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THE UNIRATIONALITY OF CERTAIN ELLIPTIC SURFACES IN CHARACTERISTIC p

TOSHIYUKI KATSURA*

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0. Introduction. Let X be a non-singular projective surface defined over an algebraically closed field k of characteristic $p \geqq 5$. X is called unirational if there exists a generically surjective rational mapping from the projective plane $\boldsymbol{P}^2$ to S. Since Zariski found an example of irrational unirational algebraic surfaces in positive characteristics (cf. Zariski [26]), many such surfaces were found and investigated by Artin (cf. [1]), Shioda (cf. [21], [22], [23] and [24]), Miyanishi (cf. [9], [10] and [11]), Rudakov and Shafarevich (cf. [16]), Blass (cf. [2]) and others. In the previous note [6], we introduced the notion of a unirational elliptic surface of base change type, and characterized irrational unirational elliptic surfaces of base change type with sections. In that case, we considered two classes of elliptic surfaces defined by the following equations:

$$\text{(I)} \qquad y^2 = 4x^3 - t^3(t-1)^3(t-\alpha)^3 x\ ,$$

$$\text{(II)} \qquad y^2 = 4x^3 - t^5(t-1)^5(t-\alpha)^5(t-\beta)^5(t-\gamma)^5\ ,$$

where t is a local coordinate of an affine line in $\boldsymbol{P}^1$, and α, β, γ are arbitrary elements of the field k. We proved the following theorem.

THEOREM I. *The elliptic surface defined by the equation* (I) *is unirational if and only if* $p \equiv 3 \bmod 4$.

Moreover, we proved that if $p \equiv 2 \bmod 3$, then the elliptic surface defined by the equation (II) is unirational. In this note, we prove the following theorem which we conjectured in [4].

THEOREM II. *The elliptic surface defined by the equation* (II) *is unirational if and only if* $p \equiv 2 \bmod 3$.

Next, we denote by E_t the elliptic curve over $k(t)$ defined by the equation (I) (resp. (II)), and s the number of points where E_t has bad reduction. We denote by $r(E_t)$ the rank of the abelian group of rational points over $k(t)$ of E_t. Using these notations, as a corollary to Theorems I

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and II, we will show the following:

(i) $r(E_t) = 0$ if $p \equiv 1 \bmod 4$ (resp. $p \equiv 1 \bmod 3$),

(ii) $r(E_t) = 2s - 4$ if $p \equiv 3 \bmod 4$ (resp. $p \equiv 2 \bmod 3$).

To prove Theorem II, we examine the eigenvalues of the Frobenius mapping on the second l-adic étale cohomology group. We will show that some eigenvalues are not powers of p if $p \equiv 1 \bmod 3$. By this fact, we conclude the non-unirationality. This method comes from Shioda [21] and [22]. As for the ranks of the elliptic curves over $k(t)$, phenomenon similar to the above can be found in Shioda [18] and [19].

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1. Notations and some lemmas. Throughout this paper, we fix an algebraically closed field k of characteristic $p > 0$. Let X be a projective variety of dimension n defined over k. Then, we use the following notations:

$k(X)$: the function field of X,
$H^i(X, O_X)$: the i-th cohomology group of the structure sheaf O_X of X,
$\chi(O_X) = \sum_{i=0}^{n} (-1)^i \dim_k H^i(X, O_X)$,
$\rho(X)$: the Picard number of X,
$\bar{F}_p$: an algebraic closure of the finite field F_p with p elements,
$\bar{Q}_l$: an algebraic closure of the l-adic number field Q_l for a prime l different from p,
$H^i(X, Q_l)$: the i-th l-adic étale cohomology group of X,
$H^i(X, \bar{Q}_l) = H^i(X, Q_l) \otimes_{Q_l} \bar{Q}_l$,
$B_i(X) = \dim_{Q_l} H^i(X, Q_l)$: the i-th Betti number of X,
$c_2(X)$: the second Chern number of X.

Let G be a finite subgroup of the automorphism group of X. We denote by X/G the quotient variety of X by G.

LEMMA 1.1 (Harder and Narashimhan [5]). *Let l be a prime number which is prime to both p and the order of G. Then, under the above notations, $H^i(X/G, Q_l)$ is isomorphic to the subspace $H^i(X, Q_l)^G$ of G-invariants in $H^i(X, Q_l)$:*

$$H^i(X/G, Q_l) \cong H^i(X, Q_l)^G . \tag{1.1}$$

For the proof, see [5, Proposition 3.2.1].

Now, let S be a non-singular projective surface defined over k.

DEFINITION 1.2. $\lambda(S) = B_2(S) - \rho(S)$ is called the Lefschetz number of S.

It is easy to see that $\lambda(S)$ is a birational invariant of S (cf. Shioda [21]). To make clear the situation in characteristic p, we formulate the following lemma which is essentially stated in Brieskorn [3].

LEMMA 1.3. *Let σ be an automorphism of order n of S, and G the cyclic group generated by σ. Assume that G has only a finite number of fixed points and that n is prime to p. Let P be a fixed point of σ, O the local ring of S at P, and σ^* the action of σ on O. Then, there exists a regular system of parameters (u, v) of O such that*

$$\sigma^*(u) = \eta u , \qquad \sigma^*(v) = \eta' v , \tag{1.2}$$

where η and η' are some primitive n-th roots of unity.

