Abstract. We compute the $E_2$-term of Borel's spectral sequence for certain holomorphic fibrations. Among some of the applications considered are the representation of automorphic cohomology of a flag domain, and the derivation of new cohomology vanishing theorems for certain compact projective varieties.

1. Introduction. In this paper we consider diagrams $E \to X \to Y$ where $X \to Y$ is a holomorphic fibre bundle (with compact fibre $F$) and $E \to X$ is a holomorphic vector bundle; the problem then is to relate the Dolbeault cohomology of $X$ with coefficients in $E$ to suitable cohomologies of $Y$ and $F$. For general $E$ there does not seem to be any way of achieving this for the space $H^{p,q}(X, E)$ with $p > 0$ in a manner accessible to explicit computation. However if $E$ is assumed to be locally trivial over $Y$ the problem is more tractable: in this case there is the (generalized) Borel spectral sequence relating $H^*_i(Y)$ and a suitable fibre cohomology to $H^*_i(X, E)$ and a convenient form of the $E_2$-term of this spectral sequence (or, more accurately, family of spectral sequences) can be found by the techniques of [1], [3], [12], [13] and [14]. In all generality the $E_2$-terms are determined by holomorphic vector bundles $H^{r,s}(E)$, associated with $E \to X$, whose fibres are suitable $(r, s)$-cohomologies of the fibres of $X \to Y$; for $p = 0$ one concludes the bundles $H^{r,s}(E)$ "represent" the direct images of the sheaf $\mathcal{O}(E)$ which thus are locally free.

The "cohomology bundles" $H^{r,s}(E)$ thus are crucial for the description of the $E_2$-terms of the Borel spectral sequence and merit some attention; we present the calculation of such bundles in some important special cases and also indicate some applications. As an example if $X$ and $Y$ are homogeneous spaces of a Lie group $G$ and if $E \to X$ is a homogeneous vector bundle, it is locally trivial over $Y$ and the cohomology bundles $H^{r,s}(E)$ are homogeneous as well. This interesting fact has, among others, the following application: Let $\mathcal{Z}_D \to M_D$ be the linear deformation space of a maximal compact subvariety of a flag domain $D$. In [26] Wells and Wolf show that under suitable conditions $M_D$ is a Stein manifold and establish a representation of the automorphic cohomology of $D$ (with
respect to a discrete subgroup $\Gamma$) in the space of $\Gamma$-invariant holomorphic sections of a certain bundle over $M_D$, cf. [26, Theorem 3.4.7]. We show below that if $L$ is the stabilizer of the compact subvariety $Y$, then this bundle over $M_D$ can be described as an associated vector bundle of a canonical principal $L$-bundle $A \to M_D$, induced by the action of $L$ on the cohomology of $Y$. In a sense, this result is “best possible” since it is known that, in general, $M_D$ is not a quotient of Lie groups.

In Section 4, we investigate the “transform” of $H^{\ast\cdot}$ under a discontinuous action of a group $\Gamma$ on a diagram $E \to X \to Y$; when $E$ is trivial over $Y$, this “transform” is determined by an automorphic factor which we compute explicitly in Theorem 4.4.

In Section 5, this result is combined with the Borel-Bott-Weil theorem and results of [27], [28] to derive new vanishing theorems for the cohomology of compact normal projective varieties $\Gamma \setminus G_0/T$; here $G_0$ is a connected, non-compact semi-simple Lie group, $T$ is a Cartan subgroup of $G_0$ contained in some maximal compact subgroup $K \subset G_0$ such that $G_0/K$ is Hermitian symmetric, and finally $\Gamma$ is a discrete subgroup of $G_0$ acting freely on $G_0/K$. Theorems 5.24 and 5.27 are the main results of this paper which also subsumes a note announced in [3] as “Construction of cohomology bundles in the case of an open real orbit in a complex flag manifold”.

2. Borel-Le Potier diagrams. Let $X, Y$ be complex manifolds and assume that $\pi: X \to Y$ is a holomorphic fibre bundle with compact fibre $F$. Moreover let $E \to X$ be a holomorphic vector bundle (with fibre $E$) with projection $\sigma$. One says that $E$ is locally trivial over $Y$ if there exists a holomorphic vector bundle $E_0 \to F$ (with fibre $E$) such that $\pi \circ \sigma: E \to Y$ is a holomorphic fibre bundle with fibre $E_0$ and group $GL(E_0)$, the group of all holomorphic automorphisms of $E_0$ (i.e., all fibrewise linear biholomorphisms $E_0 \to E_0$); it is known that $GL(E_0)$ is a complex Lie group, cf. e.g., [13]. In this case,

$$E \to X \to Y$$

will be called a Borel-Le Potier diagram (BL-diagram). More explicitly this means that each $y \in Y$ has an open neighbourhood $U$ over which there is a holomorphic trivialization $\phi_U: \pi^{-1}(U) \cong U \times F$ which is covered by a holomorphic isomorphism $\psi_U$ of the vector bundle $E|\pi^{-1}(U)$ onto $U \times E_0$:

$$\begin{array}{ccc}
E|\pi^{-1}(U) & \xrightarrow{\psi_U} & U \times E_0 \\
\downarrow & & \downarrow \\
\pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\
& & \downarrow \\
& & U
\end{array}$$
the diagram commutes and $\psi_U$ is fibrewise linear. The following are some examples:

(i) Given $\pi: X \to Y$, a holomorphic bundle with compact fibre, and the holomorphic vector bundle $W \to Y$, $\pi^*W \to X \to Y$ is a BL-diagram; this is the case originally considered in [1].

(ii) Let $H$ be a complex Lie group, $L \subset H$ a closed complex subgroup and suppose that $\rho: Z \to Y$ is a holomorphic principal $H$-bundle. Set $X = Z/L$ and suppose that $X$ has a complex structure such that the natural map $\sigma: Z \to X$ is a holomorphic principal $L$-bundle. Lastly, assume that $H/L$ is compact. The natural map $\pi: X \to Y$ then yields a holomorphic bundle with fibre $H/L$ such that $\pi \circ \sigma = \rho$. If $\lambda: H \to GL(E)$ is a finite-dimensional holomorphic representation, then we can form the holomorphic vector bundles $E_i = Z \times_H E \to X$ and $E_0 = H \times_L E \to H/L$. Under these conditions

$$E_i \to X \to Y$$

is a BL-diagram: If $U \subset Y$ is a sufficiently small open set, there is a holomorphic section $s: U \to Z$ of $\rho$ and this section is used to construct both $\phi_U$ and $\psi_U$ in the following manner: For $(z, e) \in Z \times E$ and $(h, e) \in H \times E$ let $[z, e], [h, e]$ be their respective equivalence classes in $E_i, E_0$. Then set $\phi_U^\gamma(y, hH) = \sigma(s(y)h)$ for $(y, h) \in U \times H$; this yields a trivialization of $Z|U = \rho^{-1}(U)$. A covering isomorphism $\psi_U$ in the sense of (2.2) then is obtained by setting $\psi_U^\gamma(y, [h, e]) = (\phi_U^\gamma(y, hH), [s(y)h, e])$.

(iii) Let $P \subset SL(4, C) = G$ be the parabolic subgroup defined by $a_{21} = a_{31} = a_{41} = a_{32} = a_{42} = 0$; let $V = SU(2, 2) \cap P = SU(1) \times SU(1) \times SU(2))$ and $K = SU(2) \times SU(2)$, so that $K$ is a maximal compact subgroup of the real form $G_0 = SU(2, 2)$ of $G$. Set $F_1^+ = G_0/V$, $M^+ = G_0/K$; thus there is the “double fibration”

$$P_3^+ \overset{\alpha}{\leftarrow} F_1^+ \overset{\beta}{\to} M^+$$

where $P_3^+ \subset P_3(C)$ is the “projective twister space”, of importance in mathematical physics in connection with the so-called Penrose transform, cf. [24] for some details. Let $H \to P_3^+$ be the restriction of the hyperplane bundle of $P_3(C)$. Then it can be shown that for every integer $m \alpha^*H^* \to F_1^+ \to M^+$ is a BL-diagram.

