Tôhoku Math. Journ.
36 (1984), 269-291.

SOME REMARKS ON THE INSTABILITY FLAG

S. RAMANAN AND A. RAMANATHAN

(Received June 13, 1983)

Introduction. Let G be a reductive group acting on a projective variety $M \subset \boldsymbol{P}(V)$. Mumford raised the question of associating a canonical parabolic subgroup $P(m)$ of G to any nonsemistable point $m \in M$ ([9, p. 64]). This would have the interesting consequence that if the action of G as well as the point m are rational over a perfect field, then $P(m)$ would (by Galois descent) be rational. Recently this problem was solved in the affirmative by Kempf [7] and Rousseau [16].

Besides, one can also associate to m, a conjugacy class of 1-parameter subgroups (1-PS's) in $P(m)$. Let λ be a 1-PS in this class and let $V = \bigoplus_{i \in \boldsymbol{Z}} V_i$ be the decomposition of V where V_i is the space of weight vectors of weight i for λ. Let $m = m_0 + m_1$ with $0 \neq m_0 \in V_j$ and $m_1 \in \bigoplus_{i>j} V_i$. Then we show by a refinement of Kempf's arguments that $P(m_0) = P(m)$ (Proposition 1.9). Moreover for the natural action of P/U on V_j, where U is the unipotent radical of P, m_0 becomes semistable after the polarisation is replaced by a multiple and the action of P/U is twisted by a dominant character (Proposition 1.12).

In Section 2 we investigate the existence of the instability flag over non-perfect fields. We prove that if G acts separably (see Definition 2.1) on M over k and m is a k-rational nonsemistable point then $P(m)$ is defined over k (Theorem 2.3).

In Section 3 we apply these results to the study of bundles on projective nonsingular varieties. If E is a semistable vector bundle on a projective curve X defined over an algebraically closed field of characteristic 0 then it follows from the characterisation of stable bundles in terms of unitary representation of Fuchsian groups ([10]) that any bundle associated to E by exension of structure group is also semistable. An analogous result is also valid for G-bundles, thanks to [14]. An algebraic proof of this has been given in [5], [8]. We give another algebraic proof using the existence of parabolic subgroups mentioned above. Our ideas are close to those in [1].

We will now give an outline of our proof. For simplicity we assume that X is a curve. A principal G-bundle $E \to X$ is semistable if for any reduction of structure group to a parabolic subgroup P and any dominant character on P the associated line bundle has degree $\leqq 0$ (Definitions 3.7 and 4.7). Let $\rho: G \to G'$ be a homomorphism and E' be the G'-bundle obtained by extension of structure group. A reduction of structure group of E' to the parabolic subgroup P' corresponds to a section σ of the fibre bundle $E(G'/P')$ with fibre G'/P' associated to E for the action of G on G'/P' through ρ. Let $G' \to \mathrm{GL}(V)$ be a representation giving an embedding $G'/P' \subset \boldsymbol{P}(V)$. The line bundle $\mathcal{O}(-1)$ on $\boldsymbol{P}(V)$ gives naturally a line bundle L on $E(G'/P')$. To prove the semistability of E' we have to show that $\deg \sigma^*(L) \leqq 0$.

Suppose that for some $x \in X$, $\sigma(x)$ is semistable for the action of G on the fibre $E(G'/P')_x$ (determined up to inner conjugation). Any G-invariant polynomial on V of degree n gives naturally a linear map $\tilde{\varphi}: S^n(E(V)) \to \mathcal{O}_X$. Since $\sigma(x)$ is semistable there exists φ of degree $n > 0$, such that $\tilde{\varphi}$ restricted to $\sigma^*(L)^n$, is nonzero. This means that $\sigma^*(L)^{-n}$ has a nonzero section and hence $\deg \sigma^*(L) \leqq 0$ (Proposition 3.10).

On the other hand if x_0 is the generic point of X and $m = \sigma(x_0)$ is nonsemistable then the instability flag $P(m)$ and a corresponding instability 1-PS of G are defined over $K(X)$, the function field of X (since $\operatorname{Char} K(X) = 0$). This parabolic subgroup being defined generically gives a reduction of structure group of E to a parabolic subgroup P of G. Using the notation introduced earlier, in the generic fibre $E(V)_{x_0}$, the projection $\bigoplus_{i \geqq j} V_i \to V_j$, which takes m to m_0, gives a nonzero map $\sigma^*(L) \to L_0$ where L_0 is the line subbundle of $E(V)$ corresponding to m_0. Now m_0 is semistable for P/U for a suitable linearisation. Therefore by the previous case $\deg(L_0 \otimes L_1^{-1}) \leqq 0$ where L_1 is a line bundle associated to the P-reduction of E through a dominant character. Since E is semistable, $\deg L_1 \leqq 0$. Thus $\deg L_0 \leqq 0$ and hence $\deg \sigma^* L \leqq 0$. This proves that the associated bundle E' is semistable (Theorem 3.18).

In characteristic p the above result is false since there exist semistable vector bundles whose Frobenius twist is nonsemistable. We have given a simple proof for the existence of such vector bundles on any curve of genus $\geqq 2$ (Proposition 4.4). However the rationality results of Section 2 and the above argument yield some positive results such as:

(1) If E is a vector bundle on X such that all the Frobenius twists $E^{(r)}$ are semistable then so are $S^i(E)$, $\bigwedge^i(E)$, etc. (Theorem 3.23).

(2) If E is a vector bundle of rank 2, then $S^i(E)$ is semistable for $i \leqq p - 1$ (Theorem 3.21).

We thank the referee for drawing our attention to [20].

1. Instability Flag. In this section k will be an algebraically closed field and we will be working in the category of k-schemes of finite type.

Let M be a projective variety and L an ample line bundle on M. Let G be a reductive algebraic group acting on M and linearly on L compatible with its action on M ([9], [11]).

A point $m \in M$ is *semistable* if there is a G-invariant section $s \in H^0(M, L^r)$, for some $r > 0$, such that $s(m) \neq 0$. A semistable point is quasi-stable if in addition the G-orbit of m is closed in the affine open subset $\{x \in M \mid s(x) \neq 0\}$. If further the isotropy at m is finite, then m is said to be *stable*. These definitions are slightly different from those given in [9], [7] (cf. [11]).

For large r, there exists a G-equivariant embedding of M in $\boldsymbol{P}(V)$ where $V = H^0(M, L^r)^*$ with the obvious representation of G.

Let T be a torus of G. Then V can be decomposed uniquely as $V = \bigoplus V_l$, where l runs through the character group $X^*(T)$ and $V_l = \{v \in V \mid t \cdot v = l(t)v \ \forall\, t \in T\}$. Let $m \in M$. Following Kempf [7] define the *state* $S_T(m)$ of m with respect to T to be the set $\{l \in X^*(T) \mid V_l$ – component of m is non-zero$\}$. (When there is no likelihood of confusion we use the same letter to denote a point of $\boldsymbol{P}(V)$ and any of its lifts in V.)

Let $X_*(T)$ be the group of one parameter subgroups (abbreviated: 1-PS) of T, i.e., homomorphisms of the multiplicative group G_m into T. There is a natural perfect pairing between $X^*(T)$ and $X_*(T)$. Using this pairing we think of $X^*(T)$ as the dual of $X_*(T)$ and for $l \in X^*(T)$ and $\lambda \in X_*(T)$ we denote the value of the pairing by $l(\lambda) \in \boldsymbol{Z}$. We extend this pairing to $X_*(T) \otimes_Z \boldsymbol{R}$.

For $x \in X_*(T) \otimes \boldsymbol{R}$ define $\mu(m, x) = \operatorname{Inf}\{l(x) \mid l \in S_T(m)\}$. When x is a 1-PS in G, clearly $\mu(m, x)$ depends only on x and not on T and further $\mu(gm, gxg^{-1}) = \mu(m, x)$, $g \in G$. If we take the projective embedding of M by L^{rs} $(s > 0)$ instead of L^r the corresponding $\mu(m, x)$ gets multiplied by s.

We have the following numerical criterion: m is semistable if and only if $\mu(m, x) \leqq 0$ for every 1-PS x of G ([9, Theorem 2.1, p. 49]. The μ in this reference is the negative of our μ.).

Let T be a maximal torus of G. We will introduce a scalar product in $X_*(T) \otimes \boldsymbol{R}$. For doing this we choose a maximal torus T_0 of G and fix once for all a Weyl group invariant scalar product in $X_*(T_0) \otimes \boldsymbol{R}$ which is $\boldsymbol{Q}$-valued on $X_*(T_0)$ and which makes the central torus and the semisimple torus orthogonal (cf. [4, § 2.1, p. 63]). Now T_0 can be conjugated onto T by an element of G and the isomorphism $T_0 \to T$ is well defined up to Weyl group action on T_0. Therefore the scalar product in $X_*(T_0) \otimes \boldsymbol{R}$ determines uniquely one in $X_*(T) \otimes \boldsymbol{R}$. For any 1-PS λ of G we then have a well defined norm $\|\lambda\|$, with $\|\lambda\|^2 \in \boldsymbol{Q}$. Using this scalar product we identify $X_*(T) \otimes \boldsymbol{Q}$ with its dual $X^*(T) \otimes \boldsymbol{Q}$ and we get a scalar product in $X^*(T) \otimes \boldsymbol{R}$ as well. We use the notation $\langle\ ,\ \rangle$ for these scalar products.

The function $\mu(m, x)$ clearly satisfies: $\mu(m, ax) = a\mu(m, x)$, $\forall\, a \in \boldsymbol{R}^+$. We define $\nu(m, x) = \mu(m, x)/\|x\|$.

Let $m \in M$ be a nonsemistable point of M. Then there is a 1-PS λ with $\mu(m, \lambda) > 0$. Kempf [7] has shown how to find canonically a class of such a 1-PS's. We will need a complement to his results and in proving it we shall reprove his result, for completeness.

The following lemma is essentially in [7].

