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THE FUNCTIONAL EQUATION OF ZETA DISTRIBUTIONS ASSOCIATED WITH FORMALLY REAL JORDAN ALGEBRAS

Dedicated to Prof. M. Koecher on his sixtieth birthday

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The purpose of this paper is to give an explicit expression for the Fourier transform of the zeta distributions on a certain class of prehomogeneous spaces defined by Jordan algebras.

Let V be a formally real simple Jordan algebra over $\boldsymbol{R}$. Let $\dim V = n$ and $\operatorname{rk} V = r$ (for definition, see 1.1). We fix a (positive definite) inner product on V defined by

$$(1) \qquad \langle x, y\rangle = \frac{r}{n}\operatorname{tr}(T_{xy}) \quad (x, y \in V),$$

where T_x denotes the linear transformation of V defined by $T_x(y) = xy$. The "structure group" of V, $G = \operatorname{Str}(V)$ (see 1.2), is then self-adjoint with respect to $\langle\ \ \rangle$, and hence is a reductive algebraic group. It is well-known that the pair (G, V) is a (real) prehomogeneous vector space in the sense of Sato-Shintani [6], i.e. if one denotes by G_C and V_C the complexifications of G and V, respectively, G_C is transitive on the Zariski-open set

$$V_C^\times = \{x \in V_C \mid N(x) \neq 0\}$$

(see [5c]). Here N denotes the "reduced norm" of V, which is an absolutely irreducible homogeneous polynomial function on V of degree r, characterized by the property:

$$(2) \qquad N(1) = 1, \qquad N(gx) = \det(g)^{r/n}N(x) \quad (g \in G^\circ,\ x \in V),$$

where G° is the identity connected component of G.

The set of real invertible elements $V^\times = V \cap V_C^\times$ is decomposed into the disjoint union of $r+1$ (open) G°-orbits:

$$V^\times = \coprod_{i=0}^{r} \Omega_i,$$

where Ω_i is the set of elements of signature $(r-i, i)$ ([5c]). In particular,

Ω_0, the G°-orbit of the unit element 1, is a self-dual homogeneous cone. The gamma function associated to Ω_0 is given by

$$(3)\qquad \Gamma_{\Omega_0}(s) = \int_{\Omega_0} e^{-\langle u,1\rangle} N(u)^{s-n/r} d(u)$$
$$= (2\pi)^{\frac{1}{2}(n-r)} \prod_{i=1}^{r} \Gamma\left(s - \frac{d}{2}(i-1)\right) \quad \left(\operatorname{Re} s > \frac{d}{2}(r-1)\right),$$

where $d(u)$ is the Euclidean volume element with respect to $\langle\ \ \rangle$ and d is a positive integer given by $d = 2(n-r)/r(r-1)$.

Now, for $f \in \mathscr{S}(V)$ (the Schwartz space of V), we set

$$(4)\qquad \Phi_i(f, s) = \int_{\Omega_i} f(u)|N(u)|^s d(u) \quad (0 \leqq i \leqq r).$$

Then, it is known ([6]) that this integral is convergent for $\operatorname{Re} s > 0$, the analytic function $\Phi_i(f, s)$ has a meromorphic continuation with respect to s to the whole plane $\boldsymbol{C}$, and the map $f \mapsto \Phi_i(f, s)$ is a tempered distribution on V, called a "zeta distribution". Moreover, denoting by $\hat{f}$ the Fourier transform of f, one has a functional equation of the following form

$$(5)\qquad \Phi_i\left(\hat{f}, s - \frac{n}{r}\right) = (2\pi)^{-rs} e\left(\frac{rs}{4}\right) \Gamma_{\Omega_0}(s) \sum_{j=0}^{r} u_{ij}(s) \Phi_j(f, -s),$$

where $u_{ij}(s)$ is a polynomial in $e(-s/2)$ of degree at most r.

For the cases $V = \mathrm{Her}_r(\boldsymbol{C})$ and $\mathrm{Sym}_r(\boldsymbol{R})$, explicit expressions for $u_{ij}(s)$ were obtained by Sato-Shintani [6] and Shintani [7]. For the case $r = 2$, the functional equations of the corresponding zeta functions were obtained by Siegel [8] (cf. also [2]). Other cases were treated by Muro [4] by using the micro-local analysis (cf. [10]). Here we will give a direct and unified way of computing the Fourier transform based on the theory of Jordan algebras, generalizing the method of [6], [7].

