\boldsymbol{Spin}(2n-2)$ have no complex irreducible representation of degree $2n$, $\boldsymbol{Spin}(2n)$ has only one complex irreducible representation ρ_{2n}^{C} of degree $2n$ for $n \geqq 5$, $\boldsymbol{Spin}(8)$ has just three complex irreducible representations ρ_8^C, Δ_8^+ and Δ_8^- of degree 8, and $\boldsymbol{Spin}(7)$ has only one complex irreducible representation Δ_7 of degree 8. But ρ_{2n}^C, Δ_8^+, Δ_8^- and Δ_7 have real forms, and hence they have no quaternion structure.

Suppose $G^* = \boldsymbol{Sp}(r)$, $3 \leqq r < n$. Then we obtain $r = n-2$ or $n-1$. But $\boldsymbol{Sp}(r)$ has no complex irreducible representation of degrees $2r+2$ and $2r+4$.

This completes the proof of Lemma 1.1.

PROOF OF LEMMA 1.2. By Weyl's formula, we see that there is no complex irreducible representation of $\boldsymbol{Sp}(r)$ of degree $< 8r$ except for the natural inclusion $(\nu_r)_C \colon \boldsymbol{Sp}(r) \to \boldsymbol{U}(2r)$. This fact assures the desired result. q.e.d.

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