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HOMOGENEOUS MINIMAL HYPERSURFACES IN THE UNIT SPHERES AND THE FIRST EIGENVALUES OF THEIR LAPLACIAN

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1. Introduction. The geometry of minimal closed submanifolds of the unit sphere is closely related to the eigenvalue problem of the Laplacian. In this paper, we study the first eigenvalue of the embedded minimal hypeasurfaces in the unit sphere.

Let (M, g) be an n-dimensional compact connected Riemannian manifold without boundary and Δ its (non-negative) Laplacian acting on C^∞-functions on M. Let $\{0 = \lambda_0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \uparrow \infty\}$ be its spectrum. λ_1 is called the first eigenvalue of Δ. Let f be an isometric immersion of (M, g) into the N-dimensional standard unit sphere $S^N(1)$ of the Euclidean space $\boldsymbol{R}^{N+1}$ with the coordinate $(x^0, x^1, \cdots, x^N)$. Then it is known (cf. [13]) that the $N+1$ functions $x^i \circ f$ $(i = 0, 1, \cdots, N)$ on M are the eigenfunctions of Δ with the eigenvalue n if and only if $f(M)$ is minimal in $S^N(1)$. Therefore the first eigenvalue λ_1 of Δ of an n-dimensional minimally isometrically immersed Riemannian manifold (M, g) in $S^N(1)$ is not greater than n. In particular, for the great sphere $S^n(1)$ and the generalized Clifford torus $S^p(\sqrt{p/n}) \times S^q(\sqrt{q/n})$ $(p + q = n)$ of $S^{n+1}(1)$, the first eigenvalue λ_1 is just n. In this connection, Ogiue [10] posed the following problem:

PROBLEM (A). What kind of embedded minimal hypersurfaces of $S^{n+1}(1)$ have n as the first eigenvalue of its Laplacian?

Yau [18] posed independently a similar problem. In this paper, we consider a little more restricted problem:

PROBLEM (B). Is n the first eigenvalue of the Laplacian for the embedded homogeneous minimal hypersurfaces of $S^{n+1}(1)$?

In this paper we give a partial answer to the Problem (B) using the classification (cf. [5]) of homogeneous hypersurfaces of the unit sphere and the theory of spherical functions on a compact homogeneous space:

THEOREM. *Let (M, g) be an embedded homogeneous minimal hypersurface in the unit sphere which is diffeomorphic to one of the following*:

(i) $\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$,

(ii) $\mathrm{SU}(3)/T^2$,
(iii) $\mathrm{Sp}(2)/T^2$,
(iv) G_2/T^2,
(v) $\mathrm{SO}(m) \times \mathrm{SO}(2)/\mathrm{SO}(m-2) \times \boldsymbol{Z}_2$ $(m \geqq 2)$,

where T^2 *are two dimensional maximal tori of* $\mathrm{SU}(3)$, $\mathrm{Sp}(2)$ *or* G_2. *Here* G_2 *is the simply connected compact Lie group of type* G_2. *(See the Table in* [5], [14]). *Then the first eigenvalue of the Laplacian of* (M, g) *is equal to the dimension of* M.

REMARK 1. In case of (i), the nullity (resp. the index) of the minimally embedded hypersurface $\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ in the 4-dimensional unit sphere is 7 (resp. 20) (cf. § 5).

REMARK 2. It seems that the answer to Problem (B) is affirmative. But it seems to be difficult to compute their first eigenvalue because none of the homogeneous minimal hypersurfaces in the unit sphere except the great sphere and the generalized Clifford torus is symmetric or normal homogeneous.

The outline of this paper is as follows. In Section 2, we explain the connection between the geometry of minimal immersion and the eigenvalue problem of the Laplacian. In Section 3, we give an explicit formula for the Laplacian of a regular R-space. In Section 4, we determine the minimal orbit among regular R-spaces embedded with codimension one in the unit sphere and calculate their principal curvatures. In Section 5, we calculate the first eigenvalue of the Laplacian of the homogeneous minimal hypersurfaces and Main Theorem.

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2. Laplacian and Jacobi operator. In this section, we explain the connection between the geometry of minimal immersion and the eigenvalue problem of the Laplacian.

Let M be an n-dimensional compact orientable manifold without boundary immersed in a Riemannian manifold $(\bar{M}, h)$ of dimension $n+p$. We denote by g the Riemannian metric f^*h on M induced by the Riemannian metric h on $\bar{M}$ through the immersion f. For a C^∞-variation f_t of the immersion f, that is, a C^∞ one-parameter family of immersions $\{f_t\}$ of M into $\bar{M}$ such that $f_0 = f$, consider the variation of the volume $\mathrm{vol}(M, f_t^*h)$ of the Riemannian manifolds (M, f_t^*h). The immersion f is called *minimal* if the first variation $(d/dt)(\mathrm{vol}(M, f_t^*h))_{t=0}$ vanishes for every C^∞-variation f_t of f.

To state the second variation formula, we need some notions. For $x \in M$, we denote $f(x)$ by x, and we regard T_xM as a subspace of $T_x\bar{M}$. Let $T_xM^\perp$ be the orthocomplement of T_xM in $T_x\bar{M}$, and $TM^\perp$ the normal bundle over M. For a vector field X along M, we denote the normal component of X by $X^\perp$. Let ∇, $\bar{\nabla}$ be the Riemannian connections of (M, g), $(\bar{M}, h)$, respectively. Then for vector fields X, Y on M, $\nabla_X Y$ is the tangential component of $\bar{\nabla}_X Y$. We denote the normal component of $\bar{\nabla}_X Y$ by $B(X, Y)$ and call it the second fundamental form. For each element V in the space $\Gamma(TM^\perp)$ of all C^∞-sections of $TM^\perp$ and a vector field X on M, the tangent component $-A_V X$ of $\bar{\nabla}_X V$ satisfies

$$g(A_V X, Y) = h(B(X, Y), V) ,$$

for all vector fields X, Y on M. The operator A_V is called the shape operator. The normal component $\nabla_X^\perp V$ of $\bar{\nabla}_X V$ is called the normal connection.