REMARK. In the notation of [6], our $u_{ij}(s)$ and $\Gamma_{\Omega_0}(s)$ are equal to $(2\pi)^{-\frac{1}{2}(n-r)} u_{ij}(s)$ and $(2\pi)^{\frac{1}{2}(n-r)} \gamma\left(s - \frac{n}{r}\right)$, respectively. In our case, using the relation $N(\mathrm{grad})N(u)^s = b(s)N(u)^{s-1}$ $(u \in \Omega_0)$ and (3), it is easy to see that the "b-function" is given by

$$(6)\qquad b(s) = \prod_{i=1}^{r} \left(s + \frac{d}{2}(i-1)\right).$$

NOTATION. $\boldsymbol{R}_+$ is the semi-group of positive real numbers. For $z \in \boldsymbol{C}$, we set $e(z) = \exp(2\pi\sqrt{-1}\, z)$. For a linear transformation T of a (real) vector space V and $\alpha \in \boldsymbol{R}$, we set $V(T, \alpha) = \{v \in V \mid Tv = \alpha v\}$. The real linear subspace of V generated by a subset $\{v_1, \cdots, v_m\}$ in V is denoted by $\{v_1, \cdots, v_m\}_R$. When V is endowed with an inner product $\langle \ \rangle$, we write $S[v] = \langle v, Sv\rangle$ $(v \in V)$ for any symmetric linear transformation S. For a topological group G, G° stands for the identity connected component of G.

1. Preliminaries on Jordan algebras (cf. [1], [5a], [5c]).

1.1. Let V be a formally real simple Jordan algebra of dimension n. We choose and fix a set of primitive idempotents $\{e_i\ (1 \leqq i \leqq r)\}$ such that

$$\sum_{i=1}^{r} e_i = 1\ , \qquad e_i e_j = \delta_{ij} e_i\ ;$$

the cardinality r is uniquely determined and is called the "rank" of V. The linear transformation T_{e_i} has eigen values 0, 1/2, 1, and one has $V(T_{e_i}, 1) = \{e_i\}_R$. We put

$$V_{ij} = \begin{cases} V(T_{e_i}, 1) & \text{if} \quad i = j\ , \\ V\left(T_{e_i}, \dfrac{1}{2}\right) \cap V\left(T_{e_j}, \dfrac{1}{2}\right) & \text{if} \quad i \neq j\ . \end{cases}$$

Then, dim V_{ij} $(i \neq j)$ are all equal, and one has the Peirce decomposition:

$$V = \bigoplus_{i \leqq j} V_{ij}\ . \tag{7}$$

Hence, putting $d = \dim V_{ij}$ $(i \neq j)$, one has

$$n = r + \frac{d}{2} r(r-1)\ . \tag{8}$$

It follows that $\langle e_i, e_j\rangle = \delta_{ij}$.

1.2. Following Koecher, we use the notation

$$x \,\square\, y = T_{xy} + [T_x, T_y] \qquad \text{for} \quad x, y \in V\ .$$

By definition, the structure group $G = \mathrm{Str}(V)$ is an algebraic group given by

$$G = \{g \in GL(V) \mid g(x \,\square\, y) g^{-1} = (gx) \,\square\, ({}^t g^{-1} y)\ \ (x, y \in V)\}\ .$$

Then G is reductive, and it is known that

$$\mathfrak{g} = \mathrm{Lie}\, G = \{x \,\square\, y\ \ (x, y \in V)\}_R$$

(see e.g. [5a]). Let $K = \mathrm{Aut}(V)$ be the automorphism group of the Jordan algebra V. Then one has $K^\circ = \{g \in G^\circ \mid {}^t g^{-1} = g\}$ and K° is a maximal compact subgroup of G°. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Then one has $\mathfrak{p} = \{T_x\ (x \in V)\}$, and

$$\mathfrak{a} = \{T_{e_i}\ (1 \leqq i \leqq r)\}_{\boldsymbol{R}}$$

is a maximal (abelian) subalgebra in $\mathfrak{p}$. Thus r coincides with the (real) rank of $\mathfrak{g}$.

We put $\underline{V} = \sum_{i<j} V_{ij}$ and, for $x = \sum_{i<j} x_{ij}$ $(x_{ij} \in V_{ij})$, put

$$(9) \qquad T_x^{(+)} = \sum_{i<j} e_i \,\square\, x_{ij}\,, \qquad T_x^{(-)} = \sum_{i<j} e_j \,\square\, x_{ij} \ (= {}^tT_x^{(+)})\,.$$

Then

$$\mathfrak{n}' = \{T_x^{(+)}\ (x \in \underline{V})\}\,, \qquad \mathfrak{n} = \{T_x^{(-)}\ (x \in \underline{V})\}$$

are (mutually opposite) maximal nilpotent subalgebras of $\mathfrak{g}$ normalized by $\mathfrak{a}$. Let A, N, N' denote the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}$, $\mathfrak{n}'$, respectively. Then one has Iwasawa decompositions $G^\circ = K^\circ \cdot AN = K^\circ \cdot AN'$.