(iv) We shall give more examples later (Sections 3 and 4). Another interesting example is used by Fisher in the study of the cohomology of compact complex nilmanifolds, cf. [4].

In the situation of (2.1) we also write $X_y$ for the fibre $\pi^{-1}(y)$ at $y$. With this set for each pair of natural numbers $(r, s)$
Here $H^{r,s}$ denotes the (bundle-valued) Dolbeault cohomology of type $(r,s)$. Since $X = F$ is compact these cohomologies all are finite-dimensional and one can prove the following:

**Theorem 2.5.** $H^{r,s}(E)$ is a holomorphic vector bundle over $Y$ with fibre $H^{r,s}(F, E_0)$, associated with the bundle $\pi: X \to Y$.

Explicit local trivializations will be indicated below; cf. also [1], [3], [13]. The importance of these “cohomology bundles” lies in their use in the computation of the $E_2$-terms of the Borel spectral sequence for the $\bar{\partial}$-cohomology of holomorphic fibre bundles with compact fibre; in brief the spectral sequence is obtained as follows: Let $A^{p,q}(X, E)$ be the space of smooth $E$-valued forms of type $(p, q)$ on $X$. This space has a natural decreasing filtration “in terms of base forms”: one defines $F^rA^{p,q}(X, E)$ to be the space of those $(p, q)$-forms which may be written as finite sums of forms of the type $\pi^*\alpha \wedge \beta$ with $\alpha \in A^{a,b}(Y)$, $\beta \in A^{d,q}(X, E)$ such that $a + c = p$, $b + d = q$ and $a + b \geq r$. Then $F^rA^{p,q} \supset F^{r+1}A^{p,q}$ and $\partial(F^rA^{p,q}) \subset F^rA^{p,q+1}$. If one fixes $p$, one thus obtains a decreasing filtration of $A^{p,*}(X, E) = \bigoplus_q A^{p,q}(X, E)$ which is compatible with $\partial$ and is regular, etc. Accordingly, one obtains a spectral sequence $(E_2^{*,*})$ which converges to the $\bar{\partial}$-cohomology $H^{p,*}(X, E)$. The main result, due to Borel in the case $E = \pi^* W$ and to Le Potier in the more general case, is the following:

**Theorem 2.6.** Let $E \to X \to Y$ be a Borel-Le Potier diagram as in (2.1). For each $p \geq 0$ the $E_2$-term of the Borel spectral sequence is given by

\[
E_2^{*,*} = \bigoplus_i H^{i,*}(Y, H^{s-i,*}(E)) .
\]

For $p = 0$ in particular, one obtains $E_2^{*,*} = H^{0,*}(Y, H^{0,*}(E)) = H^*(Y, \mathcal{O}(H^{0,*}(E)))$ where $\mathcal{O}(\ldots)$ denotes the sheaf of holomorphic sections. Now for $p = 0$ the Borel spectral sequence coincides with the Leray sequence and one can show that $\mathcal{O}(H^{0,*}(E)) \cong \pi_*\mathcal{O}(E)$, establishing that these direct image sheaves here are locally free; we omit all details and refer instead to [1], [12], [14] for more information—including the case $p > 0$ where the Borel sequence no longer is the Leray sequence of any “standard” locally free sheaf over $X$.

In the situation of Example (ii) above more can be said about the cohomology bundles: Again $E_0$ is the homogeneous vector bundle $H \times \lambda E$ over $F = H/L$. In particular $H$ acts on the cohomology $H^{r,s}(F, E_0)$ "by left translations" and one now shows that $H^{r,s}(E_2)$ is associated with the
principal $H$-bundle $Z \to Y$ under this action of $H$:

$$H^{r,t}(E) = Z \times_{H} H^{r,t}(F, E) .$$

This yields:

**COROLLARY 2.9.** With the notations of Example (ii), for each $p \geq 0$ there is a spectral sequence $(pE_{r,t}^{s,t})$ which converges to $H^{r,s}(Z/L, E)$ and whose $E_{r,t}$-term is given by

$$pE_{r,t}^{s,t} = \bigoplus_{i} H^{r,s-t}(Z/H, Z \times_{H} H^{r,s-t+i}(H/L, E))$$

where $E_{o} = H \times L E$.

This corollary generalizes an earlier theorem of Bott [2] to the case $p \geq 0$. In [4], Fisher obtains a result similar to (2.8) and uses it in conjunction with (2.7) to generalize the classical Mumford-Matsushima vanishing theorem for line bundle cohomologies on a torus (cf. also [15], [18]).

**REMARK.** Given a diagram (2.2), the restrictions $\phi_{v,y} = \phi_{v}|X_{y}: X_{y} \to F$ and $\varphi_{v,y}: E|X_{y} \to E_{o}$ induce isomorphisms $H^{r,s}(F, E_{o}) \cong H^{r,s}(X_{y}, E|X_{y})$ in an obvious way and these isomorphisms yield a holomorphic trivialization of the cohomology bundle $H^{r,s}(E)$ over $U \subset Y$.

3. Remarks on a representation theorem of Wells and Wolf. In their paper [26], Wells and Wolf establish—among other things!—some conjectures of Griffiths ([6], [7]) on the geometric representation of certain automorphic cohomologies; cf. also [8], [22], [23], [25]. The framework is the following:

If $D$ is a period domain or, more generally, a flag domain and $Y \subset D$ is a maximal compact subvariety of dimension $s$ then there is a diagram

$$M_{D} \xleftarrow{\pi} \mathcal{V}_{D} \xrightarrow{\tau} D$$

where $\tau$ is holomorphic, $\pi: \mathcal{V}_{D} \to M_{D}$ is a holomorphic fibre bundle with fibre $Y$; $M_{D}$ is the space of linearly deformed compact subvarieties of dimension $s$. Wells and Wolf prove the (difficult!) result that $M_{D}$ is a Stein manifold provided that $D$ has compact isotropy, $D$ being a homogeneous space $D = G_{o}/V$, cf. below. They then establish their principal representation theorem: For non-degenerate homogeneous vector bundles $E_{1} = G_{o} \times \lambda$ over $D = G_{o}/V$, there exists a Fréchet injection

$$H^{o}(D, \mathcal{O}(E_{1})) \to H^{0}(M_{D}, R^{r}\pi_{*}(\mathcal{O}(\tau^{*}E_{1}))) .$$

In this assertion $\lambda$ is an irreducible unitary representation of $V$; cf.
The injection is $G_0$-equivariant and thus permits the representation of automorphic cohomology with respect to a discrete subgroup of $G_0$.

In this section we show that

$$\tau^*E \to \mathcal{Y}_D \to M_D$$

is, in fact, a BL-diagram; since the fibre $Y$ is compact this amounts to showing that $\tau^*E$ is locally trivial over $M_D$. We then indicate how to compute the cohomology bundles $\mathcal{H}^\cdot(\tau^*E)$. Furthermore, the direct image sheaf $\mathcal{R}\pi_*(\tau^*E)$ is locally free and coincides with $\mathcal{O}(\mathcal{H}^\cdot(\tau^*E))$; this yields an explicit description of the right-hand side of (3.2).

Some of the details are the following: $G$ is a connected complex semi-simple Lie group, $P \subset G$ a parabolic subgroup and $G_0$ a non-compact real form of $G$. We assume once and for all that $V = G_0 \cap P$ is compact.