A normal C^∞-variation f_t of minimal immersion $f: M \to \bar{M}$ is a C^∞-variation f_t whose variation vector field $V_x = (d/dt)f_t(x)|_{t=0}$, $x \in M$, belongs to $\Gamma(TM^\perp)$. It is known (cf. [11]) that for every normal variation f_t of the minimal immersion $f: M \to \bar{M}$ with the variation vector field $V \in \Gamma(TM^\perp)$, we have

$$(d^2/dt^2)(\mathrm{vol}\,(M, f_t^*h))|_{t=0} = \int_M h(JV, V)dv_g$$

where dv_g is the volume element of (M, g). Here the operator J of $\Gamma(TM^\perp)$ into itself, called the *Jacobi operator*, is a self-adjoint strongly elliptic differential operator. So it has a discrete spectrum:

$$\beta_1 \leqq \beta_2 \leqq \cdots \uparrow \infty .$$

The *index* of the minimal immersion $f: M \to \bar{M}$ is the sum of the dimensions of the eigenspaces corresponding to the negative eigenvalues. The *nullity* of $f: M \to \bar{M}$ is the dimension of the 0-eigenspace. The Jacobi operator J of $\Gamma(TM^\perp)$ is determined as follows: Define the operators $\Delta^\perp$, $\tilde{R}$ and $\tilde{A}$ of $\Gamma(TM^\perp)$ into itself by

$$\Delta^\perp(V) = \sum_{i=1}^n (\nabla_{e_i}^\perp \nabla_{e_i}^\perp - \nabla_{\nabla_{e_i} e_i}^\perp)V ,$$

$$\tilde{R}(V) = \sum_{i=1}^n (\bar{R}_{e_i, V} e_i)^\perp \quad \text{and}$$

$$\tilde{A}(V) = \sum_{i=1}^n B(A_V e_i, e_i) ,$$

for evey $V \in \Gamma(TM^\perp)$, respectively. Here $\{e_i\}_{i=1}^n$ is an orthonormal local frame field of M, and $\bar{R}$ is the curvature tensor of $(\bar{M}, h)$. Then we have (cf. [11])

$$J = -\Delta^{\perp} + \tilde{R} - \tilde{A} .$$

In the following, we always assume that $(\bar{M}, h)$ is the $(n+1)$-dimensional unit sphere $S^{n+1}(1)$ with constant sectional curvature one. Let (M, g) be an n-dimensional compact orientable Riemannian manifold isometrically and minimally immersed into $S^{n+1}(1)$. Then we have

$$\tilde{R}(V) = \sum_{i=1}^{n} (h(e_i, V)e_i - h(e_i, e_i)V)^{\perp} = -nV ,$$

for every $V \in \Gamma(TM^{\perp})$. Let N be the unit normal vector field on M. Then each element in $\Gamma(TM^{\perp})$ can be written as a multiple of N by a C^{∞}-function on M. Let $C^{\infty}(M)$ be the space of all real valued C^{∞}-functions on M. Then, since $\nabla^{\perp}_{e_i}N = 0$, $i = 1, \cdots, n$, we have

$$-\Delta^{\perp}(\phi N) = (\Delta\phi)N , \quad \phi \in C^{\infty}(M) ,$$

where Δ is the Laplacian of (M, g) acting on $C^{\infty}(M)$ given by

$$\Delta\phi = -\sum_{i=1}^{n} (\nabla_{e_i}\nabla_{e_i}\phi - \nabla_{\nabla_{e_i}e_i}\phi) \tag{1.1}$$

for ϕ in $C^{\infty}(M)$. Moreover, we have

$$\tilde{A}(\phi N) = \sigma\phi N ,$$

where σ is the square of the length of the second fundamental form B, that is,

$$\sigma = \sum_{i=1}^{n} g(A_N e_i, A_N e_i) . \tag{1.2}$$

Thus the Jacobi operator J can be expressed (cf. [3, p. 231, (2.4)]) as

$$J(\phi N) = (\Delta\phi - (n + \sigma)\phi)N , \quad \phi \in C^{\infty}(M) . \tag{1.3}$$

Therefore, when the square σ of the length of the second fundamental form B is constant, the following conditions are equivalent:

(i) ϕN, $\phi \in C^{\infty}(M)$ is the eigensection of J with eigenvalue β.

(ii) ϕ is the eigenfunction of Δ with eigenvalue $n + \sigma + \beta$.

Therefore, the determination of the index and the nullity of compact minimal hypersurfaces in the unit sphere with constant σ (in particular, compact minimal homogeneous hypersurfaces) is reduced to the eigenvalue problem of the Laplacian Δ as follows:

(iii) ϕN is a Jacobi field, i.e., $J(\phi N) = 0$, if and only if ϕ is an eigenfunction of Δ with the eigenvalue $n + \sigma$.

(iv) The nullity coincides with the multiplicity of the eigenvalue

$n+\sigma$ of the Laplacian Δ.

(v) The index is the sum of the multiplicities of the eigenvalues smaller than $n+\sigma$ of the Laplacian Δ.

REMARK 1. In fact for a compact orientable minimal hypersurface M of the unit sphere with constant σ, the Laplacian Δ has n, σ and $n+\sigma$ as eigevalues. Moreover, if $\sigma \neq 0$, then the index of M is not less than $2n+3$.

REMARK 2. (i) The eigenvalues of the Laplacian of the great sphere $S^n(1)$ in the unit sphere $S^{n+1}(1)$ are $j(j+n-1)$, $j=0,1,2,\cdots$. The nullity is $n+1$, which is the multiplicity of the eigenvalue n of Δ. The index is one, which is the multiplicity of the eigenvalue 0 of Δ. (ii) The generalized Clifford torus $S^p(\sqrt{p/n}) \times S^q(\sqrt{q/n})$, $p+q=n$, in $S^{n+1}(1)$ has

$$(n/p)j(j+p-1)+(n/q)k(k+q-1)\,, \quad j,k=0,1,2,\cdots\,,$$

as the eigenvalues of the Laplacian (cf. [2]). The first eigenvalue is n with multiplicity $n+2$. The nullity is $(p+1)(n-p+1)$ (cf. [11]), which is the multplicity of the eigenvalue $2n$ of Δ. The index is $n+3$ (cf. [11]), which is the sum of the multiplicities of the eigenvalues 0 and n of Δ.

3. Laplacian of homogeneous hypersurfaces in $S^{n+1}(1)$. Hsiang-Lawson [5] showed that every homogenous hypersurface in the unit sphere is represented as an orbit of a linear isotropy group of a Riemannian symmetric space of rank 2 (see also [14]). We follow the notations in [14].