LEMMA 1.1. *Let S be a finite nonempty set of linear forms on $\boldsymbol{R}^n$. Define, for $x \in \boldsymbol{R}^n$, $\mu_S(x) = \operatorname{Inf}\{l(x) \mid l \in S\}$. Then $\mu_S(\lambda x) = \lambda\mu_S(x)$ for $\lambda \in \boldsymbol{R}^+$. Let $S^{n-1} \subset \boldsymbol{R}^n$ be the unit sphere and $\bar{\mu}_S$ the restriction of μ_S to S^{n-1}. Suppose $\mu_S(x) > 0$ for some $x \in \boldsymbol{R}^n$. Then:*

(i) *$\bar{\mu}_S$ attains its maximum at a unique point $a \in S^{n-1}$. In fact a is the only point where μ_S attains a local maximum with positive value.*

(ii) *If $S' \subset S$ is such that $\mu_{S'}(x) = \mu_S(x)$ for every x in some neighbourhood of the maximum a,* (e.g., $\{l \in S \mid l(a) = \mu_S(a)\}$ *is such an S'), then $\bar{\mu}_{S'}$ also attains its unique maximum at a.*

(iii) *If the linear forms in S take values in $\boldsymbol{Q}$ on a lattice Γ in $\boldsymbol{R}^n$ with $\boldsymbol{Q}\cdot\Gamma = \boldsymbol{Q}^n \subset \boldsymbol{R}^n$, then $\lambda\cdot a \in \Gamma$ for some nonzero $\lambda \in \boldsymbol{R}$. We then have a unique $\lambda_0 a$, $\lambda_0 > 0$, such that $\boldsymbol{R}^+\cdot a \cap \Gamma = \{q\cdot\lambda_0 a \mid q \in \boldsymbol{Z}^+\}$.*

PROOF. (i) Let $b \in S^{n-1}$ be a point at which $\bar{\mu}_S$ takes a local maximum, with $\mu_S(b) > 0$. Then we have to prove $b = a$. For $l \in S$ we have $l(a) \geqq \mu_S(a) \geqq \mu_S(b)$ and $l(b) \geqq \mu_S(b)$. Therefore if $x = ta + (1-t)b$, $0 < t < 1$, then (1) $l(x) = tl(a) + (1-t)l(b) \geqq \mu_S(b)$. If $a \neq b$, then $\|x\| < 1$ for all t sufficiently small, so that since $\mu_S(b) > 0$ by assumption, we have (2) $\mu_S(b)/\|x\| \geqq \mu_S(b)$. Dividing (1) by $\|x\|$ and using (2) we have $l(x/\|x\|) \geqq \mu_S(b)$ which contradicts local maximality at b.

(ii) Follows immediately from (i).

(iii) By (ii), we may replace S by $\{l \in S \mid l(a) = \mu_S(a)\}$ without altering a. Thus we may as well assume that $l(a) = \mu_S(a)$ for all $l \in S$. Let $V = \{x \in \boldsymbol{R}^n \mid l(x) = l'(x)\ \forall\, l, l' \in S\}$. Then evidently V is defined over $\boldsymbol{Q}$, and $a \in V$. If $\bar{S} = \{l \mid V : l \in S\}$, then $\bar{\mu}_{\bar{S}}$ attains its maximum at a. But $\bar{S}$ consists of a single linear form l. In this case, it is clear that a is the unit vector orthogonal to $\ker l$, and since $\langle\ ,\ \rangle$ is rational, so is $\boldsymbol{R}a$.

REMARK 1.2. If all $l \in S$ vanish on a subspace $W \subset \boldsymbol{R}^n$, then for any $x \in \boldsymbol{R}^n$ with $\mu_S(x) > 0$, we have $\mu_S(x) = \mu_S(x') \leqq \mu_S(x'/\|x'\|)$, where x' is the projection of x on the orthogonal complement of W. In particular, the maximal point a is orthogonal to W.

LEMMA 1.3. *Let the notation be as in Lemma* 1.1.

(i) *Denoting by* $\langle\ ,\ \rangle$ *the standard scalar product in* $\boldsymbol{R}^n$, *for any* $x \in \boldsymbol{R}^n$ *such that* $\langle x, a\rangle < 0$ (resp. $\leqq 0$) *we have* $\mu_S(x) < 0$ (resp. $\leqq 0$).

(ii) *Let* l_a *be the linear form dual to* a, i.e., $l_a(x) = \langle a, x\rangle$. *Let* $S_1 = \{l - \mu_S(a)\cdot l_a \mid l \in S\}$. *Then* $\mu_{S_1}(x) \leqq 0$ *for* $\forall x \in \boldsymbol{R}^n$.

PROOF. (i) We shall show that $\langle x, a\rangle < 0 \Rightarrow \mu_S(x) < 0$. The second assertion then follows by continuity. Assume $\langle x, a\rangle < 0$ and consider $a + tx$ for $t > 0$. We have $\|a + tx\|^2 = 1 + 2\langle x, a\rangle t + t^2\|x\|^2$. Therefore for small t, $\|a + tx\| < 1$. If $\mu_S(x) \geqq 0$ then $l(x) \geqq 0$ $\forall\, l \in S$ and we have $l(a + tx) = l(a) + tl(x) \geqq l(a)$ so that $l((a + tx)/\|a + tx\|) \geqq l(a)/\|a + tx\| \geqq l(a)$ for small t. This would contradict maximality at a. Therefore $\mu_S(x) < 0$.

(ii) Let $x = t\cdot a + y$ with $\langle a, y\rangle = 0$. If $S' = \{l \in S \mid l(a) = \mu_S(a)\}$, then by Lemma 1.1 (ii) $\bar{\mu}_{S'}$ also has a as maximum point. Applying part (i) of the present lemma to S' we get (1) $\mu_{S'}(y) \leqq 0$. Let $S'_1 = \{l - \mu_S(a)\cdot l_a \mid l \in S'\}$. Since $S'_1 \subset S_1$ we have (2) $\mu_{S_1}(x) \leqq \mu_{S'_1}(x)$. For $l \in S'$, $(l - \mu_S(a)\cdot l_a)(x) = (l - \mu_S(a)l_a)(ta + y) = l(y)$, so that $\mu_{S'_1}(x) = \mu_{S'}(y) \leqq 0$ by (1). From (2) we get $\mu_{S_1}(x) \leqq 0$.

REMARKS 1.4. If λ is a 1-PS of G we can associate to it a parabolic subgroup $P(\lambda)$. Let T be a maximal torus of G containing λ. Then $P(\lambda)$ is generated by T and the root groups U_α corresponding to the roots α such that $\alpha(\lambda) \geqq 0$ ([9, Ch. 2, § 2, pp. 55–57]). Since $P(\lambda)$ leaves the filtration of V by the weight spaces of λ (see § 1.8 below) invariant, it has the important property that for any $g \in P(\lambda)$, $\mu(gm, \lambda) = \mu(m, \lambda)$ ([9, Proposition 2.7, p. 57]).

Again the following result is due to Kempf [7].

THEOREM 1.5. *Let* $m \in M$ *be a nonsemistable point of* M. *Then*

(a) *The function* $\lambda \mapsto \nu(m, \lambda) = \mu(m, \lambda)/\|\lambda\|$ *on the set of all* 1-PS's *of* G *attains a maximum value* B.

(b) *If* T *is a maximal torus and* $\lambda \in X_*(T)$ *is such that* (i) λ *is indivisible and* (ii) $\nu(m, \lambda) = B$ *then* λ *is the only element of* $X_*(T)$ *satisfying* (i) *and* (ii).

(c) *There exists a parabolic group* P *such that if* λ *is an indivisible* 1-PS *of* G *with* $\nu(m, \lambda) = B$ *then* $P(\lambda) = P$. *If* $\nu(m, \lambda') = B$ *then* λ *and* λ' *are conjugate in* P.

PROOF. (a) Let T be a maximal torus of G. If λ is any 1-PS of G, there exists $g \in G$ such that $g\lambda g^{-1} \in X_*(T)$ and $\nu(m, \lambda) = \nu(gm, g\lambda g^{-1})$. Therefore $\max\{\nu(m, \lambda) \mid \lambda \in X_*(G)\} = \max\{\nu(gm, \lambda) \mid \lambda \in X_*(T),\ g \in G\}$. For a fixed g, the function $x \mapsto \nu(gm, x)$ on $X_*(T) \otimes \boldsymbol{R}$ depends only on the state of gm with respect to T. Since there are only finitely many possible states it follows that as g varies over G we get only finitely many distinct functions $\nu(gm, x)$, where g belongs to a finite subset A of G. Each $\nu(gm, x)$ is constant on rays and hence attains a maximum in $X_*(T) \otimes \boldsymbol{R}$, in fact at a point of $X_*(T)$, by Lemma 1.1 (iii). Therefore $\max_{g \in A} \max\{\nu(gm, x) \mid x \in X_*(T) \otimes \boldsymbol{R}\}$, is attained at a $\lambda_0 \in X_*(T)$ and $g_0 \in A$. Then clearly $\nu(m, \lambda) \leqq \nu(m, g_0\lambda_0 g_0^{-1}) = B$ for $\forall$ 1-PS λ of G.

(b) Follows immediately from Lemma 1.1 (i), (iii).

(c) By Bruhat Lemma $P(\lambda_1) \cap P(\lambda_2)$ contains a maximal torus T. We can find $p_i \in P(\lambda_i)$ such that $p_i\lambda_i p_i^{-1} \in X_*(T)$, $i = 1, 2$. But $\nu(m, p_i\lambda_i p_i^{-1}) = \nu(m, \lambda_i) = B$ (cf. Remarks 1.4). Therefore by (b) above $p_1\lambda p_i^{-1} = p_2\lambda_2 p_2^{-1}$ and (c) follows.