If one chooses maximal compact subgroups $\tilde{M}, K$ of $G, G_0$, respectively, such that $V \subset K \subset \tilde{M}$, then $V = K \cap P = \tilde{M} \cap P$, the real orbit $G_0 \cdot 0$ of the neutral coset $0 \in G/P$ is open in the complex flag manifold $X = G/P$ and thus $D = G_0/V = G_0 \cdot 0$ inherits a complex structure. $\tilde{M}/V$ and $K/V$ also possess complex structures, being equal to $G/P$ and $K^c/K^c \cap P$.

Finally, if $\lambda: V \to GL(E)$ is an irreducible unitary representation, it extends uniquely to an irreducible holomorphic representation of $P$ and it follows that the homogeneous vector bundles $G_0 \times V E \to D, K \times V E \to K/V$ inherit holomorphic structures as holomorphic pull-backs from $G_0 \cdot 0$ and $K^c/K^c \cap P$.

We put $Y = K \cdot 0 \subset D, A = \{a \in G | aY \subset D \} = G_c(D)$ in the notations of [26] $L = \{a \in G | aY = Y \} \subset A$, a closed complex Lie subgroup of $G$, and we let $\sigma: G \to G/L, \beta: G \to G/L \cap P$ be the natural maps (which are holomorphic principal bundles). Now $A$ is open in $G$, $AL = A$; furthermore setting

$$M = M_D = \sigma A \subset G/L \quad (open)$$
$$\mathcal{Y} = \mathcal{Y}_D = \beta A \subset G/L \cap P \quad (open);$$

it is clear that e.g., $\sigma^{-1}(M) = A$ and we conclude that $\sigma | A: A \to M$ is a holomorphic principal $L$-bundle. Similarly, $\beta^{-1}(\mathcal{Y}) = A$ and $\beta | A: A \to Y$ is a holomorphic principal $(L \cap P)$-bundle. If $\varepsilon: G/L \cap P \to G/L$ is the natural fibration, $\varepsilon^{-1}(M) = \mathcal{Y}$ and the fibration $\varepsilon | \mathcal{Y}: \mathcal{Y} \to M$ is the linear deformation space of $Y$.

Setting $\tilde{A} = A/L \cap P$, let $\pi_\varepsilon: A \to \tilde{A}$ be the quotient map. It then
is clear that the map \( \beta a \to \pi a, a \in A \), identifies \( \tilde{A} \) and \( \mathcal{Y} \) and also that \( \pi_2: A \to \tilde{A} \) is a holomorphic principal \((L \cap P)\)-bundle. We are thus in the situation of Example (ii) of Section 2 (with \( H = L, L = L \cap P, Z = A, \) etc.) and any holomorphic representation \( \lambda \) of \( L \cap P \) on a finite-dimensional vector space \( E \) yields a BL-diagram

\[
(3.5) \quad \tilde{E}_1 \to \tilde{A} \to M
\]

where \( \pi: \tilde{A} \to M = A/L \) again is the natural map. If we set \( E_0 = L \times _{L \cap P} E \), then the cohomology bundles of (3.5) are given by

\[
H^r(\tilde{E}_2) = A \times _{L} H^r(L/L \cap P, E_0).
\]

In the applications \( \lambda \) will be the restriction to \( L \cap P \) of a holomorphic representation of \( P \).

By the very definition of \( A \) the natural map \( \tau: G/L \cap P \to X = G/P \) restricts to a map \( \tau: \mathcal{Y} \to D(\tau \beta a = a \cdot 0 \text{ for } a \in A) \). Let also \( i: D \to X \) be the inclusion. A direct, albeit somewhat lengthy computation then yields the following:

**Theorem 3.6.** Let \( \bar{\lambda} \) be a holomorphic representation of \( P \) on the finite dimensional vector space \( E \) and \( E_\bar{\lambda} = G \times _\bar{\lambda} E \) the corresponding homogeneous vector bundle over \( X = G/P \). Set \( \lambda = \bar{\lambda}|L \cap P \) and let \( \tilde{E}_1 \to A \) be the induced bundle. Then, under the bundle isomorphism of \( \varepsilon: \mathcal{Y} \to M \) onto \( \tilde{A} \to M \) mentioned above, the diagram

\[
(3.7) \quad \tau^* E_\bar{\lambda} \to \mathcal{Y} \to M
\]

is isomorphic to

\[
\tilde{E}_1 \to A \to M.
\]

In particular (3.7) is a BL-diagram (as claimed in (3.3)) and its cohomology bundle of type \((r, s)\) is given by

\[
H^r(\tau^* E_\bar{\lambda}) = A \times _{L} H^r(L/L \cap P, E_0)
\]

where \( E_0 = L \times _{L \cap P} E \).

One concludes that the \( E_r \)-term of the Leray spectral sequence of (3.7) is given by \( ^0E_r^{r+s} = H^r(\mathcal{Y}, A \times _{L} H^{r+s}(L/L \cap P, E_0)) \). Since we assume \( V \) to be compact, the main result of [26, Section 2.5] asserts that \( M \) is a Stein manifold; accordingly, the spectral sequence degenerates: \( ^0E_r^{r+s} = 0 \) for \( s > 0 \) and we see that

\[
H^q(\mathcal{Y}, \tau^* E_\bar{\lambda}) \cong H^q(M, A \times _{L} H^{r+s}(L/L \cap P, E_0))
\]

for \( q \geq 0 \).
Suppose, in particular, that $\bar{\chi}$ is the holomorphic extension to $P$ of an irreducible unitary representation of $V$ in $E$ and let $E_i = G_0 \times \tau E$ be the corresponding homogeneous bundle over $D$ with the holomorphic structure described earlier. Then if $E_i$ is non-degenerate in the sense of [26], the results of Schmid [21] imply that $H^*(D, E_i) = 0$ for $q \neq s = \dim Y$ and that the induced map

\[(3.10) \quad H^*(D, E_i) \to H^*(\mathcal{X}, \tau^* E_i)\]

is a Fréchet injection. Lastly, one has to argue that (3.9) is an isomorphism of Fréchet spaces (using the open mapping theorem as in [26]). (3.10) and (3.9) then imply the representation theorem (3.2).

As a by-product one obtains the following:

**Corollary 3.11.** Let $\tau^*: L \to H^*(L/L \cap P, E_0)$ be the representation of $L$ on $H^*(L/L \cap P, E_0)$ induced by left multiplication. Then the space $H^0(M_D, R^*\pi_*\mathcal{O}(\tau^* E_i))$ of (3.2) coincides with the space of all maps $f: A \to H^0(L/L \cap P, E_0)$ satisfying the conditions:

(i) $f$ is holomorphic
(ii) $f(\alpha l) = \pi^*(l^{-1})f(\alpha)$ for $(a, l) \in A \times L.$

4. **Discontinuous group actions and automorphic factors.** Let $E \to X \to Y$ be a BL-diagram and suppose that the group $\Gamma$ acts freely and properly discontinuously on $E, X$ and $Y$ such that $\pi: X \to Y$ and $\sigma: E \to X$ are equivariant and that the action on $E$ is fibrewise linear. We then show that $\Gamma \backslash E \to \Gamma \backslash X \to \Gamma \backslash Y$ again is a BL-diagram and we relate the cohomologies of the two diagrams. In the special case where $E$ is globally trivial over $Y$ (i.e., $E = X \times E_0, X = Y \times F$ in the earlier notations), the cohomology bundles of the quotient diagram are determined by an automorphic factor which we compute below; applications will follow in Section 5.