PROOF. We denote by m the maximal ideal of O. σ^* acts on the vector space m/m^2. Since n is prime to p, we can find a regular system of parameters (x, y) of O such that

$$\sigma^*(x) \equiv \eta x \bmod m^2 , \qquad \sigma^*(y) \equiv \eta' y \bmod m^2 . \tag{1.3}$$

We set

$$\begin{cases} u = x + \eta^{-1}\sigma^* x + \eta^{-2}\sigma^{*2}x + \cdots + \eta^{-n+1}\sigma^{*n-1}x , \\ v = y + \eta'^{-1}\sigma^* y + \eta'^{-2}\sigma^{*2}y + \cdots + \eta'^{-n+1}\sigma^{*n-1}y . \end{cases} \tag{1.4}$$

Then we have

$$\begin{cases} u \equiv nx \bmod m^2 , & \sigma^* u = \eta u , \\ v \equiv ny \bmod m^2 , & \sigma^* v = \eta' v . \end{cases} \tag{1.5}$$

Since the number of fixed points of G are finite, both η and η' must be primitive n-th roots of unity. q.e.d.

Let $\pi\colon \tilde{S} \to S/G$ be the minimal resolution of singularities of S/G. Then we have a natural mapping

$$\pi^*\colon H^2(S/G, Q_l) \to H^2(\tilde{S}, Q_l) . \tag{1.6}$$

LEMMA 1.4. *Under the same assumptions as in Lemma 1.2, π^* is an injective homomorphism.*

PROOF. Let D be an exceptional divisor on $\tilde{S}$ which is obtained by the minimal resolution of a singular point on S/G. Then, using Lemma 1.3, we can show that D is a connected divisor composed of rational curves without loops (cf. Brieskorn [3]). So, we have

$$H^1(D, Q_l) = 0 \tag{1.7}$$

(cf. Milne [8, Chapter III, Remark 1.26]). Therefore, by the same argument as in [6, (5.24)], we can complete our proof. q.e.d.

2. The K/k-trace. Let $f: S \to C$ be a relatively minimal elliptic surface over a non-singular projective curve C defined over k. The results in this section may be well-known. But as we cannot find suitable references, we will give their proofs.

LEMMA 2.1. *If f has at least one singular fiber composed of rational curves, then the Albanese variety* $\mathrm{Alb}(S)$ *of S is isomorphic to the Jacobian variety $J(C)$ of C.*

PROOF. Let P be a point on C such that $f^{-1}(P)$ is composed of rational curves. After choosing base points on S and C, we have the following commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & \mathrm{Alb}(S) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \theta} \\ C & \xrightarrow{\psi} & J(C)\,, \end{array} \tag{2.1}$$

where φ and ψ are morphisms such that the induced morphism θ is a homomorphism.

If $\dim \varphi(S) = 0$, then we have $\dim \mathrm{Alb}(S) = \dim J(C) = 0$, because $\varphi(S)$ (resp. $\psi \circ f(S)$) must generate $\mathrm{Alb}(S)$ (resp. $J(S)$). So, we assume $\dim \varphi(S) \geqq 1$. Suppose that there exists a fiber $f^{-1}(Q)$ with $Q \in C$ such that $\varphi(f^{-1}(Q))$ is a curve. Then, there exists a hyperplane section H of $\mathrm{Alb}(S)$ such that H intersects $\varphi(f^{-1}(Q))$ and H does not contain the point $\varphi(f^{-1}(P))$. So, the effective divisor $\varphi^{-1}(H)$ intersects $f^{-1}(Q)$ and does not intersect $f^{-1}(P)$, which contradicts the fact that $f^{-1}(P)$ and $f^{-1}(Q)$ are numerically equivalent to each other. So, under our assumption, for any point Q on C, $\varphi(f^{-1}(Q))$ must be a point. Therefore, there exists a morphism from C to $\varphi(S)$ by the uniqueness of the Stein factorization. Therefore, $\mathrm{Alb}(S)$ is isomorphic to $J(C)$ by the universality of $(\mathrm{Alb}(S), \varphi)$ and $(J(C), \psi)$. q.e.d.

We denote by K the function field $k(C)$ of the curve C, and by E_t the generic fiber of f.

PROPOSITION 2.2. *Let $f: S \to C$ be a relatively minimal elliptic surface with sections. If f has at least one singular fiber composed of* *rational curves, then the K/k-trace of the generic fiber E_t is one point.*

PROOF. Let E be the K/k-trace of E_t (cf. Lang [7]). Suppose $\dim_k E = 1$. Then E is an elliptic curve defined over k, and there exists an isogeny g defined over K from E to E_t. Since E (resp. E_t) has a rational point over k (resp. over K), we have the canonical isomorphism $E \cong \mathrm{Pic}^\circ(E)$ (resp. $E_t \cong \mathrm{Pic}^\circ(E_t)$). So, considering the dual of g, we have an isogeny:

$$\hat{g}\colon E_t \to E \tag{2.2}$$

defined over K. This means that there exists a generically surjective rational mapping h from S to $E \times C$. Therefore, we have a surjective homomorphism from $\mathrm{Alb}(S)$ to $\mathrm{Alb}(E \times C)$ by the universality of Albanese varieties. On the other hand, by the assumption on S, we have $\mathrm{Alb}(S) \cong J(C)$ by Lemma 2.1, which contradicts $\dim \mathrm{Alb}(S) \geqq \dim \mathrm{Alb}(E \times C)$. q.e.d.