DEFINITION 1.6. Let $m \in M$ be a nonsemistable point. We call $B = \max\{\nu(m, \lambda) \mid \lambda = \text{1-PS of } G\}$ the *rate of instability* of m. We call any indivisible 1-PS λ with $\nu(m, \lambda) = B$ an *instability* 1-PS for m and $P(\lambda)$ *the instability flag* (or *instability parabolic subgroup*) of m and denote it by $P(m)$.

REMARKS 1.7. (i) Note that $P(m)$ depends on the scalar product chosen on $X_*(T_0) \otimes \boldsymbol{R}$. However there are not too many Weyl group invariant scalar products on $X_*(T_0) \otimes \boldsymbol{R}$.

(ii) It easily follows from Theorem 1.5 that a maximal torus T contains an instability 1-PS for m (which then is unique) if and only if $T \subset P(m)$. In particular, any parabolic group P contains an instability 1-PS for m, since a maximal torus is contained in $P \cap P(m)$.

(iii) For $g \in G$, $P(gm) = gP(m)g^{-1}$ and hence $P(m)$ contains the (reduced) isotropy at m.

We need a complement to the above theorem, which we state below as Propositions 1.9 and 1.12. First we set up some notation.

1.8. Let λ be an instability 1-PS for $m \in M \subset \boldsymbol{P}(V)$, $V = H^0(M, L^r)^*$. Decompose V for the action of λ: $V = \bigoplus_{i \in \boldsymbol{Z}} V_i$, $V_i = \{v \in V \mid \lambda(t)\cdot V = t^i v$ for every $t \in k^\times\}$. Let $V^q = \bigoplus_{i \geqq q} V_i$. Since the unipotent radical U of $P(m) = P(\lambda)$ is generated by the root groups U_α with $\alpha(\lambda) > 0$ it acts trivially on the associated graded vector space of the $P(m)$ invariant flag $\cdots V^q \supset V^{q+1} \cdots$. Therefore the reductive group $P(m)/U$ acts naturally on $\bigoplus_{q \in \boldsymbol{Z}} V^q/V^{q+1}$.

Let $j = \mu(m, \lambda) = \min\{i \mid m$ has a non-zero component in $V_i\} = \max\{q \mid m \in V^q\}$. Let m_0 be the component of m in V_j. Then $m_0 = \lim_{t \to 0} \lambda(t)\cdot m$ (the limit taken in the projective space).

PROPOSITION 1.9. *The instability* 1-PS λ *of* m *is also an instability* 1-PS *for* m_0. *Consequently* $P(m) = P(m_0)$.

PROOF. In view of Remark 1.7 (ii), $P(m)$ contains an instability 1-PS λ' for m_0. Thus it is enough to prove that if λ' is a 1-PS of $P(m)$ then $\nu(m_0, \lambda') \leqq \nu(m_0, \lambda) \cdots (*)$. We can find a $p \in P(m)$ such that λ' and $p^{-1}\lambda p$ commute. Since $P(m) = Z(\lambda) \cdot U$, where $Z(\lambda) =$ centraliser of λ and $U =$ unipotent radical of $P(m)$, we can take $p = u \in U$. Then λ and $u\lambda' u^{-1}$ commute and hence are contained in a maximal torus T of $P(m)$. Since U acts trivially on V^j/V^{j+1} we can write $um_0 = m_0 + m_1$ with $m_1 \in V^{j+1}$. Therefore the state $S_T(um_0) = S_T(m_0) \cup S_T(m_1)$ so that $\forall x \in X_*(T) \otimes \boldsymbol{R}$

(1) $$\nu(um_0, x) \leqq \nu(m_0, x) \quad \text{for any} \quad x \in X_*(T) \otimes \boldsymbol{R} .$$

By Lemma 1.1 (ii) we see that restricted to $X_*(T)$, λ is the instability 1-PS for m_0. Therefore for $x \in X_*(T) \otimes \boldsymbol{R}$

(2) $$\nu(m_0, x) \leqq \nu(m_0, \lambda) \quad \text{for} \quad x \in X_*(T) \otimes \boldsymbol{R} .$$

Now clearly

(3) $$\nu(m_0, \lambda') = \nu(um_0, u\lambda' u^{-1}) .$$

Since $x = u\lambda' u^{-1} \in X_*(T)$ (3), (1) and (2) yield (*).

1.10. We continue with the notation of § 1.8. Let $\bar{m}$ $(= \bar{m}_0)$ be the (non-zero) projection of m (as well as m_0) in V^j/V^{j+1}. We would like to assert that $\bar{m}$ is semistable for the action of $P(m)/U$ on $\boldsymbol{P}(V^j/V^{j+1})$ but for the offending 1-PS λ. However, since the action of λ makes m non-semistable, we have to cancel off its effect by twisting the action of $P(m)/U$ by a character which restricts to $t \mapsto t^{-j}$ on λ. We will make this precise in the following proposition. First we need to make some remarks about parabolic subgroups and their characters.

REMARKS 1.11. (a) Let P be a parabolic subgroup of the reductive group G. A character on P is called *dominant* if it is trivial on $Z_0 =$ the connected component of the centre of G, and is dominant with respect to a Borel subgroup B contained in P. In other words, if $T \subset B$ is a maximal torus and $\alpha \in X^*(T)$ is a positive root with respect to B then $\langle \alpha, \chi | T \rangle \geqq 0$. This notion is independent of the choice of B in P.

(b) Any maximal torus T of a reductive group G contains the connected component Z_0 of the centre of G. If $\alpha \in X^*(T)$ then some positive multiple $r. \alpha$ extends to G if and only if $\alpha(\lambda) = 0$ $\forall \lambda \in X_*(Z_0)^\perp \subset X_*(T)$. This condition is equivalent dually to $\alpha \in \{\text{Ker}: X^*(T) \to X^*(Z_0)\}^\perp \subset X^*(T)$ (cf. [4]).

(c) Any maximal torus of P/U is the isomorphic image $\bar{T}$ of a maximal torus T of P. If $P = P(\lambda)$ for a 1-PS λ, then its image $\bar{\lambda}$ in P/U is in the centre and hence by (b) some positive multiple $r \cdot l_\lambda$ of $l_\lambda \in X^*(T) = X^*(\bar{T})$, the dual of λ, extends to a character of P/U (and of P). If $\lambda \in X_*(Z_0)^\perp \subset X_*(T)$ then rl_λ gives a dominant character on P (for, the roots α such that $U_\alpha \subset P(\lambda)$ satisfy $\alpha(\lambda) \geqq 0$). Thus λ determines on $P(\lambda)$ a character $\chi(\lambda)$ determined up to raising to a positive power. It is easy to see that, for $p \in P(\lambda)$, $n > 0$, $P(p\lambda^n p^{-1}) = P(\lambda) = P$ and the corresponding characters $\chi(p\lambda^n p^{-1})$ and $\chi(\lambda)$ on P belong to the same $\boldsymbol{Q}^+$-ray, i.e., $\chi(p\lambda^n p^{-1})^{r_1} = \chi(\lambda)^{r_2}$ for some $r_1, r_2 > 0$.

(d) In particular on $P(m)$ we have a character $\chi(m)$, determined up to raising to a positive power, corresponding to any instability 1-PS λ for m, and $\chi(m)$ is a dominant character when Z_0 acts trivially on M and L (cf. Remark 1.2).

PROPOSITION 1.12. *There exist a positive integer r and a character χ on $P(m)$ in the $\boldsymbol{Q}^+$-ray determined by m* (cf. Remarks 1.11 (c), (d) above) *such that $\bar{m}_0 \in \boldsymbol{P}(V^j/V^{j+1})$ is semistable for the natural action of $P(m)/U$ with respect to the linearisation given by the line bundle $\mathcal{O}(r) \otimes \mathcal{O}_{\chi^{-1}}$ where $\mathcal{O}(r) = \bigotimes^r \mathcal{O}(1)$ with the natural action of $P(m)/U$ given by its representation on V^j/V^{j+1} and $\mathcal{O}_{\chi^{-1}}$ is the trivial line bundle with $P(m)/U$ acting on it by the character χ^{-1}.*

PROOF. We claim that λ, or more precisely its image in $P(m)/U$, is the unique instability 1-PS for $\bar{m}_0 \in \boldsymbol{P}(V^j/V^{j+1})$ for the action of $P(m)/U$ (with respect to the linearisation $\mathcal{O}(1)$!) To prove this first note that any maximal torus of $P(m)/U$ is the image $\bar{T}$ of a maximal torus T of $P(m)$ containing λ. Therefore any 1-PS of $P(m)/U$ is the image $\bar{l}$ of a 1-PS l of $P(m)$ with $l, \lambda \in X_*(T)$, T a maximal torus of $P(m)$. Then clearly $\nu(\bar{m}_0, \bar{l}) = \nu(m_0, l)$. By Lemma 1.1 (ii), $\nu(m_0, l) \leqq \nu(m_0, \lambda)$ which proves the claim.

Let χ_1 be a character of $P(m)$ in the $\boldsymbol{Q}^+$-cone determined by m (Remarks 1.11 (c), (d)). Let $B = \nu(m_0, \lambda)$ and $s = \chi_1(\lambda) > 0$. Then $\chi_1 = s \cdot l_\lambda$, l_λ being the dual of λ. Let T be any maximal torus of $P(m)$ with $\lambda \subset T$. Let $S \subset X^*(T)$ be the state of m_0 corresponding to T. By Lemma 1.3 (ii) for the state $S_1 = \{l - (B/s\|\lambda\|)\cdot\chi_1 \,|\, l \in S\}$ we have $\mu_{S_1}(x) \leqq 0$, $\forall x \in X_*(T) \otimes \boldsymbol{R}$. Now $B/s \cdot \|\lambda\| = \mu(m_0, \lambda)/s\|\lambda\|^2$. The scalar product in $X_*(T) \otimes \boldsymbol{Q}$ being rational we have $B/s \cdot \|\lambda\| \in \boldsymbol{Q}$. Let r be a positive integer such that $rB/s\|\lambda\| = q \in \boldsymbol{Z}^+$. We take $\chi = q\chi_1$. Then if we take the linearisation given by $\mathcal{O}(r) \otimes \mathcal{O}_{\chi^{-1}}$ the state for m_0 with respect to T for this linearisation becomes $S_2 = \{l_1 + l_2 + \cdots + l_r - \chi \,|\, l_i$ run through $S\}$ (since $\mathcal{O}(r)$ corresponds to the r-tuple embedding). Since $r \cdot S_1 = \{r \cdot (l -$

$(B/s\|\lambda\|)\chi_1) \mid l \in S\} \subset S_2$, we have $\mu_{S_2}(x) \leqq r \cdot \mu_{S_1}(x) \leqq 0 \quad \forall x \in X_*(T) \otimes \boldsymbol{R}$. Since $\bar{T}$ is an arbitrary maximal torus of $P(m)/U$ and the numbers r, q and χ are independent of T we have proved the proposition.