1.3. Let $\mathscr{E}$ denote the set of all r-tuples consisting of ± 1. For $\varepsilon = (\varepsilon_j) \in \mathscr{E}$, we denote the cardinality of $\{j \mid \varepsilon_j = -1\}$ by $n(\varepsilon)$ and set

$$\mathscr{E}_i = \{\varepsilon \in \mathscr{E} \mid n(\varepsilon) = i\}\,.$$

By definition, Ω_i is the G°-orbit of $-\sum_{j=1}^i e_j + \sum_{j=i+1}^r e_j$. For $\varepsilon = (\varepsilon_j) \in \mathscr{E}$, we denote by Ω_ε (resp. Ω'_ε) the AN-orbit (resp. AN'-orbit) of $\sum_{j=1}^r \varepsilon_j e_j$. Clearly, one has $\Omega_\varepsilon, \Omega'_\varepsilon \subset \Omega_i$ if $\varepsilon \in \mathscr{E}_i$.

LEMMA 1. (i) *The Ω'_ε $(\varepsilon \in \mathscr{E})$ are mutually disjoint and $\coprod_{\varepsilon \in \mathscr{E}_i} \Omega'_\varepsilon$ is a Zariski-open subset in Ω_i.*

(ii) *For each $\varepsilon \in \mathscr{E}$, the map*

$$\boldsymbol{R}_+^r \times \underline{V} \to \Omega'_\varepsilon$$
$$(t_1, \cdots, t_r) \times x \mapsto (\exp T_x^{(+)})\Big(\sum_{j=1}^r t_j \varepsilon_j e_j\Big) = v$$

is a (bijective) homeomorphism.

For a proof, see [5c]. We have also an analogous lemma for Ω_ε. The correspondence in Lemma 1, (ii) is given explicitly as follows:

$$(10) \quad v = \sum_{i=1}^r \Big(\varepsilon_i t_i + \frac{1}{4} \sum_{k>i} \varepsilon_k t_k \xi_{ik}(x)^2\Big) e_i + \frac{1}{2} \sum_{i<j} \Big(\varepsilon_j t_j \xi_{ij}(x) + \sum_{k>j} \varepsilon_k t_k \xi_{ik}(x) \xi_{jk}(x)\Big)\,,$$

where, for $x = \sum_{i<j} x_{ij} \in \underline{V}$, we set

$$\xi_{ij}(x) = \sum_{\nu=1}^{j-i} \frac{1}{\nu!} \sum_{i<k_1<\cdots<k_{\nu-1}<j} x_{ik_1} x_{k_1 k_2} \cdots x_{k_{\nu-1} j}\,.$$

It follows that for the corresponding (Euclidean) volume elements one has

$$d(v) = 2^{r-n}\Big(\prod_{j=1}^{r} t_j^{(j-1)d} dt_j\Big)\cdot d(x) .$$

Since $N(v) = \prod_{j=1}^{r}(t_j\varepsilon_j)$, the G-invariant volume element on $V^{\times}$ is given by

$$|N(v)|^{-n/r}d(v) = 2^{r-n}\Big(\prod_{j=1}^{r} t_j^{(j-1)d-n/r} dt_j\Big)\cdot d(x) . \tag{11}$$

1.4. We set

$$V^{(k)} = \sum_{i,j\leq k} V_{ij} , \qquad V_k = \sum_{i<k} V_{ik} .$$

Then $V^{(k)} = V(T_{1^{(k)}}, 1)$ is a (simple) Jordan subalgebra with the unit element $1^{(k)} = \sum_{j=1}^{k} e_j$. We denote the reduced norm of $V^{(k)}$ by $N^{(k)}$. Note that the restriction of the "standard" inner product (1) to $V^{(k)}$ is that of $V^{(k)}$. It is known that $V^{(k)}V_{k+1} \subset V_{k+1}$ and the map

$$\rho_k(v) : V^{(k)} \ni v \mapsto 2T_v | V_{k+1} \in \mathrm{End}(V_{k+1})$$

is a (unital) Jordan algebra representation of $V^{(k)}$. For $v \in V^{(k)}$, $x \in V_{k+1}$, we put

$$v[x] = \frac{1}{2}\rho_k(v)[x] = \langle x, vx\rangle .$$

LEMMA 2. (i) *For* $v \in V^{(k)}$, *one has*

$$\det(\rho_k(v)) = N^{(k)}(v)^d . \tag{12}$$

(ii) *For* $v \in V$, $v = \sum_{i\leq j} v_{ij}$, *we write*

$$v_{ii} = \xi_i e_i , \qquad v^{(k)} = \sum_{i,j\leq k} v_{ij} , \qquad v_k = \sum_{i<k} v_{ik} .$$