Let P_i $(i = 1, \cdots, s)$ be all points such that $f^{-1}(P_i)$ are singular fibers of f. Let m_i be the number of components of $f^{-1}(P_i)$, and $r(E_t)$ the rank of the abelian group of rational points over K of the generic fiber E_t. Using Lemma 2.2, we have the following:

LEMMA 2.3 (Ogg [14] and Shafarevich [17]). *Let $f\colon S \to C$ be a relatively minimal elliptic surface with sections. Assume that f has at least one singular fiber composed of rational curves. Then, the Picard number $\rho(S)$ of S is given by*

$$\rho(S) = r(E_t) + 2 + \sum_{i=1}^{s} (m_i - 1) . \tag{2.3}$$

3. Representation of a cyclic group on étale cohomology groups. In this section, we assume $\mathrm{char.}\, k = p \geqq 5$, and we consider the nonsingular projective model C of the curve defined by the equation:

$$z^6 = f_1(t)f_2(t)^2f_3(t)^3f_4(t)^4f_5(t)^5 , \tag{3.1}$$

where t and z are variables, $f_i(t)$ $(i = 1, \cdots, 5)$ are polynomials of t which have only simple zeros and which are prime to each other. We set

$$n_i = \text{degree of } f_i(t) \qquad (i = 1, \cdots, 5) . \tag{3.2}$$

We assume that among the n_i's there exists a relation:

$$n_1 + 2n_2 + 3n_3 + 4n_4 + 5n_5 = 6n , \tag{3.3}$$

where n is a suitable positive integer. Considering C as a covering of

P^1 of degree six, and using Hurwitz's theorem, we can calculate the genus $g(C)$ of C as follows:

$$g(C) = 3n + 2n_1 + n_2 - 5 . \tag{3.4}$$

Now, we consider the automorphism σ of C defined by

$$\sigma : t \to t , \qquad z \to \zeta z , \tag{3.5}$$

where ζ is a primitive sixth root of unity. We denote by $\langle\sigma\rangle$ the group generated by σ, and by σ^* the action induced by σ on $H^1(C, Q_l)$.

PROPOSITION 3.1. *The eigenvalues of σ^* on $H^1(C, Q_l)$ are given by ζ^j $(j = 1, \cdots, 5)$ with multiplicities*

$$\begin{aligned} n_1 + n_2 + n_3 + n_4 + n_5 - 2 \quad & \textit{if} \quad j = 1 \textit{ or } 5 , \\ n_1 + n_2 + n_4 + n_5 - 2 \quad & \textit{if} \quad j = 2 \textit{ or } 4 , \\ n_1 + n_3 + n_5 - 2 \quad & \textit{if} \quad j = 3 . \end{aligned}$$

PROOF. We consider the quotient curves C_α of C by the groups $\langle\sigma^\alpha\rangle$ $(\alpha = 2 \text{ or } 3)$:

$$C_\alpha = C/\langle\sigma^\alpha\rangle \qquad (\alpha = 2, 3) . \tag{3.6}$$

We have natural morphisms:

$$\pi_\alpha : C \to C_\alpha \qquad (\alpha = 2, 3) . \tag{3.7}$$

By Hurwitz's theorem, we can calculate the genus $g(C_\alpha)$ of C_α as follows:

$$g(C_2) = (n_1 + n_3 + n_5 - 2)/2 , \tag{3.8}$$

$$g(C_3) = n_1 + n_2 + n_4 + n_5 - 2 . \tag{3.9}$$

We denote by σ_α the automorphism induced by σ on C_α. We have the following commutative diagram:

$$\begin{array}{ccc} H^1(C_\alpha, Q_l) & \xrightarrow{\sigma_\alpha^*} & H^1(C_\alpha, Q_l) \\ \downarrow{\scriptstyle \pi_\alpha^*} & & \downarrow{\scriptstyle \pi_\alpha^*} \\ H^1(C, Q_l) & \xrightarrow{\sigma^*} & H^1(C, Q_l) , \end{array} \tag{3.10}$$

where the vertical arrows are injective homomorphisms induced by π_α. Since σ_3 is an automorphism of order 3 of C_3, the eigenvalues of σ_3^* on $H^1(C_3, Q_l)$ are 1, ζ^2 or ζ^4. Since $C_3/\langle\sigma_3\rangle$ is the rational curve, we have

$$H^1(C_3/\langle\sigma_3\rangle, Q_l) = 0 . \tag{3.11}$$

So, by Lemma 1.1, the eigenvalues of σ_3^* on $H^1(C_3, Q_l)$ must be ζ^2 or ζ^4. Since the trace of σ_3^* is an integer, the multiplicities of eigenvalues ζ^2 and ζ^4 must be equal to each other. Therefore, by (3.9), the multiplicities of eigenvalues of σ_3^* are equal to $g(C_3) = n_1 + n_2 + n_4 + n_5 - 2$. Using (3.10) and Lemma 1.1, we conclude that for σ^*

(3.12) the multiplicities of eigenvalues ζ^2 and ζ^4 are equal to $n_1 + n_2 + n_4 + n_5 - 2$.

Using the same method for C_2, we conclude that for σ^*

(3.13) the multiplicities of eigenvalues ζ^3 are equal to $2g(C_2) = n_1 + n_3 + n_5 - 2$.