REMARK 1.13. The polarisation $\mathcal{O}(r) \otimes \mathcal{O}_{\chi^{-1}}$ corresponds to viewing the point m_0 as its image in $S^r(V^j/V^{j+1}) \otimes \mathcal{O}_{\chi^{-1}}$.

2. Rationality of the Instability Flag.

Let K be an arbitrary field, K_s its separable closure and $\bar{K}$ its algebraic closure. Let G be a reductive group over K. Then G has a maximal torus T_0 defined (though not split) over K ([3, Proposition 7.10, p. 480]). T_0 splits over K_s (in fact over a finite extension of K in K_s). The Galois group $\mathrm{Gal}(K_s/K)$ acts on $X_*(T_0 \otimes K_s)$ (through a finite group). We take for our scalar product in $X_*(T_0 \otimes K_s) \otimes \boldsymbol{R}$ one which is invariant under both the Weyl group and the Galois group (cf. [7, § 4, p. 312]).

Let M be a projective K-scheme and L an ample line bundle on M. Let G act K-rationally on M, L. Let m be a nonsemistable point of M. We have seen that m has an instability 1-PS λ and a unique instability parabolic subgroup $P(m) = P(\lambda)$ all defined over $\bar{K}$. We wish to investigate when these are defined over K itself.

If $P(m)$ is defined over K_s using (a) the Galois invariance of the basic scalar product and (b) the uniqueness of $P(m)$ we can conclude that $P(m)$ is already defined over K, by Galois descent. In particular if K is perfect then $P(m)$ is always defined over K. This is the theorem of Kempf ([7]).

We will give a criterion for the rationality of $P(m)$ in terms of the geometric action of G on M when K is not necessarily perfect.

DEFINITION 2.1. Let $x \in M(\bar{K})$ be a $\bar{K}$-rational point of M. Let $O(x)$ be the orbit of x taken with its reduced subscheme structure. We say that G acts *separably* at x if the orbit map $G \to O(x)$, $(g \mapsto gx)$, is separable (cf. [3, § 1.14, p. 452]). If G acts separably at every point of $\overline{O(x)}$, the closure of $O(x)$ in M, we say that G acts *strongly separably* at x.

REMARKS 2.2. If G acts separably at x, then the isotropy H at x is (absolutely) reduced and the natural map $G/H \to O(x)$ is an isomorphism. If further x is a K-rational point then the latter is a K-isomorphism ([2, Theorem 17.3, p. 75, Proposition 6.7, p. 180]).

THEOREM 2.3. *Let m be a K-rational nonsemistable point of M. If G acts strongly separably at m, then the instability flag $P(m)$ is defined over K. There is an instability* 1-PS *for m also defined over K.*

PROOF. By what we have remarked above, it is enough to prove that $P(m)$ is defined over K_s. Further, once we have proved that $P(m)$ is defined over K, it has a maximal torus T defined over K, which splits over K_s. Hence the instability 1-PS contained in T (Remark 1.7 (ii)) is defined over K_s and is invariant under $\mathrm{Gal}(K_s/K)$, because of uniqueness. It follows that it is defined over K.

We can find a $g \in G(\bar{K})$ such that $gP(m)g^{-1} = P$ is defined over K_s [4]. Let $x = g\cdot m \in M(\bar{K})$. Then clearly $P = P(x)$. We need the following lemma.

LEMMA 2.4. *There is a K_s-subscheme $M(P)$ of the K-scheme $O(m)$, whose $\bar{K}$-rational points are precisely those of $O(m)$ which have $P(x)$ as their instability flag. When the action of G is strongly separable at m, the scheme $M(P)$ is absolutely reduced.*

We will now assume the lemma and complete the proof of the theorem.

Since we have assumed the action at m to be strongly separable, the K_s-scheme $M(P)$ is absolutely reduced. Therefore $M(P)$ has a K_s-rational point y ([2, § 13.3, p. 52]). Again because of the separability of the action the natural map $G/H \to O(m)$ is a K-isomorphism, where H is the isotropy at m. The map $G(K_s) \to (G/H)(K_s)$ is surjective (since $G \to G/H$ is locally trivial for etale topology, [2, pp. 182–183]). Therefore y, m being rational points of $O(m)$, we can find $h \in G(K_s)$ such that $y = hm$. Since $P(y) = P(x)$ is defined over K_s and $P(m) = hP(y)h^{-1}$ it follows that $P(m)$ is defined over K_s.

It now remains to prove the lemma.

PROOF OF LEMMA 2.4. Since P is defined over K_s and G is split over K_s, P has a maximal torus split over K_s ([4, §4.2, p. 85; Theorem 4.13 (c), p. 90]). Then the instability 1-PS λ of m in this split torus is defined over K_s. We have $M \subset \boldsymbol{P}(V)$, $V = H^0(M, L^r)^*$ with V a representation space for G over K. Decompose V for the action of λ: $V = \bigoplus_{i\in Z} V_i$, $V_i = \{v \in V \mid \lambda(t)\cdot v = t^i v,\ t \in K_s^*\}$. Since λ is defined over K_s all the subspaces V_i are defined over K_s. Let $j = \inf\{i \mid x$ has a non-zero component in $V_i\}$ and $V^j = \bigoplus_{i\geqq j} V_i$. We define $M(P)$ to be the scheme-theoretic intersection of the K_s-subschemes V^j and $O(m)$ of V. Let y be a $\bar{K}$-rational point of $M(P)$. Since $y \in V^j$, we have (1) $\mu(y, \lambda) \geqq j$. On the other hand $y \in O(m)(\bar{K}) = O(x)(\bar{K})$ so that y has the same rate of instability as x, which is $j/\|\lambda\|$. Therefore (2) $\mu(y, \lambda) \leqq j$. Now (1) and (2) show that λ is an instability 1-PS for y as well so that $P(y) = P$. Conversely for some $y \in O(m)(\bar{K})$ suppose $P(y) = P$. Then λ is an instability 1-PS for y (for: if $y = g\cdot x$ then $P(y) = gPg^{-1}$ so that $P(y) = P$ gives $g \in P$ and hence

$g\lambda g^{-1}$ is in P). Therefore $\mu(y, \lambda) = \mu(x, \lambda) = j$. Therefore $y \in V^j(\bar{K})$.

We now proceed to show that $M(P)$ is absolutely reduced when G acts strongly separably at m. Note that this is a geometric problem and therefore we can work over $\bar{K}$.

Let $\varphi: G \to O(x)$ $(=O(m))$ be the orbit map, $g \mapsto gx$. Let $\varphi^{-1}(M(P)) = N$ be the pull back. Since $P(\lambda) = P(x)$ leaves both the subschemes $O(x)$ and V^j invariant it acts on the scheme $M(P)$, and hence on its pull back N. We claim $P(\bar{K})$ acts transitively on $N(\bar{K})$. In fact we shall show that $P(\bar{K}) = N(\bar{K})$ so that $N_{\mathrm{red}} = P$. $g \in N(\bar{K}) \Leftrightarrow gx \in M(P)(\bar{K}) \Leftrightarrow g \in P(\bar{K})$.

If G acts separably at x, then $\varphi: G \to O(x)$ $(\approx G/M_x)$ is a smooth morphism. Therefore to show that $M(P)$ is reduced it is enough to show that so is N.

Because of the homogeneity of N it is enough to show that N is reduced at the identity of G. For this it is enough to show that the Zariski tangent space $T_e(N)$ to N at identity coincides with the Lie algebra of P. It is easy to see that $T_e(N) = \{X \in T_e(G) \mid X(x) \in V^j\}$. Let now $X \in T_e(N)$. We may write $X = \sum_{i \geqq i_0} X_i$, $X_{i_0} \neq 0$, where X_i is a weight vector for λ with weight $i \in \boldsymbol{Z}$. *Suppose* $i_0 < 0$. Then $X(x) = (\sum X_i)(x) \equiv X_{i_0}(x_0) \bmod V^{j+i_0+1}$. Since $X(x) \in V^j$, this implies that $X_{i_0}(x_0) = 0$. But $x_0 \in \boldsymbol{P}(V)$ is in the closure $\overline{O(x)} = \overline{O(m)}$. By assumption G acts separably at x_0. Therefore $X_{i_0}(x_0) = 0$ implies that X_{i_0} is in the Lie algebra of the isotropy at x_0. But P is the instability flag for x_0 as well (Proposition 1.9) and hence contains the isotropy at x_0 (Remarks 1.7 (iii)). Therefore X_{i_0} is in the Lie algebra of P, i.e., $i_0 \geqq 0$. *Contradiction.* This proves $T_e(N) =$ Lie algebra of P, and completes the proof of the lemma.

EXAMPLES 2.5. For parabolic subgroup P the action of G on G/P is strongly separable, as follows from the construction of the quotient G/P (see [2, Proposition 6.7, p. 180; Theorem 6.8, p. 181]). The following lemma gives another example.