First of all, we recall some well-known results (which, in any case, are easily verified): Let $X$ be a complex manifold and $\Gamma$ a group acting on $X$, say on the left, by holomorphic maps: $\Gamma \times X \to X$ maps $(\gamma, x)$ to $\gamma x$ and $x \to \gamma x$ is holomorphic; the group $\Gamma$ is considered to be discrete. The action is properly discontinuous (p.d., for short) if for each compact set $K \subset X$, the set of $\gamma \in \Gamma$ with $\gamma K \cap K \neq \emptyset$ is finite. If $\Gamma$ acts freely and properly discontinuously, then the quotient $\Gamma \backslash X$ is a complex manifold in a natural way such that the quotient map $q: X \to \Gamma \backslash X$ is a holomorphic submersion (and is, in fact, biholomorphic).

Let $E \to \Gamma \backslash X$ be a holomorphic vector bundle with fibre $E$ such that
Let \( X \times E \cong q^* E \) and let \( \phi \) be a fixed such trivialization. Since \( (q^* E)_x = E_{q(x)} = (q^* E)_{q(x)} \), the trivialization induces the linear maps \( \phi_x \cdot \phi_x \) of \( E \), denoted by \( j(\gamma, x) \). Clearly \( j(\gamma, x) \in \text{GL}(E) \) and \( x \mapsto j(\gamma, x) \) is holomorphic. Moreover \( j(\gamma \delta, x) = j(\gamma, \delta x) \cdot j(\delta, x) \) for \( \gamma, \delta \in \Gamma \) and \( x \in X \): \( j \) is an automorphic factor \( \Gamma \times X \to \text{GL}(E) \). In turn \( j \) defines a left operation of \( \Gamma \) on \( X \times E \) by: \( \gamma \cdot (x, e) = (\gamma x, j(\gamma, x)e) \) and one shows that \( E \cong \Gamma \backslash (X \times E) \) as a vector bundle over \( \Gamma \backslash X \). The action of \( \Gamma \) on \( X \times E \) is automatically free and p.d. and we also denote \( \Gamma \backslash (X \times E) \) by \( E(j) \).

**Remarks.** Given the automorphic factor \( j: \Gamma \times X \to \text{GL}(E) \) and a holomorphic map \( h: X \to \text{GL}(E) \), \( j \cdot h(\gamma, x) = h(\gamma x) \cdot j(\gamma, x) \cdot h(x)^{-1} \) defines another automorphic factor and we see that \( E(j \cdot h) \cong E(j) \) and conversely.

The holomorphic sections of \( E(j) \) coincide with those holomorphic functions \( f: X \to E \) which satisfy \( f(\gamma x) = j(\gamma, x)f(x) \) for \( (\gamma, x) \in \Gamma \times X \) (= holomorphic automorphic forms).

One now obtains the following basic result:

**Theorem 4.1.** Let \( E \to X \to Y \) be a BL-diagram, \( \sigma: E \to X \) and \( \pi: X \to Y \) the projections. Suppose that the group \( \Gamma \) acts on the left on \( E, X \) and \( Y \) by holomorphic maps such that
(a) the actions are free and properly discontinuous;
(b) the maps \( \sigma, \pi \) are equivariant;
(c) the action on \( E \) is fibrewise linear.

Then there are induced maps \( \sigma: \Gamma \backslash E \to \Gamma \backslash X \) and \( \pi: \Gamma \backslash X \to \Gamma \backslash Y \) such that
\[
\Gamma \backslash E \to \Gamma \backslash X \to \Gamma \backslash Y
\]
is a BL-diagram. Moreover the cohomology bundles of the two diagrams are related by
\[
q^* H^{r,s}_{\ast}(\Gamma \backslash E) \cong H^{r,s}(E)
\]
with \( q: Y \to \Gamma \backslash Y \) the natural map.

In the proof one uses the following fact: each \( y \in Y \) has an open neighbourhood \( U \) such that \( \gamma U \cap U = \emptyset \) for \( \gamma \neq 1 \) and then \( U \to q(U) \) is biholomorphic. This shows, e.g., that \( \pi: \Gamma \backslash X \to \Gamma \backslash Y \) is a holomorphic fibre bundle with fibre \( F(= \text{fibre of } X \to Y) \). Similar arguments then imply that \( \Gamma \backslash E \) is a holomorphic vector bundle over \( \Gamma \backslash X \) with fibre \( E \), the fibre of \( E \) and that it is also locally trivial over \( \Gamma \backslash Y \) with fibre \( E_0 \). The verifications are straightforward and are omitted here.

Let \( p: X \to \Gamma \backslash X \) be the natural projection. Then \( p_* = p|_X \) maps
the fibre $X_y = \pi^{-1}(y)$ biholomorphically onto $\tilde{\pi}^{-1}(q(y)) \subseteq \Gamma \setminus X$ and is covered by a bundle isomorphism $\tilde{E}|_{X_y} \to \Gamma \setminus E|_{\tilde{\pi}^{-1}(q(y))}$; thus it induces an isomorphism $p^*_\gamma$ of $H^r(\Gamma \setminus E)_q(y)$ onto $H^r(\tilde{E})_\gamma$ since these simply are fibre cohomologies. The maps $p^*_\gamma$ yield the isomorphism (4.2).

Next we consider the case where the basic diagram (2.1) simply is $E = X \times E_0 \to Y \times F \to Y$ where $E_0$ is a holomorphic vector bundle; in other words $E$ is globally trivial over $Y$ with fibre $E_0$. In this case $H^r(\tilde{E})_\gamma$ is the trivial bundle $Y \times H^r(F, E_0)$. (4.2) therefore yields an isomorphism

(4.3) $\phi: Y \times H^r(F, E_0) \cong q^*H^r(\Gamma \setminus E)$.

Accordingly, there is an automorphic factor $j_\phi: \Gamma \times Y \to \text{GL}(H^r(F, E_0))$ such that $H^r(\Gamma \setminus E) \cong E(j_\phi)$ and the following theorem determines $j_\phi$:

Observe, firstly, that the action of $\Gamma$ on $X = Y \times F$ necessarily is of the form $\gamma \cdot (y, f) = (\gamma y, I(\gamma, y)f)$, $f \to I(\gamma, y)f$ holomorphic in $f$ (and also in $y$). By assumption $\Gamma$ acts on $E$ by bundle automorphisms covering this action on $Y \times F$ and this implies that the holomorphic automorphism $I(\gamma, y)$ of $F$ is covered by an automorphism $\tilde{I}(\gamma, y)$ of $E_0$; $y \to I(\gamma, y)$ still is holomorphic. Accordingly, there are induced automorphisms of the vector spaces $H^r(F, E_0)$, denoted by $I(\gamma, y)^*$.

With these notations:

**Theorem 4.4.** The automorphic factor $j_\phi$ derived from (4.3) is given by

(4.5) $j_\phi(\gamma, y) = (I(\gamma, y)^{-1})^* = I(\gamma^{-1}, y)^*$

for $(\gamma, y) \in \Gamma \times Y$. The cohomology bundles $H^r(F, E_0)$ are the trivial bundles $Y \times H^r(F, E_0)$ and

(4.6) $H^r(\Gamma \setminus E) = \Gamma \setminus (Y \times H^r(F, E_0))$

where $\Gamma$ acts on the product by $\gamma \cdot (y, h) = (\gamma y, I(\gamma^{-1}, \gamma y)^*h)$.

Once again the proof is straightforward and will not be reproduced here.

5. **Vanishing theorem for projective varieties** $\Gamma \setminus G_0\backslash T$. Let $G_0$ be a connected non-compact semi-simple Lie group admitting a faithful finite-dimensional representation; $G_0$ is a real form of a connected semi-simple complex Lie group $G$. We assume here that $G$ is *simply connected*.