Finally, again using Lemma 1.1, the other eigenvalues of σ^* must be ζ or ζ^5. Since the trace of σ^* on $H^1(C, Q_l)$ is an integer, the multiplicities of eigenvalues ζ and ζ^5 must be equal to each other. Therefore, using (3.4), (3.12) and (3.13), we conclude that for σ^*

(3.14) the multiplicities of eigenvalues ζ and ζ^5 are equal to $g(C) - (g(C_2) + g(C_3)) = n_1 + n_2 + n_3 + n_4 + n_5 - 2$. q.e.d.

REMARK 3.2. Let C be the non-singular projective model of the curve defined by the equation:

$$z^4 = f_1(t)f_2(t)^2f_3(t)^3 , \tag{3.15}$$

where z and t are variables, $f_i(t)$ $(i = 1, 2, 3)$ are polynomials of t which have only simple zeros and which are prime to each other. We set

$$n_i = \text{degree of } f_i(t) , \tag{3.16}$$

and we assume that there exists an integer n such that

$$n_1 + 2n_2 + 3n_3 = 4n . \tag{3.17}$$

Then we can calculate the genus $g(C)$ of C as follows:

$$g(C) = 2n + n_1 - 3 . \tag{3.18}$$

Let σ be an automorphism of C defined by

$$\sigma\colon t \to t , \qquad z \to \xi z , \tag{3.19}$$

where ξ is a primitive fourth root of unity. Then, by the same method as in Proposition 3.1, we have the following:

PROPOSITION 3.3. *Under the same notations as above, the eigenvalues of* σ^* *on* $H^1(C, Q_l)$ *are given by* ξ^j $(j = 1, 2, 3)$ *with multiplicities* $n_1 + n_2 + n_3 - 2$ *if* $j = 1$ *or* 3, *and* $n_1 + n_3 - 2$ *if* $j = 2$.

4. Structure of some elliptic surfaces.

In this section, we assume char. $k = p \geqq 5$, and we consider the relatively minimal elliptic surface

$$f\colon S \to \boldsymbol{P}^1 \tag{4.1}$$

defined by the equation:

$$y^2 = 4x^3 - f_1(t)f_2(t)^2f_3(t)^3f_4(t)^4f_5(t)^5 , \tag{4.2}$$

where t is a local coordinate of an affine line in $\boldsymbol{P}^1$, and $f_1(t), \cdots, f_5(t)$ satisfy the same assumptions as in (3.1), (3.2) and (3.3). We see that under these assumptions $f: S \to \boldsymbol{P}^1$ has a regular fiber over the point of $t = \infty$ and has at least one singular fiber composed of rational curves. We denote by E_t the generic fiber of f, and by $r(E_t)$ the rank of the abelian group of rational points over $k(t)$ of E_t. We denote by s the number of singular fibers of f. Then we have

$$s = n_1 + n_2 + n_3 + n_4 + n_5 . \tag{4.3}$$

LEMMA 4.1. *Under the same notations as above, $r(E_t)$ is given by*

$$r(E_t) = 2s - 4 - B_2(S) + \rho(S) . \tag{4.4}$$

In particular, the following inequality holds:

$$\lambda(S) = B_2(S) - \rho(S) \leqq 2s - 4 . \tag{4.5}$$

PROOF. The discriminant $\Delta(t)$ of the Weierstrass minimal model of the equation (4.2) is given by

$$\Delta(t) = f_1(t)^2f_2(t)^4f_3(t)^6f_4(t)^8f_5(t)^{10} . \tag{4.6}$$

Therefore, we have

$$c_2(t) = \sum_{P \in P^1} \mathrm{ord}_P \Delta(t) = 2n_1 + 4n_2 + 6n_3 + 8n_4 + 10n_5 \tag{4.7}$$

(cf. Ogg [15]). By Lemma 2.1, we have $B_1(S) = 2 \dim \mathrm{Alb}\,(S) = 0$. Therefore, we have

$$B_2(S) = c_2(S) + 2B_1(S) - 2 = 2n_1 + 4n_2 + 6n_3 + 8n_4 + 10n_5 - 2 . \tag{4.8}$$

In our case, by Néron [13], the number of the singular fibers of f according to the types is as follows:

$$\begin{aligned} &n_1(\text{type } C1) , \quad n_2(\text{type } C3) , \quad n_3(\text{type } C4) , \\ &n_4(\text{type } C6) , \quad n_5(\text{type } C8) . \end{aligned} \tag{4.9}$$

Therefore, using Lemma 2.3, we have

$$\rho(S) = r(E_t) + 2 + 2n_2 + 4n_3 + 6n_4 + 8n_5 . \tag{4.10}$$

By (4.8), (4.10) and (4.3), we have

$$B_2(S) - \rho(S) = 2s - 4 - r(E_t) \leqq 2s - 4 . \tag{4.11}$$

q.e.d.

Now, we consider the non-singular projective model C of the curve defined by the equation:

$$z^6 = f_1(t)f_2(t)^2f_3(t)^3f_4(t)^4f_5(t)^5 \ . \tag{4.12}$$

As in Section 3, C has the automorphism σ defined by

$$\sigma\colon t \to t \ , \qquad z \to \zeta z \ , \tag{4.13}$$

where ζ is a primitive sixth root of unity. For the coordinate (x, y) in (4.2), we set

$$X = x/z^2 \ , \qquad Y = y/z^3 \ . \tag{4.14}$$

Then, (4.2) becomes an elliptic curve E defined by

$$Y^2 = 4X^3 - 1 \ . \tag{4.15}$$

We consider the automorphism τ of this elliptic curve defined by

$$\tau\colon X \to (1/\zeta^2)X \ , \qquad Y \to -Y \ , \tag{4.16}$$

and we set

$$g = \sigma \times \tau \ . \tag{4.17}$$

Then, g is an automorphism of order six of $C \times E$. We denote by G the cyclic group generated by g. Then, the quotient surface $(C \times E)/G$ is birationally equivalent to the original elliptic surface S.