LEMMA 2.6. *Let* $\operatorname{Char} k = p$. *Let V be the 2-dimensional space on which* SL (2) *acts naturally. Let W be a subspace of $S^m(V)$, the m-th symmetric power of V. For $m \leqq p-1$ the stabiliser of W in* SL (2) *is reduced. In other words the action of* SL (2) *on the Grassmannian is strongly separable.*

PROOF. The isotropy group scheme I of W in SL (2) has as its tangent space at identity, $\operatorname{Lie} I = \{X \in \mathrm{sl}(2) \mid \rho(X)(W) \subset W\}$ where sl (2) is the Lie algebra of SL (2) and ρ is the representation of SL (2) (and of sl (2)) on $S^m(V)$. For any $X \in \operatorname{Lie} I$ the semisimple part X_s and the nilpotent part X_n of X also belong to $\operatorname{Lie} I$ since $\rho(X_s)$ and $\rho(X_n)$ can be written as polynomials in $\rho(X)$ without constant term ([2, Proposition 4.2, p. 143]). So it is enough to prove that X_s and X_n are in Lie I_{red}, for then Lie $I_{\mathrm{red}} =$ Lie I which implies $I = I_{\mathrm{red}}$.

First we assume X is nilpotent. Then the map $t \mapsto 1 + tX$ is a homomorphism of G_a, the additive group of k, into SL(2) which has X as tangent. Therefore to show that $X \in \mathrm{Lie}\, I_{\mathrm{red}}$ it is enough to show that $\rho(1 + tX)(W) \subset W$. But then, for $m \leqq p - 1$, an easy checking shows that $\rho(1 + tX) = \exp(t\rho(X))$, where the latter is a polynomial of degree m in $\rho(X)$. Therefore $\rho(1 + tX)(W) \subset W$.

Now let us assume X is semisimple and non-zero. Then X is tangential to a maximal torus of SL(2). The eigenvalues of $\rho(X)$ are m, $m - 2, \cdots, -m$ all taken mod p. Since $m \leqq p - 1$ these are distinct as elements of the field k, for $p \neq 2$ (if $p = 2$ then $m = 1$ and there is nothing to prove). Therefore the torus corresponding to X also has the same eigenvectors as X and hence has the same invariant subspaces as X in $S^m(V)$, namely subspaces spanned by the eigenvectors.

3. Associated Bundles and Semistability. We fix some notation regarding principal bundles.

If $\pi: E \to X$ is a principal bundle on X with structure group G, (or a G-bundle for short) we recall that G operates on E on the right and π is G-invariant and isotrivial, i.e., locally trivial in the étale topology ([17]; [6, exposé XI, Definition 4.1]).

If F is a quasi projective scheme on which G operates (on the left) the associated bundle is denoted by $E(F)$. Recall $E(F)$ is the quotient of $E \times F$ under the action of G given by $g(e, f) = (e\cdot g, g^{-1}\cdot f)$, $e \in E$, $f \in F$, $g \in G$ ([17]; [6, exposé V, § 1]).

3.1. Any G-equivariant map $F_1 \to F_2$ gives naturally a morphism $E(F_1) \to E(F_2)$. A section $\sigma: X \to E(F)$ is given by a morphism $\tilde{\sigma}: E \to F$ such that $\tilde{\sigma}(e\cdot g) = g^{-1}\cdot\sigma(e)$: $\sigma(x) = (e, \tilde{\sigma}(e))$, $e \in E$, $g \in G$, $x \in X$.

3.2. If H is a closed subgroup of G (or more generally $H \to G$ a homomorphism) an H-bundle E_H together with an isomorphism of the associated G-bundle $E_H(G)$ (for the action of H on G by left translations) with E is said to give a *reduction of structure group of E to H*. Conversely E is said to be the G-bundle obtained from E_H by *extension of structure* group. For $H \subset G$ the quotient E/H is naturally isomorphic to the associated bundle $E(G/H)$. Note that $E \to E/H$ is an H-bundle and a section $\sigma: X \to E/H$ gives the H-bundle $\sigma^*(E)$ with a natural isomorphism $\sigma^*(E)(G) \cong E$. Thus we get a bijective correspondence between sections of $E/H \to X$ and reductions of structure group of E to H.

3.3. Let G act on itself by inner conjugation. Then the associated bundle $E(G) \to X$ is naturally a group scheme over X (the fibres have group structure, since the action of G preserves the group structure). If G acts on $G \times F$ by inner conjugation on the first factor and a given action $\varphi: G \times F \to F$ on F then φ is G-equivariant and hence (§ 3.1) the X-group scheme $E(G)$ acts naturally on the X-scheme $E(F)$.

3.4. Let $\boldsymbol{G} \to X$ be a reductive group scheme over X (i.e., every geometric fibre is reductive). Let $\mathbf{Par}(\boldsymbol{G}/X)$ be the functor, on the category of X-schemes, which associates to $f: S \to X$ the set of subgroup schemes $P \to f^*(\boldsymbol{G})$ (over S) such that P_s is a parabolic subgroup of $f^*(\boldsymbol{G})_s$ $\forall\, s \in S$. This functor is representable by an X-scheme $\mathbf{Par}(\boldsymbol{G}/X)$ smooth and projective over X ([6, exposé XXVI, § 3, pp. 443–446]). For the reductive group scheme $E(G) \to X$ (G is a reductive group $/k$) it is easy to see using the corresponding functors, that $\mathrm{Par}(E(G)/X)$ is naturally isomorphic to $E(\mathrm{Par}(G/k))$, where G acts on $\mathrm{Par}(G/k)$ by inner conjugation. If P is a parabolic subgroup of G then the map $G/P \to \mathrm{Par}(G/k)$ given by $gP \mapsto gPg^{-1}$ is a G-equivariant isomorphism of G/P onto the connected component $\mathrm{Par}_P(G/k)$ of $\mathrm{Par}(G/k)$ consisting of parabolic subgroups conjugate to P ([6, exposé XXVI, Cor. 3.6, p. 446]). Therefore we have a natural isomorphism $E(G/P) \approx E(\mathrm{Par}_P(G/k)) \subset \mathrm{Par}(E(G)/X)$ and hence reductions of structure group to P are in bijective correspondence with parabolic subgroup schemes of $E(G)$ of type P (i.e., each geometric fibre is conjugate to P).

From now on in this section we assume that X is an irreducible nonsingular projective curve over the algebraically closed field k (see Section 4 for higher dimensional base X). Let $K = K(X)$ be the function field of X. Let x_0 be the generic point of X. We will denote the generic fibre $E(f)_{x_0}$ by $E(F)_0$ or by F_0 when E remains fixed. In particular $G_0 = E(G)_0$ will be the generic group scheme over K.

3.5. Since $\mathrm{Par}(E(G)/X)$ is projective over the curve X any generic section extends uniquely to the whole of X. Therefore (using 3.4) there is a natural bijection between reductions of structure group of E to P and K-rational points of $\mathrm{Par}_P(G_0/K)$. The latter is the set of parabolic subgroups of G_0 defined over K, and of type P (by the representability of the functor $\mathbf{Par}(E(G)/X)$).

3.6. Any reductive group over $K = K(X)$ is quasi-split ([18]) and hence $E(G_0)$ being an inner form and quasi-split is actually split. This implies E is locally trivial in the Zariski topology. We will not make use of this fact (which holds only when X is a curve).

DEFINITION 3.7 (cf. [14], [15]). A G-bundle $E \to X$ is said to be *semistable* (resp. *stable*) if for any reduction of structure group to a parabolic subgroup P, and any dominant character χ on P we have $\deg L_\chi \leqq 0$ (resp. $\deg L_\chi < 0$) where L_χ is the line bundle associated by the character χ to the reduced P-bundle.

REMARKS 3.8. (i) For definition of dominant see Remark 1.11 (a). By $\deg L$, the degree of the line bundle L on X, we mean the degree of the divisor of any rational section of L.

(ii) The line bundle on G/P associated to the P-bundle $G \to G/P$ by a dominant character is the *inverse* (i.e., *dual*) of an ample bundle.

Let G act on the projective scheme M linearly with respect to an ample line bundle L. Then $E(L) \to E(M)$ is a line bundle and the X-group scheme $E(G)$ acts on the X-schemes $E(L)$ and $E(M)$ compatibly. If x is any point of the scheme X then $E(G)_x$ acts on $E(M)_x$ linearly with respect to the line bundle $E(L)_x$.

DEFINITION 3.9. For a section $\sigma: X \to E(M)$ define $\deg \sigma$ to be $\deg \sigma^*(E(L))$.

PROPOSITION 3.10. (i) *If $\sigma(x_0)$ is semistable, then* $\deg \sigma \geqq 0$.

(ii) *If $\sigma(x_0)$ is semistable and* $\deg \sigma = 0$ *then σ is actually a section of $E(C) \subset E(M)$ where C is a fibre of the quotient map $M^{ss} \to M/G$, $M^{ss} =$ open set of semistable points of M.*

PROOF. By taking G-equivariant embedding $M \hookrightarrow \boldsymbol{P}(V)$, $V = H^0(X, L^r)^*$, $r \gg 0$, we may replace M by $\boldsymbol{P}(V)$.

Any G-invariant $s \in S^m(V^*)^G$ gives a section $\tilde{s} \in H^0(X, E(S^m(V^*)))$ (corresponding to the constant map $E \to S^m(V^*)$, $e \mapsto s \ \forall e \in E$, cf. § 3.1).

Since E is étale trivial $G_0 \times K_s$, ($K_s =$ separable close of K), is isomorphic to $G \times K_s$ by an isomorphism determined up to inner conjugation. Over K_s, the representation of G_0 on V_0 is equivalent to the representation of G on V. Therefore the space of invariants $S^m(V_0^*)^{G_0}$ comes from $S^m(V^*)^G$ (see [9, Proposition 1.14 and its proof, pp. 42–43]). Therefore $\sigma(x_0)$ being semistable there is an $s \in S^m(V^*)^G$ such that $\tilde{s}_{x_0}(\sigma(x_0)) \neq 0$. The natural map $E(V^*) \to \sigma^*(E(L^r))$ (given by the functorial property of $\boldsymbol{P}(V)$: a point of $\boldsymbol{P}(V)$ is a 1-dimensional quotient of V^*) gives $H^0(X, S^m(E(V^*))) \to H^0(X, \sigma^*(S^m E(L^r)))$ and $\tilde{s}$ then maps to a non-zero element of the latter. Therefore $H^0(X, \sigma^*(E(L))^{mr}) \neq 0$ which gives $\deg \sigma \geqq 0$. This proves (i).