If $v^{(k-1)}$ *is invertible* (i.e. *if* $N^{(k-1)}(v^{(k-1)}) \neq 0$), *one has*

$$N^{(k)}(v^{(k)}) = N^{(k-1)}(v^{(k-1)})(\xi_k - {v^{(k-1)}}^{-1}[v_k]) . \tag{13}$$

(*For* $k = 1$, *we make a convention that* $v^{(0)} = 1$, $v_1 = 0$.)

PROOF. To prove (12), we may assume $k = r - 1$. Then, for any $v \in V^{(r-1)}$, there exists $g \in K$ such that

$$ge_r = e_r , \qquad gv = \sum_{i=1}^{r-1} t_i e_i \quad \text{with} \quad t_i \in \boldsymbol{R}$$

([5c]). Then g leaves V_r invariant, and one has

$$\rho_{r-1}(v) = (g | V_r)^{-1}\rho_{r-1}\Big(\sum_{i=1}^{r-1} t_i e_i\Big)(g | V_r) .$$

Hence

$$\det(\rho_{r-1}(v)) = \det\Big(\rho_{r-1}\Big(\sum_{i=1}^{r-1} t_i e_i\Big)\Big) = \Big(\prod_{i=1}^{r-1} t_i\Big)^d = N^{(r-1)}(v)^d ,$$

which proves our assertion.

To prove (13), we may again assume $k = r - 1$. Then, in our notation, one has

$$\begin{aligned} v &= v^{(r-1)} + v_r + \xi_r e_r \\ &= \exp(e_r \square x)(v^{(r-1)} + \xi'_r e_r) , \end{aligned}$$

where

$$\begin{cases} x = 2\rho_{r-1}(v^{(r-1)})^{-1} v_r , \\ \xi'_r = \xi_r - v^{(r-1)^{-1}}[v_r] \end{cases}$$

(see [5c]). It follows that

$$\begin{aligned} N(v) &= N(v^{(r-1)} + \xi'_r e_r) \\ &= N^{(r-1)}(v^{(r-1)})\xi'_r , \end{aligned} \qquad \text{q.e.d.}$$

We denote the projection map $v \mapsto v^{(k)}$ by P_k and, when $P_{k-1}(v) = v^{(k-1)}$ is invertible, set

$$\begin{aligned} \chi_k(v) &= N^{(k)}(P_k(v))/N^{(k-1)}(P_{k-1}(v)) \\ &= \xi_k - v^{(k-1)^{-1}}[v_k] . \end{aligned} \tag{14}$$

(We set $\chi_1(v) = \xi_1$ for any v.) Then, when all $P_k(v) = v^{(k)}$ $(1 \leqq k \leqq r)$ are invertible, one has

$$N(v) = \prod_{k=1}^{r} \chi_k(v) . \tag{15}$$

Similarly, the projection map onto $V(T_{1^{(k)}}, 0)$ is denoted by P'_k, i.e. $P'_k(v) = \sum_{i,j \geqq k+1} v_{ij}$. We denote the reduced norm of $P'_k(V) = V(T_{1^{(k)}}, 0)$ by $N^{(k)\prime}$ and, when $P'_k(v)$ is invertible, set

$$\chi'_k(v) = N^{(k-1)\prime}(P_{k-1}{}'(v))/N^{(k)\prime}(P'_k(v)) .$$

Then, in the notation of Lemma 1, (ii), it is clear that, for $v \in \Omega'_\varepsilon$, one has

$$P'_k(v) = \exp(T^{(+)}_{P'_k(x)})\Big(\sum_{i=k+1}^{r} t_i \varepsilon_i e_i\Big) ,$$

whence follows that

$$N^{(k)\prime}(P'_k(v)) = \prod_{i=k+1}^{r} (t_i \varepsilon_i) \quad \text{and} \quad \chi'_k(v) = t_k \varepsilon_k .$$

Thus one has $v \in \Omega'_\varepsilon$, if and only if all $P'_k(v)$ $(1 \leqq k \leqq r)$ are invertible and the sign of $\chi'_k(v)$ is ε_k ([5c]). One has also an analogous result for Ω_ε, P_k, and χ_k.