Now, we consider the automorphism g' of $C \times E$ defined by

$$g' = \sigma^5 \times \tau \ , \tag{4.18}$$

and denote by G' the cyclic group generated by g'. Then, by (4.13) and (4.18), g' is given by

$$g'\colon \begin{cases} t \to t \ , & z \to \zeta^5 z \ , \\ X \to (1/\zeta^2)X \ , & Y \to -Y \ . \end{cases} \tag{4.19}$$

We set

$$\begin{cases} x' = f_1(t)z^4X/f_3(t)f_4(t)^2f_5(t)^3 \ , \\ y' = f_1(t)^2f_2(t)z^3Y/f_4(t)f_5(t)^2 \ . \end{cases} \tag{4.20}$$

We see that x' and y' are G'-invariant. By (4.15) and (4.20), we have

$$y'^2 = 4x'^3 - f_1(t)^5f_2(t)^4f_3(t)^3f_4(t)^2f_5(t) \ . \tag{4.21}$$

We denote by $f'\colon S' \to \boldsymbol{P}^1$ the relatively minimal elliptic surface defined by (4.21). Then we can show that $(C \times E)/G'$ is birationally equivalent to S'. Summing up these results, we have the following:

PROPOSITION 4.2. *Under the same notations as above, there exists the following diagram:*

$$
\begin{tikzcd}
\tilde{S} \arrow[d] \arrow[dr, "\pi"] & & C \times E \arrow[dl, "h"'] \arrow[dr, "h'"] & & \tilde{S}' \arrow[d] \arrow[dl, "\pi'"'] \\
S & (C \times E)/G & & (C \times E)/G' & S' ,
\end{tikzcd}
\tag{4.22}
$$

where $\pi: \tilde{S} \to (C \times E)/G$ (resp. $\pi': \tilde{S}' \to (C \times E)/G'$) *is the minimal resolution of singularities of* $(C \times E)/G$ (resp. $(C \times E)/G'$), h *and* h' *are projections, and vertical arrows are birational morphisms.*

REMARK 4.3. Let $f: S \to \boldsymbol{P}^1$ be the relatively minimal elliptic surface defined by the equation:

$$y^2 = 4x^3 - f_1(t)f_2(t)^2f_3(t)^3x \ , \tag{4.23}$$

where t is a local coordinate of an affine line in $\boldsymbol{P}^1$, and $f_1(t)$, $f_2(t)$, $f_3(t)$ satisfy the same assumptions as in (3.15), (3.16) and (3.17). Let C be the non-singular projective model of the curve defined by the equation

$$z^4 = f_1(t)f_2(t)^2f_3(t)^3 \ . \tag{4.24}$$

C has the automorphism σ defined by

$$\sigma: t \to t \ , \qquad z \to \xi z \ , \tag{4.25}$$

where ξ is a primitive fourth root of unity. Let E be the elliptic curve defined by the equation:

$$Y^2 = 4X^3 - X \ . \tag{4.26}$$

E has the automorphism τ defined by

$$\tau: X \to -X \ , \qquad Y \to -\xi Y \ . \tag{4.27}$$

Let $f': S' \to \boldsymbol{P}^1$ be the relatively minimal elliptic surface defined by the equation:

$$y^2 = 4x^3 - f_1(t)^3f_2(t)^2f_3(t)x \ . \tag{4.28}$$

We set

$$g = \sigma \times \tau \ , \qquad g' = \sigma^3 \times \tau \ . \tag{4.29}$$

Then, g and g' are two automorphisms of $C \times E$. We denote by G (resp. G') the cyclic group generated by g (resp. g'). Then we can prove that $(C \times E)/G$ (resp. $(C \times E)/G'$) is birationally equivalent to S (resp. S'), and we have the diagram which is similar to (4.22).

5. Proof of Theorem II.

In this section, we again assume char. $k = p \geqq 5$, and we first prove the following theorem.

THEOREM 5.1. *Let* $f: S \to \boldsymbol{P}^1$ *be the relatively minimal elliptic surface* *defined by the equation* (II). *Let s be the number of singular fibers, and assume $p \equiv 1 \bmod 3$. Then, the Picard number $\rho(S)$ of S is given by*

$$B_2(S) - \rho(S) = 2s - 4 . \tag{5.1}$$

PROOF. Taking a suitable coordinate t of an affine line in $\boldsymbol{P}^1$, we can rewrite the equation (II) as follows:

$$y^2 = 4x^3 - t^5(t-1)^5(t+1)^5(t-a)^5(t-b)^5(t-c)^5 , \tag{II$'$}$$

with suitable elements a, b and c of k. Then, over the point $t = \infty$, f has a regular fiber. For some special values of a, b and c, the equation (II′) is not a minimal Weierstrass model (cf. Ogg [15]). So, taking the minimal Weierstrass model of the equation (II′), we can write the equation defining this elliptic surface $f: S \to \boldsymbol{P}^1$ as in the form (4.2), where in our case $f_i(t)$ $(i = 1, \cdots, 5)$ are polynomials of t which have only simple zeros at -1, 0, 1, a, b or c, and which are prime to each other. Moreover, in (3.3), we have $2 \leqq n \leqq 5$ for any specialization of a, b and c. Moreover, we can check that for any specialization of a, b and c, we have

$$5n_1 + 4n_2 + 3n_3 + 2n_4 + n_5 = 6 , \qquad n_1 = 0 . \tag{5.2}$$

We are now in the same situation as in Section 4 with some additional conditions. So, we use the notations in Section 4.