Now if further $\deg \sigma = 0$, then $\sigma^*(E(L))^{mr}$ must be the trivial bundle since it has a non-zero section. This means $\tilde{s}_x(\sigma(x))$ is non-zero for every

$x \in X$. Therefore σ is actually a section of $E(D(s))$ where $D(s)$ is the affine open subset of $\boldsymbol{P}(V)$ defined by $s \neq 0$. The G-invariant quotient map $D(s) \to D(s)/G$ gives

$$\begin{array}{ccl} E(D(s)) & \xrightarrow{p} & E(D(s)/G) = (D(s)/G) \times X \\ {\scriptstyle\sigma}\uparrow\downarrow & \swarrow & \\ X & & \end{array}$$

Since $D(s)$ and $D(s)/G$ are affine ([9], [11]) the map $X \to D(s)/G$ given by $p \cdot \sigma$ is a constant. This proves (ii).

REMARK 3.11. From the above proof it follows that for a section $\sigma\colon X \to E(M)$, $\sigma(x_0)$ is semistable if and only if $\sigma(x)$ is semistable for all x in an open set of X. We also have that, $\sigma(x_0)$ is nonsemistable if and only if $\sigma(x)$ is nonsemistable for all x in X.

PROPOSITION 3.12. *Let* $\operatorname{Char} k = 0$. *We keep the notation as in* 3.10. *Let σ be a section of $E(M)$ such that $\sigma(x_0)$ is semistable and* $\deg \sigma = 0$. *Then there is a section σ_0 of $E(C^{qs})$ and an isomorphism $\sigma_0^* L^q \to \sigma^* L^q$, for some $q > 0$ where C^{qs} is the unique quasistable orbit of the fibre C. Moreover if the bundle E is stable σ is already a section of $E(C^{qs})$.*

PROOF. In Section 1 we outlined Kempf's method of getting an instability 1-PS for a nonsemistable point m which can be viewed as a method for finding an 1-PS which takes m to the closed G-invariant set 0 at the fastest rate. In [7] Kempf has done this more generally: we can find a 1-PS which takes m fastest to a given closed G-invariant subset S when $m \notin S$ and $S \cap \overline{O(m)} \neq \varnothing$. In this case also because of uniqueness if the base field is perfect such a 1-PS can be chosen to be rational ([7, Theorem 4.2]).

Now C_0^{qs} being the unique minimal dimensional orbit in C_0, it is left invariant by $\operatorname{Gal}(\bar{K}/K)$ and hence is defined over K. If $\sigma(x_0) \in C_0^{qs}$ we have nothing to prove. Suppose $\sigma(x_0) \notin C_0^{qs}$. Then by what we have said above there is a 1-PS λ of G_0, defined over K, corresponding to $\sigma(x_0)$.

Recall from Proposition 3.10 that $C \subset D(s)$ with $s \in S^m(V^*)^G$. Let $q = r \cdot m$. Let $W = H^0(M, L^q)^*$. Then $M \subset \boldsymbol{P}(W)$. Let W^+ (resp. W^0) be the space generated by the eigenvectors of λ with non-negative characters (resp. trivial character) on G_m. Clearly W^+, W^0 define subbundles still denoted by W^+, W^0, of $E(W)$. By construction $\sigma(x) \in W^+$ for every $x \in X$. The projection $\pi\colon W^+ \to W^0$ map σ into a non-zero vector and defines a section σ_0 of $E(C^{qs})$. Moreover π induces a non-zero map $\sigma_0^* L^q \to \sigma^* L^q$. By Proposition 3.10 (i) $\deg \sigma_0^* L^q \geqq 0$. But $\deg \sigma^* L^q = 0$. Therefore $\deg \sigma_0^* L^q = 0$ and $\sigma_0^* L^q \to \sigma^* L^q$ is an isomorphism.

To prove the second assertion we suppose $\sigma(x_0) \notin C_0^{qs}$ and E is stable and arrive at a contradiction. Since $\sigma_0(x_0) \in C_0^{qs}$ the section σ_0 considered as a map $\sigma_0: E \to C$ (cf. § 3.1) actually maps into $C^{qs} = G/I$, I the isotropy at a point of C^{qs}. Thus σ_0 gives a section of $E(G/I)$ and hence a reduction of structure group of G to I. Since $G/I = C^{qs}$ is affine I must be reductive (cf. [11, p. 51] and the references given therein). But in G_0 we have $(E_I(I))_0 = I_0 \subset P(\lambda)$ (by the uniqueness of $P(\lambda)$, cf. Remark 1.7 (iii) and [7]). Therefore we have a reduction of structure group of the bundle E to the reductive subgroup I contained in a parabolic subgroup P of G. Since Char $k = 0$, $I \subset R =$ a Levi subgroup of P. When E is stable this is not possible since R is also contained in an opposite parabolic group P^0 of P and there are dominant characters on P and P^0 which restrict to inverses of each other on R (cf. [4, § 4.8, p. 88]).

PROPOSITION 3.13. *Suppose $Z_0 =$ the connected component of the centre of G acts trivially on M and L. Let E be semistable. If $\sigma(x_0)$ is nonsemistable and has an instability* 1-PS *defined over K, then* $\deg \sigma \geqq 0$.

COROLLARY 3.14. *E is semistable $\Leftrightarrow$ for all M, L, σ, with Z_0 acting trivially on M, L and σ (x_0) strongly separable, we have* $\deg \sigma \geqq 0$.

PROOF OF PROPOSITION. Let λ be an instability 1-PS of $\sigma(x_0)$ defined over K. The instability flag $P(\lambda) = P_0$ is also defined over K. By the remarks in Sections 3.4 and 3.5, P_0 gives a parabolic subgroup scheme $P \subset E(G)$ and a reduction of structure group to a parabolic subgroup $P \subset G$ with $P = E_P(P)$ (where E_P is the reduced P-bundle). Recall $M \subset \boldsymbol{P}(V)$, $V = H^0(X, L^r)^*$. Let V_0^q be the subspace of $E(V)_0 = V_0$ generated by the eigenvectors of λ with eigenvalue $\geqq q \in \boldsymbol{Z}$ (cf. § 1.8). Note that V_0^q is defined over K. Let V^q be the subbundle of $E(V)$ whose generic fibre is V_0^q. Since P_0 leaves V_0^q invariant the group scheme $E_P(P) = P$ leaves the subbundle V^q invariant. Then it is easy to see that there is a subspace W^q of V left invariant by P such that $V^q = E_P(W^q) \subset E(V)$ (such a W^q is obtained by identifying a geometric fibre of $E_P(V)$ with V, the identification being well defined up to action by P).

Let $\mu(\sigma(x_0), \lambda) = j$. Then $\sigma(x_0) \in V_0^j$ and has a non-zero image in V_0^j/V_0^{j+1}. Therefore $\sigma(x) \in V^j$ $\forall x \in X$ and we have a non-zero homomorphism $\sigma^*(E(L^r)^*) \to V^j/V^{j+1}$. Let L' be the line subbundle generated by the image of this map. Then $L_0' = \text{Image}\, \overline{\sigma(x_0)}$ of $\sigma(x_0)$ in V_0^j/V_0^{j+1}. By Proposition 1.12 (cf. also Remark 1.13) $\overline{\sigma(x_0)}$ is semistable for the action of P_0/U_0 ($U_0 =$ Unipotent radical of P_0) as a point of $\boldsymbol{P}(S^m(V_0^j/V_0^{j+1}) \otimes \mathcal{O}_{\chi^{-1}})$, $m > 0$. Since Z_0 acts trivially on M, L, χ is trivial on Z_0 and is a dominant character on P_0 (Remark 1.11 (d)). Further since $P_0 \times K_s$ and $P \times K_s$ are conjugate in $G_0 \times K_s \approx G \times K_s$ the character χ gives a well defined dominant character on P, again denoted by χ. Now $S^m(V_0^j/V_0^{j+1}) \otimes \mathcal{O}_{\chi^{-1}}$ is the generic fibre of the bundle associated to E_P through the action of P/U on $S^m(W^j/W^{j+1}) \otimes \mathcal{O}_{\chi^{-1}}$. Therefore by Proposition 3.10 (i), applied to the P/U-bundle $E_P(P/U)$ and the section $\bar{\sigma}$ given by $\overline{\sigma(x_0)}$, we have (1) $\deg(L'^m \otimes L(\chi)) \geqq 0$, where $L(\chi)$ is the line bundle associated to E_P through the character χ on P. We now use the fact E is semistable to get (2) $\deg L(\chi) \leqq 0$. From (1) and (2), $\deg L' \geqq 0$. Since there is a non-zero homomorphism $\sigma^*(E(L^r)^*) \to L'$ we have $\deg \sigma^*(E(L^r)^*) \leqq 0$ which gives $\deg \sigma \geqq 0$.

PROOF OF COROLLARY 3.14. We have only to note that the action of G on $G/P = M$ is separable and use the above proposition and Proposition 3.10 (i).

DEFINITION 3.15. A reduction of structure group of $E \to X$ to a parabolic subgroup P is called *admissible* if for any character χ on P which is trivial on Z_0 the line bundle associated to the reduced P-bundle E_P has degree zero.

DEFINITION 3.16. A G-bundle $E \to X$ is *quasi-stable* if it has a reduction to a Levi-component R of a parabolic subgroup P such that the reduced R-bundle E_R is stable and the extended P-bundle $E_R(P)$ is an admissible reduction of structure group of G to P.