Let $(\mathfrak{u},\theta)$ be an effective orthogonal symmetric Lie algebra of compact type and let (U,K) be a symmetric pair associated to $(\mathfrak{u},\theta)$ (cf. [4]). Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{u}=\mathfrak{k}+\mathfrak{p}$ the orthogonal decomposition of $\mathfrak{u}$ with respect to a fixed $\mathrm{Ad}(U)$-invariant inner product $(\ ,\ )$ on $\mathfrak{u}$. We choose a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$ and denote by Σ the set of all roots of $(\mathfrak{u},\theta)$ with respect to $\mathfrak{a}$. We fix a linear order in Σ and denote by Σ_+ the set of all positive elements in Σ. For $\lambda \in \Sigma_+$, set

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k};\, (\mathrm{ad}\,H)^2X = -\lambda(H)^2X \text{ for all } H \in \mathfrak{a}\}\,,$$
$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p};\, (\mathrm{ad}\,H)^2X = -\lambda(H)^2X \text{ for all } H \in \mathfrak{a}\}\,.$$

It is known that $\dim \mathfrak{k}_\lambda = \dim \mathfrak{p}_\lambda$, say $m(\lambda)$, for each $\lambda \in \Sigma_+$. Let $\mathfrak{k}_0$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{k}$ and $\mathfrak{p}$ have the following orthogonal decompositions:

$$\mathfrak{k} = \mathfrak{k}_0 + \sum_{\lambda \in \Sigma_+} \mathfrak{k}_\lambda\,, \qquad \mathfrak{p} = \mathfrak{a} + \sum_{\lambda \in \Sigma_+} \mathfrak{p}_\lambda\,.$$

For a fixed unit vector $H \in \mathfrak{a}$, let L be the stabilizer of the adjoint action of K at H whose Lie algebra $\mathfrak{l}$ is $\{X \in \mathfrak{k}; \mathrm{ad}(X)(H) = 0\}$. Define an embedding Φ_H of K/L into $\mathfrak{p}$ by $\Phi_H(kL) = \mathrm{Ad}(k)H$. The homogeneous space K/L is called an *R-space* and Φ_H its *standard embedding* (cf. [16]). The image $N(H)$ of Φ_H is a submanifold of the unit sphere S in $\mathfrak{p}$ with respect to the inner product $(\, , \,)$. Here we regard $\mathfrak{p}$ also as Euclidean space with the inner product $(\, , \,)$ and identify the tangent space of $\mathfrak{p}$ with $\mathfrak{p}$ itself. Consider the Riemannian metric on $N(H)$ induced by the inner product $(\, , \,)$ of $\mathfrak{p}$. Then the tangent space $T_HN(H)$ of $N(H)$ at H is the direct sum of $\mathfrak{p}_\lambda$, $\lambda \in \Sigma_H$, and the orthogonal complement $T_HN(H)^\perp$ of $T_HN(H)$ in T_HS is the direct sum of $H^\perp$ and $\mathfrak{p}_\lambda$, $\lambda \in \Sigma_+ - \Sigma_H$, under the above identification. Here $\Sigma_H = \{\lambda \in \Sigma_+; \lambda(H) \neq 0\}$ and $H^\perp = \{H' \in \mathfrak{a}; (H', H) = 0\}$.

In the following we assume $H \in \mathfrak{a}$ to be a regular element, that is, $\Sigma_+ = \Sigma_H$. In this case $N(H)$ or K/L is called a *regular R-space*. Then we have

$$T_HN(H) = \sum_{\lambda \in \Sigma_+} \mathfrak{p}_\lambda \, , \qquad T_HN(H)^\perp = H^\perp \, ,$$

and the Lie algebla $\mathfrak{l}$ is $\mathfrak{k}_0$. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{k}$ with respect to $(\, , \,)$. Then we have $\mathfrak{m} = \sum_{\lambda \in \Sigma_+} \mathfrak{k}_\lambda$ and the $\mathrm{Ad}(\mathrm{L})$-invariant decomposition $\mathfrak{k} = \mathfrak{l} + \mathfrak{m}$.

Let g be the Riemannian metric on K/L induced by the embedding Φ_H: $K/L \to N(H)$. Under the identification of the tangent space $T_o(K/L)$ of K/L at the origin $o = \{L\}$ with $\mathfrak{m}$ by $\mathfrak{m} \ni X \mapsto X_o \in T_o(K/L)$, we get an inner product $\langle \, , \, \rangle$ on $\mathfrak{m}$ defined by $\langle X, Y \rangle = g(X_o, Y_o)$, $X, Y \in \mathfrak{m}$. Since $\langle X, Y \rangle = ([X, H], [Y, H])$ for all $X, Y \in \mathfrak{m}$ (cf. [16, p. 208]), $\{|\lambda(H)|^{-1}X_{\lambda,i}; \lambda \in \Sigma_+, i = 1, \cdots, m(\lambda)\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle \, , \, \rangle$. Here, for each $\lambda \in \Sigma_+$, $\{X_{\lambda,i}; i = 1, \cdots, m(\lambda)\}$ (resp. $\{Y_{\lambda,i}; i = 1, \cdots, m(\lambda)\}$ is an orthonormal basis of k_λ (resp. $\mathfrak{p}_\lambda$) with respect to $(\, , \,)$ such that

$$[H', X_{\lambda,i}] = \lambda(H')Y_{\lambda,i} \quad \text{and}$$
$$[H', Y_{\lambda,i}] = -\lambda(H')X_{\lambda,i} \, , \quad \text{for all} \quad H' \in \mathfrak{a} \, .$$

Therefore the Laplacian Δ of the Riemannian manifold $(K/L, g)$ is given as follows: Let $\tilde{g}$ be a left invariant metric on K induced by the inner product on $\mathfrak{k}$, defined in such a way that the restriction to $\mathfrak{m}$ coincides with $\langle \, , \, \rangle$ and that $\mathfrak{m}$ and $\mathfrak{l}$ are mutually orthogonal. Then the canonical projection π: $K \ni k \mapsto kL \in K/L$ is a Riemannian submersion (cf. [2]) of $(K, \tilde{g})$ onto $(K/L, g)$ all of whose fibers are totally geodesic. Let $\tilde{\Delta}$ be the Laplacian of $(K, \tilde{g})$. Then we have the following proved in [7, p. 477].

PROPOSITION 3.1. *For every smooth function f on K/L, we have*

$$\pi^*(\Delta f) = \tilde{\Delta}(\pi^* f) = -\sum_{\lambda \in \Sigma_+, i=1,\cdots,m(\lambda)} \lambda(H)^{-2} L^2_{X_{\lambda,i}}(\pi^* f) .$$

Here $\pi^* f$ is a function on K defined by $\pi^* f(k) = f(\pi(k)) = f(kL)$, $k \in K$, and L_X ($X \in \mathfrak{m}$) denotes the Lie derivation on K with respect to the left invariant vector field X.