First, we assume that a, b, c are contained in a finite field $\boldsymbol{F}_q$ with $q = p^\delta$ for some integer δ. Replacing δ by a large enough integer, we can assume that the automorphisms g, g' are defined over $\boldsymbol{F}_q$ with $q = p^\delta$, and that the subspace generated by algebraic cycles in $H^2(C \times E, \bar{Q}_l)$ has a basis consisting of elements which are represented by $\boldsymbol{F}_q$-rational algebraic cycles on $C \times E$. Let F_X in general be the Frobenius morphism of an algebraic variety X relative to $\boldsymbol{F}_q$. We denote by F_X^* the homomorphism induced by F_X on the étale cohomology groups. We denote by e_1 and e_2 the eigenvectors of τ^* in $H^1(E, \bar{Q}_l)$. Using Lemma 1.1 and the same method as in Section 3, we see that the eigenvalues of τ^* are ζ and ζ^5, where ζ is a primitive sixth root of unity. Since $\tau^* \circ F_E^* = F_E^* \circ \tau^*$, we have the following:

$$\begin{cases} \tau^*(e_1) = \zeta e_1 , & \tau^*(e_2) = \zeta^5 e_2 , \\ F_E^*(e_1) = \alpha_1 e_1 , & F_E^*(e_2) = \alpha_2 e_2 , \end{cases} \tag{5.3}$$

where α_1 and α_2 are eigenvalues of F_E^*. Since $p \equiv 1 \bmod 3$, the elliptic curve E defined by (4.15) is not supersingular (cf. Deuring [4]). Therefore, any powers of α_i $(i = 1, 2)$ are not powers of p, and $\alpha_1\alpha_2$ is equal to q (cf. Mumford [12], for instance). As for the action of σ^* on $H^1(C, \bar{Q}_l)$, since $\sigma^* \circ F_C^* = F_C^* \circ \sigma^*$, using Proposition 3.1 and (4.3), we have the following:

$$\sigma^*(f_{ij}) = \zeta^j f_{ij}\,, \qquad F_C^*(f_{ij}) = \beta_{ij} f_{ij}\,, \tag{5.4}$$

where f_{ij} ($1 \leqq i \leqq s-2$ if $j=1$ or 5, $1 \leqq i \leqq n_1+n_2+n_3+n_4-2$ if $j=2$ or 3, and $1 \leqq i \leqq n_1+n_3+n_5-2$ if $j=3$) are common eigenvectors in $H^1(C, \bar{Q}_l)$ of σ^* and F_C^*, and β_{ij} are eigenvalues of F_C^*. We consider the following canonical isomorphism (Künneth formula):

$$H^2(C \times E, \bar{Q}_l) = H^2(C, \bar{Q}_l) \oplus H^2(E, \bar{Q}_l) \oplus \{H^1(C, \bar{Q}_l) \otimes_{\bar{Q}_l} H^1(E, \bar{Q}_l)\}\,. \tag{5.5}$$

The group $G = \langle g \rangle$ in (4.17) acts on this space. Using Lemmas 1.1, 1.4 and Proposition 4.2, we have the natural injection:

$$\begin{aligned} &H^2(C, \bar{Q}_l) \oplus H^2(E, \bar{Q}_l) \oplus \{H^1(C, \bar{Q}_l) \otimes_{\bar{Q}_l} H^1(E, \bar{Q}_l)\}^G \\ &\quad \cong H^2(C \times E/G, \bar{Q}_l) \xhookrightarrow{\pi^*} H^2(\tilde{S}, \bar{Q}_l)\,. \end{aligned} \tag{5.6}$$

By (5.3) and (5.4), the basis of $\{H^1(C, \bar{Q}_l) \otimes_{\bar{Q}_l} H^1(E, \bar{Q}_l)\}^G$ are given by

$$f_{i1} \otimes e_2\,, \qquad f_{i5} \otimes e_1 \qquad (1 \leqq i \leqq s-2)\,. \tag{5.7}$$

We have

$$\begin{aligned} &F_{C\times E}^*(f_{i1} \otimes e_2) = \beta_{i1}\alpha_2(f_{i1} \otimes e_2)\,, \qquad F_{C\times E}^*(f_{i5} \otimes e_1) = \beta_{i5}\alpha_1(f_{i5} \otimes e_1) \\ &(1 \leqq i \leqq s-2)\,. \end{aligned} \tag{5.8}$$