REMARK 3.17. Let E_P be an admissible reduction of structure group to a parabolic subgroup P. Then E is semistable if and only if the P/U bundle $E_P(P/U)$ is semistable. This follows fairly easily from the relation between parabolic subgroups of P/U and G ([4, § 4.4, p. 86], [15, § 2]).

THEOREM 3.18. *Let* Char $k = 0$. *Let H be a reductive group and $\rho: G \to H$ a homomorphism which maps the connected component Z_0 of the centre of G into that of H. Then if E is a semistable G-bundle then the extended H-bundle $E(H)$ is semistable. If E is quasi-stable then so is $E(H)$.*

PROOF. Let P be a parabolic subgroup of H. Let $M = H/P$ and $L \to M$ be the line bundle associated to the inverse of a dominant character on P (cf. Remark 3.8 (ii)). Then G acts on M, L through ρ with Z_0 acting trivially. Since Char $k = 0$ all actions are separable. Therefore for any section σ of $E(H)(M) = E(M)$, $\deg \sigma \geqq 0$ by Corollary 3.14. Thus $E(H)$ is semistable.

To prove the second assertion, clearly we can assume E is stable. If $E(H)$ has no admissible reductions at all to any proper parabolic subgroup of H then $E(H)$ is stable and we have nothing to prove. So assume $E(H)$ has an admissible reduction to a parabolic subgroup P. Let σ be an admissible reduction to P with $\dim P$ being minimal for admissible reduction. This reduction gives a section σ of $E(H)(M) = E(M)$ with $\deg \sigma = 0$. By Proposition 3.12 we then have a quasi-stable G-orbit C^{qs} in M such that $\sigma \in E(C^{qs})$. Let I be the isotropy subgroup of G for a point of C^{qs}. Since we are in characteristic zero the natural bijective morphism $G/I \to C^{qs}$ is an isomorphism and $\sigma \in E(G/I)$ gives a reduction of structure group of E to I. Thus we get a further reduction of structure group of the reduced P-bundle $E(H)_P$ to the subgroup $\rho(I) \subset P$. Now C^{qs} being affine I is reductive ([11, p. 51]). Therefore $\rho(I)$ is reductive. Again since $\operatorname{Char} k = 0$, $\rho(I) \subset R =$ a Levi subgroup of P. Therefore $E(H)$ admits an admissible reduction of structure group to R, with reduced R-bundle $E(H)_R$. Since any proper parabolic subgroup of R is the intersection with R of a proper parabolic subgroup of G contained in P ([4, Proposition 4.4, p. 86]), it follows easily that an admissible reduction of $E(H)_R$ would give an admissible reduction of structure group of $E(H)$ to a proper subgroup of P. This contradicts the minimality of the reduction to P. Therefore $E(H)_R$ is a stable R-bundle proving $E(H)$ is quasi-stable.

THEOREM 3.19. *Let* $\operatorname{Char} k = 0$. *Let E be a stable G-bundle. Then E has a unique minimal reduction to a (not necessarily connected) reductive subgroup. This reduction is minimal in the sense that it does not admit a further reduction of structure group to a (not necessarily connected) reductive subgroup.*

PROOF. Clearly we can find some minimal reduction to a reductive subgroup. We only have to prove the uniqueness.

LEMMA 3.20. *Let H be a (not necessarily connected) reductive subgroup of G. Then there is a representation $\rho\colon G \to \mathrm{GL}(V)$ and a quasi-stable point (for the action of G on V) $\bar{m} \in \boldsymbol{P}(V)$ such that the isotropy subgroup of G at $\bar{m}$, as well as at any point $m \in V$ which projects to m, is H.*

PROOF OF LEMMA. Follows easily from ([2, Chapter II, §§5.1, 5.5]).

Now from the above lemma and Proposition 3.12 it follows easily that reductions to (not necessarily connected) reductive subgroups are equivalent to sections σ of $E(M)$ with $\deg \sigma = 0$ and $\sigma(x_0)$ semistable. Thus now let σ_1 and σ_2 be sections of $E(M_1)$ and $E(M_2)$ giving two minimal reductions to reductive subgroups H_1 and H_2, with $\deg \sigma_i = 0$ and $\sigma_i(x_0)$ semistable. Then $\sigma_1 \times \sigma_2$ is a section of $E(M_1 \times M_2)$ with the same properties and gives further reduction to both σ_1 and σ_2 to a smaller reductive group. This proves uniqueness.

For $\operatorname{Char} k = p$ (> 0) we have the following results 3.21 and 3.23.

THEOREM 3.21. *Let* $\operatorname{Char} k = p$. *Let* E *be a semistable* SL (2)*-bundle. Let* V *be the* 2*-dimensional space on which* SL (2) *acts naturally. For* $m \leqq p - 1$, *the associated bundle* $E(S^m(V))$ *is semistable.*

PROOF. The proof is the same as the first part of the proof of Theorem 3.18 where for the separability of the action we have to use Lemma 2.6 instead of characteristic 0.

Before stating the next result we have to recall a few facts about the Frobenius map.

Let k be a field of characteristic $p > 0$ and $n > 0$ an integer. Let $\varphi: X \to \operatorname{Spec} k$ be a scheme over k. The p^n-th power map $\mathcal{O}_X \to \mathcal{O}_X$ given by $f \to f^{p^n}$ is a homomorphism and gives rise to a morphism $F_X: X \to X$ called the (absolute) Frobenius. Since $\varphi F_X = F_k \varphi$, we have a commutative diagram

$$\begin{array}{ccc} F_k^*(X) & \xrightarrow{A} & X \\ {\scriptstyle F_k^*\varphi}\downarrow & \overset{g}{\nwarrow} \; X \; \nearrow & \downarrow{\scriptstyle \varphi} \\ & \swarrow \qquad \searrow & \\ \operatorname{Spec} k & \xrightarrow{F_k} & \operatorname{Spec} k \end{array}$$

If k is a perfect field, F_k and A are isomorphisms.

If $\boldsymbol{F}_q \subset k$ is the finite field with $q = p^n$ elements then F_k is identity on $\operatorname{Spec} \boldsymbol{F}_q$. Suppose $\boldsymbol{X} \to \operatorname{Spec} \boldsymbol{F}_q$ is an $\boldsymbol{F}_q$-scheme and $X = \boldsymbol{X} \times_{\boldsymbol{F}_q} \operatorname{Spec} k$. Then $F_k^*(X)$ is naturally isomorphic to X. In this case the morphisms A and g in (4) give morphisms $X \to X = F_k^*(X)$ called respectively the *arithmetic Frobenius* and the *geometric Frobenius* (with reference to the $\boldsymbol{F}_q$-structure $\boldsymbol{X}$).

Let $\pi: E \to X$ be a G-bundle. Pulling back by the Frobenius we get a G-bundle $F_X^*(E) \to X_F$ on X_F; (where we take the k-structure on $F_X^*(E)$ to be the one defined by the composite $F_X^*(E) \to X_F \xrightarrow{F_k \cdot \varphi} \operatorname{Spec} k$). If k is a perfect field we can change the k-structure of $F_*(E)$, X_F and G by composing their structure morphisms with $F_k^{-1}: \operatorname{Spec} k \to \operatorname{Spec} k$ to get a bundle $F_*(E) \to X$ with structure group $F_k^*(G)$ (in (4) replacing X by G we see that A gives a k-isomorphism of $F_k^*(G)$ with G, the latter having the k-structure changed by F_k^{-1}).

REMARK 3.22. It can be easily checked that this $F_k^*(G)$ bundle is the same as the bundle obtained from E by the extension of structure group $g: G \to F_k^*(G)$.

Now let a group scheme $\boldsymbol{G} \to \operatorname{Spec} \boldsymbol{F}_q$ over $\boldsymbol{F}_q$ such that $G = \boldsymbol{G} \times_{\boldsymbol{F}_q} \operatorname{Spec} k$ be given. Then $F_k^*(G) \approx G$. So the $F_k^*(G)$-bundle $F_X^*(E) \to X$ gives a G-bundle called the *Frobenius twist* of E (with reference to the $\boldsymbol{F}_q$-structure on $\boldsymbol{G}$).

If we take a different $\boldsymbol{F}_q$-structure $\boldsymbol{G}'$ for G then the isomorphism of $F_k^*(G)$ with G given by $\boldsymbol{G}'$ obviously differs from that given by $\boldsymbol{G}$ by an automorphism of G. Therefore the Frobenius twist of E with reference to $\boldsymbol{G}'$ is obtained from that with reference to $\boldsymbol{G}$ by an extension of structure group through an automorphism $G \to G$. It follows easily from the definition that if a G-bundle is semistable then the G-bundle obtained from it by extension of structure group through an automorphism is also semistable. Therefore the notion of Frobenius twist of E being semistable is independent of the $\boldsymbol{F}_q$-structure chosen for G. If G is a reductive group over $k = \bar{k}$, then G has a $\boldsymbol{Z}$-structure and hence an $\boldsymbol{F}_q$-structure. It is easy to see that if $F_X^*(E)$ is semistable then E is semistable.

THEOREM 3.23. *Let* $\operatorname{Char} k = p$. *Let $E \to X$ be a semistable G-bundle such that the Frobenius twists $(F_X^m)^*(E)$ are semistable for all $m \geqq 0$. Then for any homomorphism $\rho: G \to H$, H reductive and ρ maps the connected component of the centre of G into that of H, the H-bundle $E(H)$ is semistable.*

PROOF. Let $M = H/P'$, P' a parabolic subgroup of H, and L the line bundle on M associated to the inverse of a dominant character on P'. Let σ be a section of $E(M)$. We have only to show that $\deg \sigma \geqq 0$.

If $\sigma(x_0)$ is semistable then by Proposition 3.10 (i), we are through. Assume $\sigma(x_0)$ is nonsemistable. Then we can find an instability 1-PS λ of $\sigma(x_0)$ defined over $K^{p^{-n}} = \{x \in \bar{K} \mid x^{p^n} \in K\}$ for some $n \geqq 0$ (cf. begining of § 2).