Now let us assume that $(\mathfrak{u}, \theta)$ is of rank 2, that is, $\dim \mathfrak{a} = 2$. Then for every regular element H in $\mathfrak{a}$, we have $\dim H^\perp = 1$ and $N(H)$ is a hypersurface of the unit sphere S in $\mathfrak{p}$. Moreover by Hsing-Lawson [5] every compact homogeneous hypesurface in the unit sphere can be obtained as $N(H)$ in this manner. In the following we always assume that $\dim \mathfrak{a} = 2$ and $N(H)$ or K/L is a regular R-space, i.e., $H \in \mathfrak{a}$ is regular.

4. Minimal homogeneous hypersurfaces and principal curvatures. Let the situation be as in Section 3. In this section, we use the result in [8], [14] to determine the minimal hypersurfaces among the family of regular R-spaces $N(H)$.

Let $\Sigma_+^* = \{\lambda \in \Sigma^+; \lambda/2 \notin \Sigma\}$ (say $\Sigma_+^* = \{\lambda_0, \cdots, \lambda_{p-1}\}$, $p = \#\Sigma_+^*$). For a fixed regular element H in $\mathfrak{a}$, choose Z in $H^\perp \cap S$ to be a unit normal vector of $N(H)$ at H. Let H_i, $i = 0, \cdots, p-1$, be unit vectors in $\mathfrak{a}$ satisfying $\lambda_i(H_i) = 0$, $i = 0, \cdots, p-1$, and $0 < \theta_i < \pi$, the angles between the vectors H and H_i, $i = 0, \cdots, p-1$. We may choose λ_i, $i = 0, \cdots, p-1$, in such a way that

$$0 < \theta_0 < \theta_1 < \cdots < \theta_{p-1} .$$

Then by [14], the distinct principal curvatures k_i of the regular R-space $N(H)$ with respect to the normal vector Z are $-\lambda_i(Z)/\lambda_i(H)$, $i = 0, 1, \cdots, p-1$, the multiplicity m_i of k_i is equal to $m(\lambda_i) + m(2\lambda_i)$, $i = 0, 1, \cdots, p-1$. By the definition of θ_i, we have

$$-\lambda_i(Z)/\lambda_i(H) = \cot(\theta_i) \quad i = 0, 1, \cdots, p-1 .$$

Münzner [8] showed that

$$\theta_i = \theta_0 + i\pi p^{-1} , \quad i = 0, \cdots, p-1 , \tag{4.1}$$

$$m_i = m_{i+2} , \quad \text{where indices are considered mod } p . \tag{4.2}$$

Moreover he showed in [9] that

$$p \in \{1, 2, 3, 4, 6\} . \tag{4.3}$$

If $p = 1$, then $N(H)$ is a small or great sphere of the unit sphere. If $p = 2$, then $N(H)$ is the generalized Clifford torus $S^{m_0}(r_0) \times S^{m_1}(r_1)$, $n = m_0 + m_1$, $r_0^2 + r_1^2 = 1$. In the following we treat the case $p \geqq 3$. It is known (cf. [1] and [14]) that $m_0 = m_1$ if $p = 3, 6$. Therefore the mean curvature h of $N(H)$ with respect to Z is given (cf. [8, p. 64]) by

$$(4.4)\qquad \begin{cases} \dim(N(H))h = m_0 p \cot(p\theta_0)\,, & \text{if} \quad p = 3\,, \\ \dim(N(H))h = (m_0 p/2)\cot(p\theta_0/2) - (m_1 p/2)\tan(p\theta_0/2)\,, & \\ & \text{if} \quad p = 4, 6\,. \end{cases}$$

Therefere by (4.1), (4.4), we have:

PROPOSITION 4.1. *Suppose* $\dim \mathfrak{a} = 2$, $H \in \mathfrak{a}$ *is regular and that the regular R-space* $N(H)$ *is a minimal hypersurface in the unit sphere* S *in* $\mathfrak{p}$. *Then we have the following*:

(I) *Determination*:

(i) *If* $p = 3$ *or* 6, *then the angle* θ_0 *between the vectors* H *and* H_0 *is given by* $\pi/(2p)$.

(ii) *If* $p = 4$, *then* θ_0 *satisfies the equation*

$$\tan\theta_0 = (-\sqrt{m_1} + \sqrt{m_0 + m_1})/\sqrt{m_0}\,.$$

(II) *The distinct principal curvatures* k_i *of the minimal regular R-space* $N(H)$ *are*:

(i) $\sqrt{3},\ 0,\ -\sqrt{3}\,,$ *if* $p = 3$.

(ii) $(\sqrt{m_0 + m_1} + \sqrt{m_1})/\sqrt{m_0}\,,\ (\sqrt{m_0} + \sqrt{m_1} - \sqrt{m_0 + m_1})/(\sqrt{m_0} + \sqrt{m_0 + m_1} - \sqrt{m_1})\,,\ -(\sqrt{m_0 + m_1} - \sqrt{m_1})/\sqrt{m_0}\,,$
$-(\sqrt{m_0 + m_1} - \sqrt{m_1} + \sqrt{m_0})/(\sqrt{m_0} + \sqrt{m_1} - \sqrt{m_0 + m_1})\,,$
if $p = 4$, *and*

(iii) $2 + \sqrt{3}\,,\ -2 + \sqrt{3}\,,\ 1\,,\ -1\,,\ 2 - \sqrt{3}\,,\ -2 - \sqrt{3}\,,$
if $p = 6$.

5. Proof of Main Theorem.

5.1. In this section, we prove Main Theorem calculating the eigenvalues of the Laplacian in each case. We first explain our method of computing the eigenvalues of the Laplacian for the compact minimal homogeneous Riemannian manifold $(K/L, g)$. Recall the spherical representation theory for the coset space K/L (cf. [15]). Let $D(K)$ be the set of all finite dimensional inequivalent irreducible unitary representations (ρ, V^ρ) of K and $D(K, L)$ the set of all spherical representations in $D(K)$ for the pair (K, L), that is,

$$D(K, L) = \{(\rho, V^\rho) \in D(K);\ V_L^\rho \neq \{0\}\}\,,$$

where

$$V_L^\rho = \{v \in V^\rho;\ \rho(l)v = v \text{ for all } l \in L\}\,.$$

Let $((\ ,\ ))$ be a Hermitian inner product on V^ρ invariant under the action of $\rho(K)$ and $\{v_i;\ i = 1, \cdots, \dim V^\rho\}$ an orthonormal basis of V^ρ with respect to $((\ ,\ ))$ so that $\{v_j;\ j = 1, \cdots, \dim V_L^\rho\}$ is a basis of V_L^ρ for $(\rho, V^\rho) \in D(K, L)$. Then by Peter-Weyl's theorem,

$$\{\rho_{ij};\ 1 \leqq j \leqq \dim V_L^\rho,\ 1 \leqq i \leqq \dim V^\rho,\ (\rho, V^\rho) \in D(K, L)\}\ ,$$

is a complete orthogonal system of the space $C_C^\infty(K/L)$ of all C^∞-complex valued functions of K/L with respect to the following inner product:

$$((\phi_1, \phi_2)) = \int_{K/L} \phi_1(xL)\overline{\phi_2(xL)}dv_g\ , \quad \phi_1, \phi_2 \in C_C^\infty(K/L)\ . \tag{5.1}$$