Suppose that there exists an β_{i1} such that $\beta_{i1} = \alpha_1$. Then we consider the group $G' = \langle g' \rangle$ in (4.18). As was shown in Section 4, the quotient surface $(C \times E)/G'$ is birationally equivalent to the surface S' defined by (4.21). The discriminant $\Delta'(t)$ of the Weierstrass minimal model (4.21) is given by

$$\Delta'(t) = f_1(t)^{10} f_2(t)^8 f_3(t)^6 f_4(t)^4 f_5(t)^2\,. \tag{5.9}$$

Using (5.2), we have

$$c_2(S') = \sum_{P \in P^1} \mathrm{ord}_P\, \Delta'(t) = 12 \tag{5.10}$$

(cf. Ogg [15]). Therefore, S' is a rational surface (cf. Katsura [6], for instance). Hence $H^2(S', \bar{Q}_l)$ must be spanned by algebraic cycles, that is, we have

$$\lambda(S') = 0\,. \tag{5.11}$$

On the other hand, using Lemmas 1.1 and 1.4, we have the natural injection:

$$\begin{aligned} &H^2(C, \bar{Q}_l) \oplus H^2(E, \bar{Q}_l) \oplus \{H^1(C, \bar{Q}_l) \otimes_{\bar{Q}_l} H^1(E, \bar{Q}_l)\}^{G'} \\ &\quad \cong H^2(C \times E/G', \bar{Q}_l) \xhookrightarrow{\pi'^*} H^2(\tilde{S}', \bar{Q}_l)\,. \end{aligned} \tag{5.12}$$

We also have the element $f_{i1}\otimes e_1$ in $H^1(C, \bar{Q}_l)\otimes_{\bar{Q}_l} H^1(E, \bar{Q}_l)$, such that

$$(5.13)\qquad g'^*(f_{i1}\otimes e_1) = f_{i1}\otimes e_1\ ,\qquad F^*_{C\times E}(f_{i1}\otimes e_1) = \alpha_1^2 f_{i1}\otimes e_1\ .$$

Using (5.12), we can regard $f_{i1}\otimes e_1$ as an element of $H^2(\tilde{S}', \bar{Q}_l)$. Since any power of α_1 is not a power of p, $f_{i1}\otimes e_1$ is not an algebraic cycle (cf. Tate [25]). Therefore we have

$$(5.14)\qquad \lambda(S') = \lambda(\tilde{S}') \geqq 1\ ,$$

which contradicts (5.11). Therefore, we have

$$(5.15)\qquad \beta_{i1} \neq \alpha_1 \quad \text{for}\ \ 1 \leqq i \leqq s-2\ .$$

By the same method, we have

$$(5.16)\qquad \beta_{i5} \neq \alpha_2 \quad \text{for}\ \ 1 \leqq i \leqq s-2\ .$$

Using $\alpha_1\alpha_2 = q$, (5.15) and (5.16), we have

$$(5.17)\qquad \beta_{i1}\alpha_2 \neq q\ ,\qquad \beta_{i5}\alpha_1 \neq q \qquad (1 \leqq i \leqq s-2)\ .$$

Regarding $f_{i1}\otimes e_2$ and $f_{i5}\otimes e_1$ as elements of $H^2(\tilde{S}, \bar{Q}_l)$ by (5.6), we conclude that neither $f_{i1}\otimes e_2$ nor $f_{i5}\otimes e_1$ $(1 \leqq i \leqq s-2)$ are algebraic cycles in $H^2(\tilde{S}, \bar{Q}_l)$ by (5.8) and (5.17). Therefore, we have

$$(5.18)\qquad B_2(S) - \rho(S) = \lambda(S) = \lambda(\tilde{S}) \geqq 2s-4\ .$$

Hence, using Lemma 4.1, we have $B_2(S) - \rho(S) = 2s-4$.

Finally, we consider the case where some of a, b, c in the equation (II′) are not contained in $\bar{\boldsymbol{F}}_p$. Let $f_0\colon S_0 \to \boldsymbol{P}^1$ be a specialization of $f\colon S \to \boldsymbol{P}^1$ such that $f_0\colon S_0 \to \boldsymbol{P}^1$ is defined over a finite field and the types of singular fibers of f_0 are the same as those of f. Since $B_2(S) = B_2(S_0)$ and $\rho(S) \leqq \rho(S_0)$, we have

$$(5.19)\qquad B_2(S) - \rho(S) \geqq B_2(S_0) - \rho(S_0)\ .$$

We have now $B_2(S_0) - \rho(S_0) = 2s-4$ as above. Therefore, using Lemma 4.1, we get $B_2(S) - \rho(S) = 2s-4$. q.e.d.

Now we prove Theorem II. To prove the "only if" part, we assume $p \equiv 1 \bmod 3$. Suppose that there exists a unirational surface $f\colon S \to \boldsymbol{P}^1$ in Class (II). Then, by Shioda [21], S is a supersingular surface, that is, $B_2(S) = \rho(S)$. On the other hand, by Theorem 5.1, we have $B_2(S) - \rho(S) = 2s-4$, where s is the number of singular fibers of $f\colon S \to \boldsymbol{P}^1$. In our case, we have at least three singular fibers, that is, singular fibers over the points of $t = 0$, 1 and ∞. Therefore, we have $B_2(S) - \rho(S) \geqq 2$, which contradicts the supersingularity of S. The "if" part is proved in Katsura [6]. By the base change by a purely inseparable morphism of degree p, we get a rational elliptic surface which has a generically surjective rational mapping to S. q.e.d.

COROLLARY 5.2. *Let $f: S \to \boldsymbol{P}^1$ be an elliptic surface in Class* (II), *and E_t the generic fiber of f. Then, the following six conditions are equivalent*:

(i) *S is unirational elliptic surface of base change type.*
(ii) $p \equiv 2 \bmod 3$.
(iii) *E_t is a supersingular elliptic curve over $k(t)$.*
(iv) *S is a supersingular surface.*
(v) *S is a Zariski surface.*
(vi) *S is a unirational surface.*

PROOF. This follows from Theorems 5.1 and II. For the details, see Katsura [6]. q.e.d.