Let $K \subseteq G_0$ be a maximal compact subgroup such that $G_0/K$ has a $G_0$-invariant complex structure (thus is a Hermitian symmetric space). Since $G_0$ and $K$ now have the same rank, we can choose a Cartan subgroup $T$ of $G_0$ such that $T \subseteq K$; $G, G_0$ and $K$ satisfy the assumptions of Section 3.
Let $g$, $f$ and $t$ be the complexifications of the Lie algebras $g_0$, $f_0$ and $t_0$ of $G_0$, $K$ and $T$, respectively, and for a Cartan decomposition $g_0 = f_0 \oplus p_0$, set $p = pf$; here $p_0 = f_0^c$ with respect to the Killing form $(, )$ of $g_0$. Let $A$ be the set of non-zero roots of $(g, t)$ and let $A_\pm, A_k$ be the sets of those roots $\alpha \in A$ whose root spaces $g_\alpha$ satisfy $g_\alpha \subset p$ respectively $g_\alpha \subset f$ (compact, non-compact roots). Choose a system of positive roots compatible with the complex structure of $G_0/K$, i.e., such that the following holds: If $A_\pm = A^+ \cap A$, and if $p = p^+ \oplus p^-$ is the splitting of the complexified tangent space at $0 \in G_0/K$ induced by the complex structure, then

$$p^\pm = \Sigma\{g_\alpha | \alpha \in A_\pm\}.$$  

The compatibility condition on $A^+$ may be rephrased as follows: Every non-compact root $\alpha \in A^+$ is totally positive: this means that if $\beta \in A_k$ is such that $\alpha + \beta \in A$, then in fact $\alpha + \beta \in A_k$. Equivalently one can say that $p^\pm$ are $K$-stable abelian subalgebras.

With $A'_\pm = A^+ \cap A_k$, set $b_\pm = \mathfrak{f} \oplus \mathfrak{g}(\pm \alpha | \alpha \in A'_\pm)$, $u = \mathfrak{f} \oplus \mathfrak{p}^-$, $b = \mathfrak{f} \oplus \mathfrak{g}(\pm \alpha | \alpha \in A^+)$, and let now $K^c$, $P^\pm$, $U, B_k$ and $B$ be the closed complex subgroups of $G$ corresponding to these Lie algebras. $B_k \subset K^c$ is a Borel subgroup such that $K \cap B_k = T$, $B_k = K^c \cap B$ and we set

$$F = K/T = K^c/B_k;$$

in the notations of Section 3, $V = T$ for the choice $P = B$. The following is fundamental:

**Theorem 5.3.** (Harish-Chandra [9], [19], [31]). The subgroups $K^c$, $P^\pm$ and $U$ are closed in $G$ and $P^\pm$ are simply connected. The exponential maps $p^\pm \to P^\pm$ are diffeomorphisms, $K^c$ normalizes $P^\pm$ and $U = K^cP^-$, a semi-direct product, is a parabolic subgroup of $G$ such that $G_0 \cap U = K$. The map $(x, k, y) \to (\exp x)k(\exp y)$ of $p^+ \times K^c \times p^-$ into $G$ is a biholomorphism onto a dense open subset $\Omega = P^+K^cP^-$ in $G$ containing $G_0$. Given $a \in \Omega$ let

$$a = a^+k(a)a^-$$

be the corresponding decomposition, $k(a) \in K^c$. In particular, $(ak)^+ = a^+$, $k(ak) = k(a)k$ for $a, k \in K$. Then the map $\zeta: \Omega \to p^+$ given by

$$\zeta(a) = \log(a^+)$$

induces a biholomorphism of $G_0/K$ onto $\zeta(G_0)$; $\zeta(G_0)$ is a bounded domain in $p^+$.  

Now set $Y = G_0/K$ and define $J: G_0 \times Y \to K^c$, following Satake [16], [20], by
\[ J(a, y) = k(a \exp \zeta(y)) \]

one has \( J(ab, y) = J(a, by)J(b, y) \) for \( a, b \in G_0 \) and letting \( 0 = 1K \) be the neutral coset, \( J(a, 0) = k(a) \), in particular: \( J(k, 0) = k \). \( J(a, y) \) is \( C^\infty \) in \( (a, y) \) and holomorphic in \( y \) and is called the canonical automorphic factor of \( Y \). If moreover \( \tau: K^c \to GL(E) \) is a holomorphic representation, we set \( j = \tau \circ J \) and obtain what is called the canonical automorphic factor of type \( \tau \) \((16)\).

With the notations introduced above, \( B \subset G \) is a Borel subgroup such that \( G_0 \cap B = T \); hence \( G_0/T \) inherits a complex structure as the open (real) orbit \( G_0 \cdot 0 \subset G/B \). Similarly, the complex structure of \( Y = G_0/K \) is the one of the orbit \( G_0 \cdot 0 \subset G/U \).

From \([10; \text{Lemma 2}]\), one obtains the following:

**Proposition 5.7.** The map \( \phi(aT) = (aK, J(a, 0)B) = (aK, k(a)B_k) \) of \( G_0/T \) onto \( Y \times F \) is biholomorphic and the action of \( G_0 \) on \( G_0/T \) transforms into the following action on \( Y \times F \):

\[ a(y, f) = (ay, J(a, y)f) \]

Since the argument in \([10]\) appears to be somewhat incomplete we include a proof of the assertion: \( \phi \) is injective since \( K \cap B_k = T \) and \( k(a)k = k(a)k \) for \( a \in G_0, k \in K \). Next, \( k(a)^{-1}kB_k \in F \) for \( a \in G_0, k \in K^c \) and so we can write \( k(a)^{-1}k = k_0b_0 \) with \( k_0 \in K, b_0 \in B_k \). With this \( \phi(ak_0T) = (ak, kB_k) \) and \( \phi \) is surjective. Using once more that \( J(a, 0)k_0 = k(a)k_0 = kb_0^{-1} \), one derives (5.8) by a direct computation. Note also that \( \phi \) certainly is \( C^\infty \).

Next, by the definition of the holomorphic structure of \( G_0/T \), \( \phi^{-1} \) will be holomorphic if and only if the composite map \( (aK, kB_k) \to ak_0B \in G_0 \cdot 0 \subset G/B \) is holomorphic. Since \( B_k \subset B \) and \( K^c \) normalizes \( P \subset B \), we have \( ak_0B = ak(a)^{-1}kB = a^+kB \) (cf. Theorem 5.3) and by (5.5), \( aK \to a^+ \) is holomorphic and, of course, so is \( kB_k \to kB \). Accordingly, \( (ak, kB_k) \to a^+kB \) is holomorphic and maps \( Y \times F \) to \( G_0 \cdot 0 \subset G/B \); hence \( \phi^{-1} \) is holomorphic. Thus \( \phi \) is a diffeomorphism such that \( \phi^{-1} \) is holomorphic and, therefore, \( \phi \) itself is holomorphic. This completes the argument.

Now we fix a \( C^\infty \) character \( \lambda \) of \( T \) and form the line bundle \( L_\lambda = G_0 \times T C \to G_0/T \); since \( \lambda \) extends uniquely to a holomorphic character of \( B \), \( L_\lambda \) has the structure of a holomorphic line bundle over \( G_0/T(\subset G/B) \). Also define \( E_0 = K^c \times b_k C \to F = K^c/B_k \). Then:

**Proposition 5.9.** Let again \( Y = G_0/K \). Then \( L_\lambda \to G_0/T \to Y \) is a BL-diagram with cohomology bundles \( H^{r,s}(L_\lambda) = Y \times H^{r,s}(F, E_0) \).