Here dv_g is the volume element of $(K/L, g)$ and

$$\rho_{ij}(x) = ((\rho(x)v_j, v_i))\ , \quad x \in K\ , \quad 1 \leqq j \leqq \dim V_L^\rho\ , \quad 1 \leqq i \leqq \dim V^\rho \tag{5.2}$$

are regarded as functions on K/L since $\rho(l)v_j = v_j$, $l \in L$, implies $\rho_{ij}(xl) = \rho_{ij}(x)$ for all $x \in K$, $l \in L$. We identify ρ_{ij} with $\pi^*\rho_{ij}$.

Now since the Laplacian Δ of $(K/L, g)$ is expressed in terms of the Lie algebra $\mathfrak{k}$ (cf. Proposition 3.1), we have

$$(\Delta\rho_{ij})(x) = ((\rho(x)\rho(D)v_j, v_i))\ , \quad 1 \leqq j \leqq \dim V_L^\rho\ , \quad 1 \leqq i \leqq \dim V^\rho\ .$$

Here $\rho(D)$ is the endomorphism of V^ρ given by

$$\rho(D) = -\sum_{\lambda \in \Sigma_+, i=1,\cdots,m(\lambda)} \lambda(H)^{-2}\rho(X_{\lambda,i})^2\ , \tag{5.2}$$

where $\{X_{\lambda,i};\ \lambda \in \Sigma_+,\ i = 1, \cdots, m(\lambda)\}$ is as in Section 3 and $\rho(X)$, $X \in \mathfrak{k}$, is the infinitesimal representation of ρ on V^ρ. We note that $\rho(D)$ leaves V_L^ρ invariant, since $\{\lambda(H)^{-1}X_{\lambda,i};\ \lambda \in \Sigma_+,\ i = 1, \cdots, m(\lambda)\}$ is an orthonormal basis of $\mathfrak{m}$ with respect to the $\mathrm{Ad}(L)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ induced by the invariant Riemannian metric g on K/L. Therefore if we find all the eigenvalues of the endomorphism $\rho(D)$ on the finite dimensional space V_L^ρ for each $(\rho, V^\rho) \in D(K, L)$, then these eigenvalues exhaust all the eigenvalues of the Laplacian Δ of $(K/L, g)$. Except in the case $\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, however, it is very difficult to find all the eigenvalues of $\rho(D)$. In the remaining cases, we will determine only the first eigenvalue λ_1 of the Laplacian Δ of $(K/L, g)$ by estimating all the eigenvalues of $\rho(D)$.

5.2. Case of $\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. In this case, let $(\mathfrak{u}, \mathfrak{k}) = (\mathfrak{su}(3), \mathfrak{so}(3))$, $\mathfrak{p} = \{\sqrt{-1}X;\ X\ 3 \times 3\text{-real matrices},\ X - {}^tX = 0,\ \mathrm{tr}\,X = 0\}$, and $(U, K) = (\mathrm{SU}(3), \mathrm{SO}(3))$. Here ${}^tX$ (resp. $\mathrm{tr}\,X$) is the transpose (resp. the trace) of X. Then the $\mathrm{Ad}(K)$ action on $\mathfrak{p}$ is given by $\mathrm{SO}(3) \times \mathfrak{p} \ni (k, X) \mapsto kXk^{-1} \in \mathfrak{p}$. Put $\mathfrak{a} = \{\sqrt{-1}\,\mathrm{diag}\,(y_1, y_2, y_3);\ y_i \in \boldsymbol{R},\ \sum_{i=1}^3 y_i = 0\}$. Here we denote by $\mathrm{diag}\,(y_1, y_2, y_3)$ the diagonal 3×3 matrix whose diagonal entries are y_1, y_2 and y_3. Put

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

and $H = \sqrt{-1}\,\mathrm{diag}(-1, 1, 0) \in \mathfrak{a}$. Let $(\ ,\ )$ be the $\mathrm{Ad}(U)$-invariant inner product on $\mathfrak{su}(3)$ defined by $(X, Y) = -2^{-1}\mathrm{tr}(XY)$, $X, Y \in \mathfrak{su}(3)$. Then $(H, H) = 1$. Let μ_i be the linear form on $\mathfrak{a}$ defined by $\mathfrak{a} \ni \sqrt{-1}\,\mathrm{diag}(y_1, y_2, y_3) \to y_i$. Then we have $\Sigma = \Sigma^* = \{\pm(\mu_i - \mu_j);\ 1 \leqq i < j \leqq 3\}$. Fix a lexicographic order $>$ on Σ so that $\mu_1 > \mu_2 > \mu_3$. Then $\Sigma_+^* = \{\mu_i - \mu_j;\ 1 \leqq i < j \leqq 3\}$ and $p = 3$. Put $c = 1/\sqrt{3}$ and

$$H_0 = \sqrt{-1}\,\mathrm{diag}(-2c, c, c), \qquad H_1 = \sqrt{-1}\,\mathrm{diag}(c, c, -2c),$$
$$H_2 = \sqrt{-1}\,\mathrm{diag}(c, -2c, c).$$

Let $\lambda_0 = \mu_2 - \mu_3$, $\lambda_1 = \mu_1 - \mu_2$, $\lambda_2 = \mu_1 - \mu_3$. Then we have

$$\lambda_i(H_i) = 0, \quad 0 \leqq i \leqq 2, \qquad \text{and} \qquad \theta_0 = \pi/6, \quad \theta_1 = \pi/2, \quad \theta_2 = 5\pi/6.$$

Thus by Proposition 4.1, the $\mathrm{Ad}(K)$ orbit through H is minimal in the 4-dimensional unit sphere $S = \{X \in \mathfrak{p};\ (X, X) = 1\}$. The stabilizer L of K at H under the $\mathrm{Ad}(K)$ action is a finite subgroup of $\mathrm{SO}(3)$ consisting of

$$\mathrm{diag}(1, 1, 1), \qquad \mathrm{diag}(-1, 1, -1), \qquad \mathrm{diag}(-1, -1, 1) \quad \text{and}$$
$$\mathrm{diag}(1, -1, -1).$$