REMARK 5.3. Let $f: S \to \boldsymbol{P}^1$ be the relatively minimal elliptic surface defined by the equation (I). Using Proposition 3.3 and Remark 4.3, we can prove the following theorem by the same method as in Theorem 5.1.

THEOREM 5.4. *Let s be the number of singular fibers. Assume* $p \equiv 1 \bmod 4$. *Then the Picard number $\rho(S)$ of S is given by*

$$B_2(S) - \rho(S) = 2s - 4 . \tag{5.20}$$

Using this theorem, we can give another proof of Theorem I by the same method as above. As a corollary to Theorems I and II, using Lemma 4.1, we have the following:

COROLLARY 5.5. *Let E_t be the elliptic curve defined by the equation* (I) (resp. (II)). *Then, the rank $r(E_t)$ of the abelian group of rational points over $k(t)$ of E_t is given by*

(i) $r(E_t) = 0$ *if* $p \equiv 1 \bmod 4$ (resp. $p \equiv 1 \bmod 3$),
(ii) $r(E_t) = 2s - 4$ *if* $p \equiv 3 \bmod 4$ (resp. $p \equiv 2 \bmod 3$),

where s is the number of points at which E_t has bad reduction.

REFERENCES

[1] M. ARTIN, Supersingular $K3$ surfaces, Ann. Sci. Ecole Norm. Sup. 4e serie, 7 (1974), 543-568.

[2] P. BLASS, Unirationality of Enriques' surfaces in characteristic two, Comp. Math. 45 (1982), 393-398.

[3] E. BRIESKORN, Rationale Singularitäten Komplexer Flächen, Invent. Math. 4 (1968), 336-358.

[4] M. DEURING, Die Typen der Multiplikatorenringe elliptischer Functionenkörper, Abh. Math. Sem. Univ. Hamburg 14 (1941), 197-272.

[5] G. HARDER AND M. S. NARASHIMHAN, On the cohomology group of moduli spaces of

vector bundles on curves, Math. Ann. 212 (1975), 215–248.
[6] T. KATSURA, Unirational elliptic surfaces in characteristic p, Tôhoku Math. J. 33 (1981), 521–553.
[7] S. LANG, Abelian Varieties, Interscience-Wiley, New York, 1959.
[8] J. S. MILNE, Etale Cohomology, Princeton University Press, Princeton, New Jersey, 1980.
[9] M. MIYANISHI, Unirational quasi-elliptic surfaces in characteristic 3, Osaka J. Math. 13 (1976), 513–522.
[10] M. MIYANISHI, Unirational quasi-elliptic surfaces, Japan J. Math. 3 (1977), 395–416.
[11] M. MIYANISHI, Lectures on curves on rational and unirational surfaces, Tata Institute of Fundamental Research, Bombay, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
[12] D. MUMFORD, Abelian Varieties, Oxford University Press, 1970.
[13] A. NÉRON, Modèles minimaux des variètés abéliennes sur les corps locaux at globaux, Publ. Math. Inst. Hautes Etudes Sci. 21 (1964).
[14] A. P. OGG, Cohomology of abelian varieties over function fields, Ann. of Math. 76 (1962), 185–212.
[15] A. P. OGG, Elliptic curves and wild ramification, Amer. J. Math. 89 (1967), 1–21.
[16] A. N. RUDAKOV AND I. R. SHAFEREVICH, Supersingular $K3$ surfaces over fields of characteristic 2, Math. USSR-Izv. 13-1 (1979), 147–165.
[17] I. R. SHAFAREVICH, Principal homogeneous spaces defined over a function field, Amer. Math. Soc. Trans., Series 2, 37 (1964), 85–114.
[18] T. SHIODA, On elliptic modular surfaces, J. Math. Soc. Japan 24 (1972), 20–59.
[19] T. SHIODA, On rational points of the generic elliptic curve with level N structure over the field of modular functions of level N, J. Math. Soc. Japan 25 (1973), 144–157.
[20] T. SHIODA, Algebraic cycles on certain $K3$ surfaces in characteristic p, Manifolds Tokyo 1973, Univ. of Tokyo Press, 1975.
[21] T. SHIODA, An example of unirational surfaces in characteristic p, Math. Ann. 211 (1974), 233–236.
[22] T. SHIODA, On unirationality of supersingular surfaces, Math. Ann. 255 (1977), 155–159.
[23] T. SHIODA, Some results on unirationality of algebraic surfaces, Math. Ann. 230 (1977), 153–168.
[24] T. SHIODA AND T. KATSURA, On Fermat varieties, Tôhoku Math. J. 31 (1979), 97–115.
[25] J. TATE, Algebraic cycles and poles of zeta functions, Arithmetical Algebraic Geometry, Harper and Row, New York, 1965.
[26] O. ZARISKI, On Castelnuovo's criterion of rationality $p_a=P_2=0$ of an algebraic surface, Illinois J. Math. 2 (1958), 303-315.

MATHEMATICAL INSTITUTE
STATE UNIVERSITY OF UTRECHT
3508 TA UTRECHT
THE NETHERLANDS
AND
DAPARTMENT OF MATHEMATICS
YOKOHAMA CITY UNIVERSITY
YOKOHAMA, 236
JAPAN