For the proof, observe first of all that the map \( \phi \) of Proposition 5.7
is a global trivialization of the holomorphic bundle $G_0/T \to Y$. We define a map $\psi$ from $L_1$ to $F \times E_0$ covering $\phi$ by

\[(5.10) \quad \psi([a, z]) = (aK, [k(a), z]) \]

for $(a, z) \in G_0 \times C$. Since $\lambda$ extends to $B_k$ and $k(at) = k(a)t$ for $a \in G_0$, $t \in T$, $\psi$ is well-defined. A simple verification shows that $\psi$ is a fibrewise linear bijection and it is obvious that $\psi$ covers $\phi$. There still remains to be shown that $\psi$ is holomorphic, in which case it will be a biholomorphic bundle isomorphism.

The point here is to show that $[a, z] \to [k(a), z]$ is holomorphic from $L_1$ to $E_0$ since $[a, z] \to aT \to aK$ clearly is holomorphic. Now the representation $\lambda$ extends up to $B$ and therefore $E_0$ is the bundle induced on $F$ by the bundle $G \times \_C \to G/B$ under the natural map $F = K^c/B_k \to G/B$. On the other hand, if $j: G_0/T \to G/B$ again is the natural map, the definition of $L_1$ shows that this bundle is holomorphically isomorphic to $j^*(G \times \_C)$; explicitly, these bundle isomorphisms are given by

\[i([k, z]) = (kB_k, [k, z]) , \quad (k, z) \in K^c \times C ,\]

for $E_0$, and

\[j([a, z]) = (aT, [a, z]) , \quad (a, z) \in G_0 \times C ,\]

for $L_1$.

Now the map $[a, z] \to k(a)B_k$ is the composition $[a, z] \to aT \to aK$ and thus is holomorphic. There remains the map $[a, z] \to [k(a), z]: P^- \subset [B, B]$ implies $\lambda(P^-) = 1$ and so in $G \times \_C$, one has $[k(a), z] = [(a^*)^{-1}a, z]$ where $a^+$ again is defined as in Theorem 5.3; by the same theorem, this is holomorphic in $[a, z]$ since it is holomorphic in $aK$. With this, the proposition is established.

**Corollary 5.11.** Under the isomorphism $\psi: L_1 \cong Y \times E_0$, the action of $G_0$ on $L_1$ transforms into the action $a \cdot (y, [k, z]) = (ay, [J(a, y)k, z]):$ $= (ay, J(a, y)[k, z])$ for $a \in G_0, y \in Y, (k, z) \in K^c \times C$.

In order to mention explicitly the representations involved in their construction, it will again be convenient to denote homogeneous bundles such as $L_1, E_0$, etc., by $K^c \times \_\lambda, G_0 \times \_\lambda$, etc.

Given $k \in K^c$, let $l_k$ denote left translation by $k$ in $F = K^c/B_k$ as well as, e.g., in $E_0$. With this, we set

\[(5.12) \quad I(a, y) = l_{J(a, y)}: F \to F; \quad \tilde{I}(a, y) = l_{J(a, y)}: E_0 \to E_0 \]

for $(a, y) \in G_0 \times Y$. It is clear that $\tilde{I}(a, y)$ is a bundle map over $I(a, y)$. If $l_k^*: H^r(F, E_0) \to H^r(F, E_0)$ is the induced action, then the representation
\(\pi^{r,s}\) of \(K^c\) in \(H^{r,s}(F, E_0)\) is given by \(\pi^{r,s}(k) = l_k^{r-1}\).

Recall that \(G_0\) acts on \(Y \times F\) by \(a(y, f) = (ay, I(a, y)f)\). Let now \(\Gamma\) be a discrete subgroup of \(G_0\) which acts freely on \(G_0/K = Y\). Then the action of \(G_0\) restricts to a free and p.d. action of \(\Gamma\) on \(Y \times F\) and the same holds for the action on \(Y \times E_0\); the projections \(Y \times E_0 \to Y \times F\) and \(Y \times F \to Y\) are \(\Gamma\)-equivariant. Thus, all the assumption of Theorem 4.4 are satisfied and, since \((I(\gamma, y)^{-1})^* = \pi^{r,s}(I(\gamma, y)) = j_{\nu,*,}(\gamma, y)\), one has:

**Theorem 5.13.** \(\Gamma \backslash L_1 \to \Gamma \backslash G_0/T \to \Gamma \backslash Y\) is a BL-diagram and its cohomology bundle of type \((r, s)\) is \(H^{r,s}(\Gamma \backslash L_1) = \Gamma \backslash (Y \times H_{r,s}(F, E_0))\) where \(\Gamma\) acts by \(\gamma \cdot (y, h) = (\gamma y, j_{\nu, *,}(\gamma, y)h)\).

An equivalent description of \(H^{r,s}(\Gamma \backslash L_1)\) may be obtained as follows: Suppose that \(\tau: K \to \text{GL}(E)\) is a finite dimensional representation of \(K\) in the complex vector space \(E\); \(\tau\) extends holomorphically to \(K^c\) and then to \(U = K^cP^\infty\) by requiring that \(\tau|P^\infty = 1\). Using [16], [17] and [19], one concludes that the bundles \(E(j_{\nu,1}, Y)\) and \(\Gamma \backslash (G \times \nu^\tau)|Y\) are holomorphically equivalent where the restriction to \(Y\) of \(G \times \nu\tau\) is taken with respect to the Borel embedding \(Y = G_0/K \to G_0\cdot 1U \subset G/U\). With this we have:

**Corollary 5.14.** \(H^{r,s}(\Gamma \backslash L_1) = \Gamma \backslash (G \times \nu^{\pi^{r,s}})|Y\).

Applying Theorem 2.6, we obtain:

**Corollary 5.15.** Under the assumptions of Theorem 5.13 there is for each \(p \geq 0\) a spectral sequence \((^pE_\tau^{*,*})\) which converges to \(H^{r,s}(\Gamma \backslash G_0/T, \Gamma \backslash L_1)\) and whose \(E_\tau\)-term is

\[
^pE_\tau^{*,*} = \bigoplus Y H^{r,s-\tau}(\Gamma \backslash Y, \Gamma \backslash (G \times \nu^{\pi^{r,s-\tau}})|Y).
\]

In particular \(^pE_\tau^{*,*} = H^{*,*}(\Gamma \backslash Y, \Gamma \backslash (G \times \nu^{\pi^{*,*}})|Y)\).

Next, the Borel-Weil theorem [11] implies that the representations \(\pi^{*,*}\) vanish for all \(t\) except \(t = q_0\), an integer determined by \(\lambda\) and described in detail below. Thus we conclude:

**Corollary 5.16.**

\[
H^{0,*}(\Gamma \backslash G_0/T, \Gamma \backslash L_1) = H^{0,*-q_0}(\Gamma \backslash Y, \Gamma \backslash (G \times \nu^{0,q_0})|Y)
\]

for every \(q_0\); cf. also (5.18) below.

This result was first established by Ise [10, Proposition 8] under the additional assumption that \(\Gamma \backslash Y\) is compact; we do not require this restriction here.