Thus the homogeneous space K/L is just $\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, $\mathfrak{k} = \mathfrak{m}$ and $\mathfrak{l} = \{0\}$. We note that $\{2^{-1}X_1, X_2, X_3\}$ is an orthonormal basis of $\mathfrak{so}(3)$ with respect to the inner product $\langle\ ,\ \rangle$ given by $\langle X, Y\rangle = ([X, H], [Y, H])$, $X, Y \in \mathfrak{p}$. Let π; $S^3 = \mathrm{SU}(2) \to \boldsymbol{RP}^3 = \mathrm{SO}(3)$ be the canonical projection given by $\mathrm{SU}(2) \ni x \mapsto \mathrm{Ad}(x) \in \mathrm{SO}(3) \subset \mathrm{GL}(\mathfrak{su}(2))$. Then $L' = \pi^{-1}(L)$ consists of

$$\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \pm\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \text{and}$$
$$\pm\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

Moreover put

$$H^* = \sqrt{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad 2U = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \text{and}$$
$$2V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\{4^{-1}H^*, U, V\}$ is an orthonormal basis of $\mathfrak{su}(2)$ with respect to the inner product induced by the Riemannian metric π^*g on SU(2). Here g is the invariant Riemannian metric on K/L corresponding to $\langle\ ,\ \rangle$ with respect to which Φ_H is a minimal embedding into the 4-dimensional sphere S. g is also regarded as a left invariant Riemannian metric on SO(3). Thus the Laplacian Δ_{π^*g} of the Riemannian manifold (SU(2), π^*g) is expressed as

$$\Delta_{\pi^*g} = -((1/16)L_{H^*}^2 + L_U^2 + L_V^2) ,$$

where L_X is the Lie derivative with respect to the left invariant vector field $X \in \mathfrak{su}(2)$ on SU(2). Therefore we have only to calculate the eigenvalues of Δ_{π^*g} making use of the above expression.

It is known that $D(\mathrm{SU}(2)) = \{(\rho_m, V^m);\ m$ non-negative integers$\}$. Here V^m is the vector space of all homogeneous polynomials of degree m in two complex variables z_1, z_2, and the action of SU(2) on V^m is defined by

$$\rho_m(x)f(z_1, z_2) = f(az_1 - \bar{b}z_2, bz_1 + \bar{a}z_2) , \quad \text{with} \quad x = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \mathrm{SU}(2)$$

for every $f \in V^m$. The $\rho_m(\mathrm{SU}(2))$-invariant Hermitian inner product is

$$((f_1, f_2)) = \sum_{k=0}^{m} a_k \bar{b}_k k!\,(m-k)! ,$$
$$\text{for} \quad f_1 = \sum_{k=0}^{m} a_k z_1^k z_2^{m-k} \quad \text{and} \quad f_2 = \sum_{k=0}^{m} b_k z_1^k z_2^{m-k} .$$

Then $v_k = (1/\sqrt{k!\,(m-k)!})z_1^k z_2^{m-k}$ is an orthonormal basis of V^m with respect to $((\ ,\ ))$. Moreover the subspace $V_{L'}^m$ of V^m is

$$\left\{f \in V^m;\ \rho_m\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} f = \rho_m\begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix} f \right.$$
$$\left. = \rho_m\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} f = \rho_m\begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix} f = f \right\} .$$

Then $V_{L'}^m \neq \{0\}$ if and only if m is even.
If we let $m = 2p$, then the subspace $V_{L'}^m$ is given as follows:

(i) If $p = 2l$, then $V_{L'}^m$ is generated by $\{w_i\}_{i=1}^{l+1}$. Here w_i are given by

$$w_i = 2^{-1}(v_{2(i-1)} + v_{4l-2(i-1)}) , \quad i = 1, \cdots, l+1 .$$

(ii) If $p = 2l + 1$, then $V_{L'}^m$ is generated by $\{w_i'\}_{i=1}^{l}$. Here w_i' are given by

$$w_i' = 2^{-1}(v_{2i-1} - v_{4l-2i+3}) , \quad i = 1, \cdots, l .$$

The endomorphism $\rho_m(D)$ on $V_{L'}^m$, defined by (5.2), satisfies the following:

(i) If $p = 2l$, then

$$\rho_m(D)w_i = \{(1/16)(4(i-1)-2p)^2 + 4(i-1)(p-i+1) + p\}w_i\,, \quad i = 1, \cdots, l+1\,.$$

(ii) If $p = 2l+1$, then

$$\rho_m(D)w_i' = \{(1/16)(4i-2-2p)^2 + (2i-1)(2p-2i+1) + p\}w_i'\,, \quad i = 1, \cdots, l\,.$$

PROPOSITION 5.1. *The spectrum of the Laplacian of the compact Riemannian manifold* $(\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2, g)$, *minimally embedded in the 4-dimensional unit sphere is given as follows: The eigenvalues are*

(i) $(i-1-l)^2 + (2i-2)(4l-2i+2) + 2l\,,$
$l = 0, 1, 2, \cdots\,; \quad i = 1, 2, \cdots, l+1\,,$ *and*

(ii) $(i-1-l)^2 + (2i-1)(4l-2i+3) + 2l+1\,,$
$l = 1, 2, \cdots\,; \quad i = 1, 2, \cdots, l\,.$

The multiplicity of the eigenvalues (i) (resp. (ii)) *is* $4l+1$ (resp. $4l+3$).

Therefore, by Proposition 5.1, we see that the first eigenvalue of $(\mathrm{SO}(3)/\boldsymbol{Z}_2 \times \boldsymbol{Z}_2, g)$ is 3, while the index (resp. the nullity) is 20 (resp. 7).

5.3. Case of $\mathrm{SU}(3)/T^2$. Let $(\ ,\ )'$ be the inner product of $\mathfrak{su}(3)$ defined by $(X, Y)' = -2^{-1}\operatorname{tr}(XY)$, $X, Y \in \mathfrak{su}(3)$. Let $\mathfrak{h} = \{X \in \mathfrak{su}(3);\ X$ diagonal$\}$, $\mathfrak{m}'$ the orthocomplement of $\mathfrak{h}$ in $\mathfrak{su}(3)$ with respect to $(\ ,\ )'$. Then $T^2 = \{k \in \mathrm{SU}(3);\ k$ diagonal$\}$ is a maximal torus of $\mathrm{SU}(3)$.