Pulling back by the Frobenius F_X we have that the action of the generic fibre $F_X^* E(G)_0 = F_K^*(E(G)_0)$ on $F_X^* E(M)_0 = F_K^*(E(M)_0)$ is the base change by the Frobenius F_K of the action of $E(G)_0$ on $E(M)_0$. The Frobenius F_K factors through an isomorphism:

$$\begin{array}{ccc} \operatorname{Spec} K & \xrightarrow{F_K} & \operatorname{Spec} K \\ & \searrow^{\approx} & \uparrow \\ & & \operatorname{Spec} K^{p^{-n}} \end{array}$$

where $\operatorname{Spec} K^{p^{-n}} \to \operatorname{Spec} K$ is given by the inclusion $K \subset K^{p^{-n}}$.

Therefore for the action of $F_K^*(E(G)_0)$, $(F^*\sigma)(x_0)$ has an instability 1-PS $F_K^*(\lambda)$ which is defined over K. Therefore $F^*(E)$ being semistable we can now apply Proposition 3.13 to get that $\deg F^*\sigma \geqq 0$. Now $(F_X^*\sigma)^*(F_X^*E(L)) = F_X^*(\sigma)^*(E(L))) = (\sigma^*(E(L)))^{p^n}$ (noting that for a line bundle L, $F_X^*(L) = L^{p^n}$). Therefore $\deg \sigma = (1/p^n) \deg F^*\sigma \geqq 0$.

4. Some Examples and Remarks.

4.1. If V is a vector bundle then $S^p(V)$ contains $F^*(V)$ and hence for a semistable V, we cannot expect $S^p(V)$ to be semistable in general even if $rkV = 2$. In fact, more generally, for $j \geqq p$, $S^j(V)$ is not semistable if $F^*(V)$ is not. For, $S^j(V)$ has a subbundle of the form $\rho_*(V)$ where ρ is the irreducible representation with highest weight $j\cdot\lambda$, λ being the highest weight of the natural representation. It is enough to show that $\rho_*(V)$ is not semistable if $j \geqq p$. From the tensor product theorem of Steinberg, we conclude that $\rho_*(V) = \rho'_*(V) \otimes \rho''_*(F^*V)$ where ρ', ρ'' are irreducible representations. Since $F^*(V)$ is not semistable by assumption, neither can $\rho_*(V)$ be.

4.2. Theorem 3.21 does not hold for $\mathrm{SL}(n)$-bundles for $n \geqq 3$. In fact if E is a semistable $\mathrm{SL}(2)$-bundle then $V = S^{n-1}(E)$ be an $\mathrm{SL}(n)$-bundle which is semistable if $n \leqq p$ by Theorem 3.21. However if $j(n-1) \geqq p$ and $F^*(E)$ is not semistable then $S^j(V)$ is not semistable since $S^j(V)$ has $S^{j(n-1)}(E)$ as a quotient which is not semistable by 4.1.

4.3. In view of Remark 3.22 the assumption that the Frobenius twists are semistable in Theorem 3.23 is necessary. Moreover it can be proved that there exist bundles of which all Frobenius twists are semistable. For example this follows from the theorem of Nori ([12, Proposition 3.4, p. 36]) to the effect that a bundle E satisfying an algebraic relation of the form $\sum n_i E^{\otimes i} \approx \sum m_i E^{\otimes i}$ $(n_i, m_i \geqq 0)$ is semistable.

PROPOSITION 4.4. *Let X be a nonsingular projective curve of genus $g \geqq 2$ over a field of characteristic $p > 0$. For every semistable subsheaf V of $F_*\mathcal{O}_X$ with $\mu(V) > 0$, we have F^*V is nonsemistable.*

PROOF. In fact, by the projection formula, for any two locally free sheaves V_1, V_2 we have $F_*(F^*V_1 \otimes V_2) \cong V_1 \otimes F_* V_2$ and hence $H^0(X, \mathrm{Hom}(F^*V_1, V_2)) \cong H^0(X, \mathrm{Hom}(V_1, F_*V_2))$. In particular, the inclusion $V \to F_*\mathcal{O}_X$ gives a nonzero map $F^*V \to \mathcal{O}_X$ and since $\mu(F^*V) = p\cdot\mu(V) > 0$, this implies that F^*V is not semistable.

REMARK 4.5. For any coherent sheaf $\mathcal{F}$, we have $H^i(X, F_*\mathcal{F}) \approx H^i(X, \mathcal{F})$. Hence $\chi(F_*\mathcal{O}_X) = \chi(\mathcal{O}_X)$ and using Riemann-Roch, we see that $\deg F_*\mathcal{O}_X = (p-1)(g-1) > 0$. Therefore there do exist semistable subbundles of $F_*\mathcal{O}_X$ with positive μ. In any case, it seems probable that $F_*\mathcal{O}_X$ is itself semistable. In this connection we note that Raynaud [20, Theorem 4.1.1] has shown that the cokernel of the inclusion $\mathcal{O}_X \to F_*\mathcal{O}_X$ is semistable.

For higher dimensional projective nonsingular X we make the following definitions.

DEFINITION 4.6. A *rational G-bundle* on X is a G-bundle on an open subscheme U of X with $\operatorname{codim}(X-U) \geqq 2$.

Let H be a given ample line bundle on X. Let $\dim X = n$. We define $\deg L$ (*with respect to* H) to be the intersection number $c_1(L)\cdot c_1(H)^{n-1}$, where c_1 denotes the first Chern class.

If L is a line bundle on an open subscheme of X with $\operatorname{codim}(X - U) \geqq 2$, then L extends uniquely to a line bundle on X and hence $\deg L$ makes sense.

DEFINITION 4.7. A rational G-bundle $E \to U \subset X$, with U open and $\operatorname{codim}(X-U) \geqq 2$, is said to be *stable* (resp. *semistable*) *with respect to the polarisation* H if for any reduction to a parabolic subgroup P of E over any open subset $U' \subset U$, $\operatorname{codim}(X-U') \geqq 2$, the line bundle associated to any dominant character on P has degree < 0 (resp. $\leqq 0$).

If V is a torsion free sheaf on X then for some open set $U \subset X$, $\operatorname{codim}(X-U) \geqq 2$, $V \mid U$ is a vector bundle. Then the above definition of stability and semistability of the $\mathrm{GL}(r)$-bundle corresponding to $V \mid U$ is equivalent to the usual definition of stability and semistability (see [8]).

Using the notation of Sections 3.4, and 3.5, any generic section of $\operatorname{Par}(E(G)/U))$ extends to an open subset $U' \subset U$ with $\operatorname{codim}(X-U') \geqq 2$ (by the valuative criterion for properness). Therefore we again have a 1-1 correspondence between reduction of structure group to P over open subschemes of codim $\geqq 2$ and the $K(X)$-rational points of $\operatorname{Par}_P(G_0/K(X))$. Then it is easy to check that the proofs of Theorems 3.18 and 3.23 go through for X of arbitrary dimension. Therefore Theorems 3.18 and 3.23 hold for rational G-bundles over X.

REFERENCES

[1] F. A. BOGOMOLOV, Holomorphic tensors and vector bundles on projective manifolds (in Russian), Izvestija Akademii Nauk SSR, Ser Mat. 42 (1978), 1227–1287.

[2] A. BOREL, Linear Algebraic Groups, W. A. Benjamin, New York, 1969.

[3] A. Borel and T. A. Springer, Rationality properties of linear algebraic groups II, Tohoku Math. J. 20 (1968), 443-497.

[4] A. Borel and J. Tits, Groupes réductifs, Pub. Math. I.H.E.S. No. 27 (1965), 55-150.

[5] D. Giesekar, On a theorem of Bogomolov on Chern classes of stable bundles, Amer. J. Math. 101 (1979), 77-85.

[6] A. Grothendieck and M. Demazure, Schemas en groupes I, II, III (SGA 3), Lecture Notes in Math. 151, 152, 153, Springer-Verlag, Berlin-Hidelberg-New York, 1976.

[7] G. Kempf, Instability in invariant theory, Ann. of Math. 108 (1978), 299-316.

[8] M. Maruyama, The theorem of Grauert-Mülich-Spindler, Math. Ann. 255 (1981), 317-333.

[9] D. Mumford, Geometric Invariant Theory, Springer-Verlag, Berlin-Heidelberg-New York, 1965.

[10] M. S. Narasimhan and C. S. Seshadri, Stable and unitary vector bundles on a compact Riemann surface, Ann. of Math. 82 (1965), 540-567.

[11] P. E. Newstead, Lectures on introduction to moduli problems and orbit spaces, Tata Institute Lecture Notes, Springer-Verlag, 1978.

[12] M. V. Nori, On the representation of the fundamental group, Compositio Math. 33 (1976), 29-41.

[13] M. V. Nori, Characterisations of the representations of the algebraic fundamental group of a variety, Thesis, University of Bombay, 1980.

[14] A. Ramanathan, Stable principal bundles on a compact Riemann surface, Math. Ann. 213 (1975), 129-152.

[15] A. Ramanathan, Stable principle bundles on a compact Riemann surface—Construction of moduli space, Thesis, University of Bombay, 1976.

[16] G. Rousseau, Immersible sphériques et théorie des invariants, C.R. Acad. Sci. Paris, 286 (1978), 247-250.

[17] J.-P. Serre, Espaces fibrés algèbriques, in: Anneau de Chow et applications, Seminaire Chevalley, 1958.

[18] R. Steinberg, Regular elements of semisimple algebraic groups, Publ. Math. I.H.E.S. No. 25, (1965).

[19] H. Tango, On the behaviour of extensions of vector bundles under the Frobenius map, Nagoya Math. J. 48 (1972), 73-89.

[20] M. Raynaud, Sections des fibrés vectoriels sur une courbe, Bull. Soc. Math. France 110 (1982), 103-125.

School of Mathematics
Tata Institute of Fundamental Research
Bombay 400 005
India