Next we investigate when the spaces in Corollary 5.16 vanish: Identify \(\lambda\) with an integral element \(\lambda\) in the dual \(t^*\) of \(t\), i.e., a linear
form $\lambda$ such that 

$$\frac{2(\lambda, \alpha)}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}, \quad \alpha \in \Delta;$$

recall that $(\cdot, \cdot)$ denotes the Killing form of $\mathfrak{g}$. Also set $2\delta = \Sigma_{\Delta^+} \alpha$, $2\delta_k = \Sigma_{\Delta^+_k} \alpha$, $2\delta_s = \Sigma_{\Delta^+_s} \alpha = 2\delta - 2\delta_k$ and let $W_k$ be the subgroup of the Weyl group $W$ of $(\mathfrak{g}, \mathfrak{t})$ generated by the compact root reflections. Since $\Delta^+$ is compatible with the complex structure of $G_{/\mathbb{C}}$, one knows that $wA^+_s = A^+_w$ for $w \in W_k$ and $(\delta_s, \alpha) = 0$ for $\alpha \in \Delta^+_s$. A linear form $\gamma \in \mathfrak{t}^*$ is said to be $\Delta$-regular ($\Delta_k$-regular) if $(\gamma, \alpha) \neq 0$ for $\alpha \in \Delta (\alpha \in \Delta_k)$. With this, we define $F_{\Delta_k} \subset \mathfrak{t}^*$ and $P^{(\Delta)} \subset \mathfrak{t}$ as follows: $A \in F_{\Delta_k}$ if and only if $A$ is integral, $A + \delta$ is $\Delta$-regular and

$$(5.17) \quad (A + \delta, \alpha) > 0 \text{ for } \alpha \in \Delta_k^+; \quad \alpha \in P^{(\Delta)} \text{ for } \alpha \in \Delta \text{ if and only if } (A + \delta, \alpha) > 0 \quad \text{whenever } A + \delta \text{ is } \Delta \text{-regular.}$$

Thus $P^{(\Delta)}$ is a system of positive roots corresponding to the $\Delta$-regular element $A + \delta$. The Borel-Weil theorem states that $\pi^{\alpha, t}$ vanishes for all $t$ if there is $\alpha \in \Delta_k^+$ such that $(\lambda + \delta_k, \alpha) = 0$; if this is not the case, then $\lambda + \delta_k$ is $\Delta_k^+$-regular and the value of $q_0$ in Corollary 5.16 is

$$(5.18) \quad q_0 = |\{ \alpha \in \Delta_k | (\lambda + \delta_k, \alpha) < 0 \}| = |w(-\Delta_k) \cap \Delta_k^+|$$

where $w \in W_k$ is the unique element such that $(w(\lambda + \delta_k), \alpha) > 0$ for every $\alpha \in \Delta_k^+$ and where $|s|$ denotes the cardinality of the set $s$.

Moreover $\pi^{\alpha, t_0}$ is an irreducible representation of $K$ with $\Delta_k^+$-highest weight

$$(5.19) \quad \tau(\lambda, w) = w(\lambda + \delta_k) - \delta_k.$$

With the above choice of $w$ it is a straightforward computation to prove:

**Proposition 5.20.** $\tau(\lambda, w) \in F'_0$ if and only if $\lambda + \delta$ is $\Delta$-regular. In this case $P^{(\tau(\lambda, w))} = wP^{(\Delta)}$.

At this point we nearly are in a position to apply some results of [27] and [28] to obtain vanishing theorems for the spaces $H^0(\Gamma \backslash G_{/\mathbb{C}}, T, \Gamma \backslash L_\Delta)$; however some additional notation will be needed:

Let $A$ be integral and such that $A + \delta$ is $\Delta$-regular. Given $(w_1, \tau)$ in $W \times W_k$, we define:

$$(5.21) \quad Q_A = \{ \alpha \in \Delta_k^+ | (A + \delta, \alpha) > 0 \}, \quad P_{w_1}^{(\Delta)} = P^{(\Delta)} \cap \Delta_s,$$

$$2\delta^{(\Delta)} = \Sigma \{ \alpha | \alpha \in P^{(\Delta)} \},$$

$$\Phi_{w_1}^{(\Delta)} = w_1(-P^{(\Delta)} \cap P^{(\Delta)}).$$
\[ \Phi^\alpha_\delta = \tau(-A^\alpha_\delta) \cap A^\alpha_\delta \cdot \]
\[ A_{A^\alpha_\delta, w_1} = \{ \alpha \in P_\alpha \mid w_1^{-1} \tau \alpha \in -P(\delta) \}. \]

Assume now that \( \Gamma \setminus Y \) is compact. In this case, the main theorem [28, Theorem 4.3] applies to the right-hand side of Corollary 5.16. Among other things this theorem states that if \( \pi_d \) is an irreducible \( K \)-module with \( A^\alpha_\delta \)-highest weight \( A \in F^\alpha \) and if \( H^{0,\alpha}(\Gamma \setminus Y, \Gamma \setminus (G \times \tau \pi_d)) \mid Y \neq 0 \), then there is a pair \((w, \tau) \in W \times W_k\) such that

(5.22) \[ q = |A_{A^\alpha_\delta, w_1}| - 2 |Q_d \cap A_{A^\alpha_\delta, w_1}| + |Q_d|. \]

One has \( A^\alpha_\delta \subset w_1 P(\alpha) \), \( A_{A^\alpha_\delta, w_1} = \Phi^\alpha_\delta - \Phi^\alpha_\delta \) and \( \tau(\delta + \delta - \delta(\alpha)) = w_1(A + \delta - \delta(\alpha)) \); also, \( A_{A^\alpha_\delta, w_1}, \Phi^\alpha_\delta \) and \( \{ \alpha \in P(\alpha) \mid \tau \alpha \in -P(\alpha) \} \) are contained in \( \{ \alpha \in P(\alpha) \mid (A + \delta - \delta(\alpha), \alpha) = 0 \} \), with \( \Phi^\alpha_{\delta-1} \subset \{ \alpha \in A^\alpha_\delta \mid (A + \delta - \delta(\alpha), \alpha) = 0 \} \).

We now assume that \( \lambda \in \mathfrak{t}^* \) is integral and such that \( \lambda + \delta \) is \( \Delta \)-regular; one notes that \( \lambda + \delta \) is \( A^\alpha_\delta \)-regular, so that the Borel-Weil theorem gives the highest weight \( \tau(\lambda, w) \) of (5.19) and Proposition 5.20 yields \( \pi^\lambda = wP(2) \). One concludes that \( P^\lambda = wP(2) \) and hence that

(5.23) \[ A_{\tau(\lambda, w), \tau w} = wA_{\lambda, \tau w}. \]

Similar arguments show that \( Q_{\tau(\lambda, w)} = wQ_\lambda \), \( \tau(\lambda, w) + \delta - \delta(\tau(\lambda, w)) = w(\lambda + \delta - \delta(\lambda)), \Phi^\lambda(\tau(\lambda, w)) = w_1 w(-P(2)) \cap wP(2) \). Hence Corollary 5.16 and the equation (5.22) yield:

THEOREM 5.24. Let \( \lambda \in \mathfrak{t}^* \) be integral, \( L_1 \to G_0/T \) the corresponding holomorphic line bundle. Suppose that the discrete subgroup \( \Gamma \subset G_0 \) acts freely on \( Y = G_0/K \) such that \( \Gamma \setminus Y \) is compact. If \( \lambda + \delta \) is not \( \Delta \)-regular, then \( H^{0,\lambda}(\Gamma \setminus G_0/T, \Gamma \setminus L_1) = 0 \) for every \( q \). On the other hand if \( \lambda + \delta \) is \( \Delta \)-regular then \( \lambda + \delta \) is \( A^\alpha_\delta \)-regular and there is a unique element \( w \in W_k \) such that \( \tau(\lambda + \delta), \alpha > 0 \) for every \( \alpha \in A^\alpha_\delta \). Then for every \( q \)

\[ H^{0,\lambda}(\Gamma \setminus G_0/T, \Gamma \setminus L_1) = H^{0,\lambda-q_0}(\Gamma \setminus Y, \Gamma \setminus (G \times \tau \pi_{\lambda-q_0}) \mid Y) \]

where \( \pi_{\lambda-q_0} \) is the representation of \( K \) with \( A^\alpha_\delta \)-highest weight \( w(\lambda + \delta) - \delta \) and \( q_0 \) is given by (5.18). If \( H^{0,\lambda}(\Gamma \setminus G_0/T, \Gamma \setminus L_1) \neq 0 \) there is a pair \((w, \tau) \in W \times W_k\) such that the following hold:

(i) \[ q = q_0 + |A_{\lambda, \tau w, w_1}| - 2 |Q_\lambda \cap A_{\lambda, \tau w, w_1}| + |Q_\lambda|; \]

(ii) \[ A^\alpha_\delta \subset w_1 wP(2), wA_{\lambda, \tau w, w_1} = \tau^{-1} w_1 (-P(2)) \cap (wP(2) - \Phi^\alpha_{\delta-1}), \tau w(\lambda + \delta - \delta(2)) = w_1 w(\lambda + \delta - \delta(2)) = w(\lambda + \delta - \delta(2)), \Phi^\alpha_{\delta-1} \subset \{ \alpha \in A^\alpha_\delta \mid (w(\lambda + \delta - \delta(2), \alpha) = 0) \}; \]

(iii) \( wA_{\lambda, \tau w, w_1}, w_1 w(-P(2) \cap wP(2)) \) and \( \{ \alpha \in wP(2) \mid \tau \alpha \in -wP(2) \} \) are contained in \( \{ \alpha \in P(2) \mid (w(\lambda + \delta - \delta(2), \alpha) = 0) \}. \)
Because of its generality this theorem—like [28, Theorem 4.3]—has several corollaries of which we mention the following:

Firstly, assume that \((\lambda + \delta - \delta^{(\alpha)}) \neq 0\) for every \(\alpha \in P^{(\lambda)}_n\). By (iii), one has \(A_{1_{\text{m, w}}, w} = \emptyset\) and so (i) gives \(q = q_0 + |Q_1|\):

**Corollary 5.25.** If \(\lambda\) is integral, \(\lambda + \delta\) is \(\Delta\)-regular and \((\lambda + \delta - \delta^{(\alpha)})\) is \(\neq 0\) for every \(\alpha \in P^{(\lambda)}_n\), then \(H^0,0(\Gamma \setminus G_0/T, \Gamma \setminus L_2) = 0\) for \(q \neq q_0 + |Q_1|\).

Next suppose that \(\lambda\) is \(\Delta^+_n\)-dominant. Then we must have \(w = 1\) and so \(q_0 = 0\). If moreover \((\lambda + 2\delta, \alpha) < 0\) for \(\alpha \in \Delta^+_n\), one finds that \(Q_1 = \emptyset\) and \((\lambda + \delta - \delta^{(\alpha)})\) is \(< 0\) for \(\alpha \in \Delta^+_n\); this yields the following known result:

**Corollary 5.26.** If \(\lambda\) is \(\Delta^+_n\)-dominant integral and \((\lambda + 2\delta, \alpha) < 0\) for \(\alpha \in \Delta^+_n\), then \(H^0,0(\Gamma \setminus G_0/T, \Gamma \setminus L_2) = 0\) for \(q \neq 0\).

This result can also be obtained directly from the Kodaira vanishing theorem. Another specialization of \(\lambda\) leads to the following result:

**Theorem 5.27.** Let \(\lambda\) be integral such that \(\lambda + \delta\) is \(\Delta\)-regular and suppose that \(P^{(\lambda)}\) is compatible with an invariant complex structure on \(Y = G_0/K\) (cf. the beginning of this section). If \(H^0,0(\Gamma \setminus G_0/T, \Gamma \setminus L_2) \neq 0\), there exists a parabolic subalgebra \(\theta = r \oplus u\) of \(g\), \(r\) the reductive and \(u\) the unipotent part of \(\theta\), such that if \(\theta_{\alpha, n}\) denotes the set of non-compact roots in \(u\) and \(\Delta(r)\) the set of all roots in \(r\), then

(i) \(q = q_0 + 2|\theta_{\alpha, n} \cap wQ| + |\Delta^+_n - wQ| - |\theta_{\alpha, n}|\) with \(q_0, w\) as in Theorem 5.24;

(ii) \(\theta\) contains the Borel subalgebra \(t + \Sigma(g_\alpha | \alpha \in wP^{(\lambda)})\);

(iii) \((w(\lambda + \delta - \delta^{(\alpha)}), \alpha) = 0\) for \(\alpha \in \Delta(r)\).

The result follows from Proposition 5.20, the calculation in (5.23), and [27, Theorems 5.24 and 2.3], once one observes that since \(P^{(r(1, w))} = wP^{(\lambda)}\) and \(w \in W_s\), every non-compact root in \(P^{(r(1, w))}\) actually is totally positive.

A very simple application of Theorem 5.27 is the following: Assume that \(\lambda\) actually is \(\Delta^+\)-dominant. Then \(P^{(\lambda)} = \Delta^+\) (so that every non-compact root in \(P^{(\lambda)}\) is totally positive), \(\delta^{(1)} = \delta, w = 1, q_0 = 0, Q_1 = \Delta^+_n, \theta_{\alpha, n} = wP^{(\lambda)} - \Delta(r) = \Delta^+_n - \Delta(r) \subset \Delta^+_n\); hence by (i) of Theorem 5.27, \(q = 2|\theta_{\alpha, n}| - |\theta_{\alpha, n}| = |\theta_{\alpha, n}|\).

**Corollary 5.28.** If \(\lambda\) is \(\Delta^+\)-dominant integral and if \(H^0,0(\Gamma \setminus G_0/T, \Gamma \setminus L_2) \neq 0\), then \(q = |\theta_{\alpha, n}|\) for some parabolic subalgebra \(\theta = r \oplus u \subset g\) containing \(t + \Sigma \delta^+_n g_\alpha\).

Moreover \((\lambda, \Delta(r)) = 0\). If \(G_0\) is simple then the set of numbers \(|\theta_{\alpha, n}|\)
for θ such that θ ⩾ t + Σ₄g₄ is determined completely in [27, Table 3.4].

In particular $H^q(G_0/T, L_i) = 0$ for $q < \{[\alpha \in \mathcal{A} \mid (\lambda, \alpha) > 0]\}$. We conclude with some (more or less known) remarks about the cohomology of $G_0/T$:

By Proposition 5.9 and the fact that the spectral sequence $^*E^{*,*}$ degenerates since $Y = G_0/K$ is Stein, we obtain:

**Theorem 5.29.** With the above notations, for any integral $\lambda$ and all $q \geq 0$

1. $H^q(G_0/T, L_i) \cong H^q(Y, Y \times H^q(F, E_0)) \cong H^q(Y) \otimes H^q(F, E_0)$.

Hence if there is $\alpha \in \mathcal{A}$ such that $(\lambda + \delta, \alpha) = 0$ then by the Borel-Weil theorem $H^q(G_0/T, L_i) = 0$ for all $q$. If $\lambda + \delta$ is $\mathcal{A}$-regular let $w, q_0$ be as in Theorem 5.24. Then $H^q(G_0/T, L_i) = 0$ for $q \neq q_0$ and $H^q(G_0/T, L_i) \cong H^q(Y) \otimes H^q(F, E_0)$ where $H^q(F, E_0)$ is an irreducible $K$-module with $\mathcal{A}$-highest weight $w(\lambda + \delta) - \delta$.

**Corollary 5.30.** In particular suppose that $(\lambda + \delta, \alpha) < 0$ for $\alpha \in \mathcal{A}$. Then $q_0 = |\mathcal{A}| = s = \dim_c K/T$ and hence $H^q(G_0/T, L_i) = 0$ for $q \neq s$.

Equation (i) of Theorem 5.29 may be regarded as a Hermitian version of the representation formula (3.2): in the present situation the fibration $\mathcal{D}_n \to M_n$ of (3.1) collapses to $G_0/T \to G_0/K$ by [26, Proposition 2.4.7].

**References**