In this case, we have $(\mathfrak{su}(3) \times \mathfrak{su}(3), \Delta)$, as a pair $(\mathfrak{u}, \mathfrak{k})$, where $\Delta = \{(X, X);\ X \in \mathfrak{su}(3)\}$. Put $\mathfrak{p} = \{(X, -X);\ X \in \mathfrak{su}(3)\}$, $U = \mathrm{SU}(3) \times \mathrm{SU}(3)$, and $K = \{(x, x);\ x \in \mathrm{SU}(3)\}$. Then (U, K) is the corresponding symmetric pair. Put $\mathfrak{a} = \{(X, -X);\ X \in \mathfrak{h}\}$. Let $(\ ,\ )$ be the $\mathrm{Ad}(U)$-invariant inner product on $\mathfrak{u}$ given by

$$((X, Y), (X', Y')) = (X, X')' + (Y, Y')'\,, \quad X, X', Y, Y' \in \mathfrak{su}(3)\,.$$

Fix an element $H = (H', -H') \in \mathfrak{a}$, where $H' = \sqrt{-1}\operatorname{diag}(-1, 1, 0) \in \mathfrak{h}$. Let μ_i be the linear map on $\mathfrak{a}$ defined by $\mathfrak{a} \ni (X, -X) \to \theta_i \in \boldsymbol{R}$, where $X = \sqrt{-1}\operatorname{diag}(\theta_1, \theta_2, \theta_3)$, $i = 1, 2, 3$. Then $\Sigma = \Sigma^* = \{\pm(\mu_i - \mu_j);\ 1 \leqq i < j \leqq 3\}$. Fix a lexicographic order $>$ on Σ so that $\mu_1 > \mu_2 > \mu_3$. Then $\Sigma_+^* = \{\mu_i - \mu_j;\ 1 \leqq i < j \leqq 3\}$ and $p = \sharp\Sigma_+^* = 3$. Put $H_i = (H_i', -H_i') \in \mathfrak{a}$, $i = 0, 1, 2$, where H_i' coincides with H_i as in 5.2. Define λ_i as in 5.2 using μ_i. Then $\lambda_i(H_i) = 0$, $i = 0, 1, 2$, and $\theta_0 = \pi/6$, $\theta_1 = \pi/2$, $\theta_2 = 5\pi/6$. Therefore by Proposition 4.1, the $\mathrm{Ad}(K)$-orbit $N(H)$ through H is minimal in the 7-dimensional unit sphere $S = \{(X, -X) \in \mathfrak{p};\ (X, X)' = 1\}$. The subgroup L is $\{(t, t); t \in T^2\}$ and the orthocomplement $\mathfrak{m}$ of the Lie algebra $\mathfrak{l}$ in $\mathfrak{k}$ is $\{(X, X); X \in \mathfrak{m}'\}$. Identify $\mathfrak{p}$ with $\mathfrak{su}(3)$ by $\mathfrak{p} \ni (X, -X) \mapsto X \in \mathfrak{su}(3)$. Under this identification, the $\mathrm{Ad}(K)$-action on $\mathfrak{p}$ coincides with the $\mathrm{Ad}(\mathrm{SU}(3))$-action on $\mathfrak{su}(3)$ and the homogeneous space K/L is $\mathrm{SU}(3)/T^2$. Moreover the inner product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ induced by $(K/L, g)$ coincides with the inner product $\langle\ ,\ \rangle'$ on $\mathfrak{m}'$ defined by

$$\langle X, Y\rangle' = ([X, H'], [Y, H'])' , \quad X, Y \in \mathfrak{m}' .$$

Therefore we have only to calculate the eigenvalues of the Laplacian Δ' of the Riemannian manifold $(\mathrm{SU}(3)/T^2, g')$ whose Riemannian metric g' is induced by the inner product $\langle\ ,\ \rangle'$ on $\mathfrak{m}'$. The Laplacian Δ' is given as follows: Let H_1, H_2 be any fixed orthonormal basis of $\mathfrak{h}$ with respect to $(\ ,\ )'$. Set

$$X(x_1, \cdots, x_6) = \begin{pmatrix} 0 & -\bar{z}_1 & -\bar{z}_2 \\ z_1 & 0 & -\bar{z}_3 \\ z_2 & z_3 & 0 \end{pmatrix},$$

where $z_1 = x_1 + \sqrt{-1}x_2$, $z_2 = x_3 + \sqrt{-1}x_4$, $z_3 = x_5 + \sqrt{-1}x_6$, $x_i \in \boldsymbol{R}$ $(i = 1, \cdots, 6)$. Put $X_i = X(0, \cdots, 0, 1, 0, \cdots, 0)$, with 1 in the i-th place for $i = 1, \cdots, 6$. Then $\{H_1, H_2, X_i\ (i = 1, \cdots, 6)\}$ is an orthonormal basis of $\mathfrak{su}(3)$ with respect to $(\ ,\ )'$. By the definition of $\langle\ ,\ \rangle'$, $\{(1/2)X_1, (1/2)X_2, X_3, X_4, X_5, X_6\}$ is an orthonormal basis of $\mathfrak{m}'$ with respect to $\langle\ ,\ \rangle'$.

We extend the inner product $\langle\ ,\ \rangle'$ on $\mathfrak{m}'$ to that on $\mathfrak{su}(3)$, which we denote by the same letter as $\langle X, Y\rangle' = (X, Y)'$ for $X, Y \in \mathfrak{h}$, and $\langle X, Y\rangle' = 0$ for $X \in \mathfrak{m}'$, $Y \in \mathfrak{h}$. Let $\tilde{g}$ be the left invariant Riemannian metric on $\mathrm{SU}(3)$ induced by $\langle\ ,\ \rangle'$. Then for each smooth function f on $\mathrm{SU}(3)/T^2$,

$$\pi^*(\Delta' f) = \tilde{\Delta}(\pi^* f) = ((1/4)\Delta_{g_0} - (3/4)(L_{X_3}^2 + L_{X_4}^2 + L_{X_5}^2 + L_{X_6}^2))(\pi^* f) ,$$

where $\pi^* f(k) = f(\pi(k))$, $k \in \mathrm{SU}(3)$, π is the projection of $\mathrm{SU}(3)$ onto $\mathrm{SU}(3)/T^2$, and Δ_{g_0} is the Laplacian of the bi-invariant Riemannian metric g_0 on $\mathrm{SU}(3)$ induced by the $\mathrm{Ad}(\mathrm{SU}(3))$-invariant inner product $(\ ,\ )'$ on $\mathfrak{su}(3)$. Since $(X, Y)' = -B(X, Y)/12$, for $X, Y \in \mathfrak{su}(3)$ with the Killing form B of $\mathfrak{su}(3)$, we have

$$\Delta_{g_0} = -12\Omega ,$$

where Ω is the Casimir operator of $\mathfrak{su}(3)$ (cf. [17]). Therefore, by 5.1, we have only to consider the eigenvalues of the operator

$$\rho(D) = -3\rho(\Omega) - (3/4)(\rho(X_3)^2 + \rho(X_4)^2 + \rho(X_5)^2 + \rho(X_6)^2)$$

on the subspace V_L^ρ of V^ρ for all $(\rho, V^\rho) \in D(\mathrm{SU}(3), T^2)$. Yamaguchi [17] determined the spectra of flag manifolds, in particular, $D(\mathrm{SU}(3), T^2)$ and the eigenvalues of $\rho(\Omega)$ for each $(\rho, V^{\rho}) \in D(\mathrm{SU}(3), T^2)$. By his results, there exists a bijection between $D(\mathrm{SU}(3), T^2)$ and the set $\hat{D}(\mathrm{SU}(3), T^2)$ given by

$$\hat{D}(\mathrm{SU}(3), T^2) = \{((m_1 + 2m_2)/3, (2m_1 + m_2)/3);\ m_1, m_2 \text{ non-negative integers}\} .$$

We put $p_1 = (2m_1 + m_2)/3$ and $p_2 = (m_1 + 2m_2)/3$. We denote this bijection by $\hat{D}(\mathrm{SU}(3), T^2) \ni (p_1, p_2) \to \rho_{p_1,p_2} \in D(\mathrm{SU}(3), T^2)$. Then for each $(p_1, p_2) \in \hat{D}(\mathrm{SU}(3), T^2)$, the eigenvalue of $\rho_{p_1,p_2}(\Omega)$ is

$$(1/6)(m_1p_1 + m_2p_2 + 2(p_1 + p_2)) .$$

The operator $P = -(L^2_{X_3} + L^2_{X_4} + L^2_{X_5} + L^2_{X_6})$ is positive, that is,

$$\int_{\mathrm{SU}(3)} (P\phi)\bar{\phi}\, dv_{g_0} \geqq 0 ,$$

for every smooth function ϕ on SU (3), where dv_{g_0} is the volume element of $(\mathrm{SU}(3), g_0)$. Since P commutes with Δ_{g_0}, the eigenvalues of $\rho_{p_1,p_2}(D)$ is not less than $(m_1p_1 + m_2p_2 + 2(p_1 + p_2))/2$, for each $(p_1, p_2) \in \hat{D}(\mathrm{SU}(3), T^2)$. For all (p_1, p_2) in $\hat{D}(\mathrm{SU}(3), T^2)$ except for $(p_1, p_2) = (1, 1)$, we have

$$(m_1p_1 + m_2p_2 + 2(p_1 + p_2))/2 \geqq 6 .$$

In the case of $(p_1, p_2) = (1, 1)$, we have $(m_1p_1 + m_2p_2 + 2(p_1 + p_2))/2 = 3$. But the representation $\rho_{1,1}$ coincides with the adjoint representation of SU (3) on $\mathfrak{su}(3)$. In this case, the subspace $V_L^{\rho_{1,1}}$ is just the space $\mathfrak{h}$. Hence the eigenvalues of $\rho_{1,1}(D)$ on this space are 6 and 12. Thus the first eigenvalue of the 6-dimensional minimal hypersurface $\mathrm{SU}(3)/T^2$ in the unit sphere is just 6.

In the cases of $\mathrm{Sp}(2)/T^2$, G_2/T^2 and $\mathrm{SO}(m) \times \mathrm{SO}(2)/\mathrm{SO}(m-2) \times \boldsymbol{Z}_2$, $m \geqq 2$, the assertions of Main Theorem can be proved in the same manner as in the case of $\mathrm{SU}(3)/T^2$. We omit the lengthy computations.

REFERENCES

[1] S. ARAKI, On root systems and an infinitesimal classification of irreducible symmetric spaces, J. Math. Osaka City Univ. 13 (1962), 1–34.

[2] M. BERGER, P. GAUDUCHON AND E. MAZET, Le spectre d'une variété riemannienne, Lecture Notes in Math. 194, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[3] T. HASEGAWA, Spectral geometry of closed minimal submanifolds in a space form, real or complex, Kōdai Math. J. 3 (1980), 224–252.

[4] S. HELGASON, Differential Geometry and Symmetric Spaces, Academic Press, New York, 1962.

[5] W. Y. HSIANG AND H. B. LAWSON, JR., Minimal submanifolds of low cohomogeneity, J. Differential Geom. 5 (1971), 1–36.

[6] H. MUTO, The multiplicity of the first eigenvalue of the Laplacian on SU(2), SO(3) and $\mathrm{Sp}(n+1)/\mathrm{Sp}(n)$, preprint.

[7] H. MUTO AND H. URAKAWA, On the least positive eigenvalue of Laplacian for compact homogeneous spaces, Osaka J. Math. 17 (1980), 471–484.

[8] H. F. MÜNZNER, Isoparametrische Hyperflächen in Sphären, Math. Ann. 251 (1981), 57–71.

[9] H. F. MÜNZNER, Isoparametrische Hyperfläschen in Sphären, II, Math. Ann. 256 (1981), 215–232.

[10] K. OGIUE, Open Problems, Surveys in Geometry, 1980/1981, Geometry of the Laplace Operator, edited by T. Ochiai, (in Japanese).

[11] J. SIMONS, Minimal varieties in Riemannian manifolds, Ann. of Math. 88 (1968), 62–105.

[12] M. SUGIURA, Spherical functions and representation theory of compact Lie groups, Sci. Papers College Gen. Ed. Univ. Tokyo 10 (1960), 187–193.

[13] T. TAKAHASHI, Minimal immersions of Riemannian manifolds, J. Math. Soc. Japan 18 (1966), 380–385.

[14] R. TAKAGI AND T. TAKAHASHI, On the principal curvatures of homogeneous hypersurfaces in a unit sphere, Diff. Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 469–481.

[15] M. TAKEUCHI, Modern Theory of Spherical Functions (in Japanese), Iwanami, Tokyo, 1975.

[16] M. TAKEUCHI AND S. KOBAYASHI, Minimal imbeddings of R-spaces, J. Differential Geom. 2 (1968), 203–215.

[17] S. YAMAGUCHI, Spectra of flag manifolds, Mem. Fac. Sci. Kyushu Univ. 33 (1979), 95–112.

[18] S. T. YAU, Problem Section, Seminar on Differential Geometry, Ann. Math. Studies 102, Princeton Univ. Press, 1982, p. 692.

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