1.5. For $\sigma = (\sigma_k) \in C^r$, one defines the gamma function of r variables associated to Ω_0 by

$$\Gamma_{\Omega_0}(\sigma) = \int_{\Omega_0} e^{-\langle u,1\rangle}\Big(\prod_{k=1}^{r} \chi'_k(u)^{\sigma_k}\Big) N(u)^{-n/r} d(u) .$$

Then, by changing variables as in Lemma 1, (ii), it can be shown that this integral is convergent for $\operatorname{Re}\sigma_k > \frac{d}{2}(r-k)$ and

$$\Gamma_{\Omega_0}(\sigma) = (2\pi)^{\frac{1}{2}(n-r)} \prod_{k=1}^{r} \Gamma\Big(\sigma_k - \frac{d}{2}(r-k)\Big) \tag{16}$$

(cf. [3], [5b]). If we identify $s \in \boldsymbol{C}$ with the r-tuple $(s, s, \cdots, s)$, then (3) becomes a special case of (16).

2. Computation of the Fourier transforms.

2.1. In the following computation, f is a function in $\mathscr{S}(V)$ whose support is compact and contained in the union of the sets Ω_ε $(\varepsilon \in \mathscr{E})$. For $\sigma = (\sigma_k) \in \boldsymbol{C}^r$ and $\varepsilon = (\varepsilon_k) \in \mathscr{E}$, we put

$$\Psi_\varepsilon(f, \sigma) = \int_{\Omega_\varepsilon} f(v) \prod_{k=1}^{r} |\chi_k(v)|^{\sigma_k}\, d(v) , \tag{17}$$

$$\Psi'_\varepsilon(\hat{f}, \sigma) = \int_{\Omega'_\varepsilon} \hat{f}(v) \prod_{k=1}^{r} |\chi'_k(v)|^{\sigma_k}\, d(v) \quad \Big(\operatorname{Re}\sigma_k > \frac{d}{2}(r-k) - \frac{n}{r}\Big) . \tag{18}$$

Then, if one identifies $s \in \boldsymbol{C}$ with the r-tuple $(s, \cdots, s)$, one has by Lemma 1, (i)

$$\Phi_i(f, s) = \sum_{\varepsilon \in \mathscr{E}_i} \Psi_\varepsilon(f, s) , \qquad \Phi_i(\hat{f}, s) = \sum_{\varepsilon \in \mathscr{E}_i} \Psi'_\varepsilon(\hat{f}, s) . \tag{19}$$

Hence, by computing the integral $\Psi'_\varepsilon\Big(\hat{f}, s - \frac{n}{r}\Big)$ and using Theorem 1 in [6], we will obtain the functional equation of the zeta distributions.

2.2. By Lemma 1 and (11), writing $v \in \Omega'_\varepsilon$ in the form $v = (\exp T_x^{(+)})(\sum t_k \varepsilon_k e_k)$, one has for $\operatorname{Re}\sigma_k > \frac{d}{2}(r-k)$

$$\begin{aligned}\Psi'_\varepsilon\Big(\hat{f}, \sigma - \frac{n}{r}\Big) &= \int_{\Omega'_\varepsilon} \hat{f}(v) \prod_{k=1}^{r} |\chi'_k(v)|^{\sigma_k} |N(v)|^{-n/r} d(v) \\ &= 2^{r-n} \int \Big[\int f(u) e\Big(\langle u, (\exp T_x^{(+)})\Big(\sum_{k=1}^{r} t_k \varepsilon_k e_k\Big)\rangle\Big) d(u)\Big] \\ &\quad \times \prod_{k=1}^{r} (t_k^{\sigma_k + (k-1)d - n/r} dt_k)\, d(x) ,\end{aligned}$$

where the integral is taken over $t_k \in \boldsymbol{R}_+$ $(1 \leqq k \leqq r)$, $x \in \underline{V}$ and $u \in V$. We write

$$u = \sum_{k=1}^{r} \xi_k e_k + \sum_{k<l} u_{kl}\ , \quad \xi_k \in \boldsymbol{R}\ , \quad u_{kl} \in V_{kl}\ ,$$
$$\xi_{kl}(x) = x'_{kl}\ , \qquad x' = \sum_{k<l} x'_{kl}\ ,$$
$$u_k = \sum_{i<k} u_{ik}\ , \qquad x'_k = \sum_{i<k} x'_{ik}\ .$$

Then, by (10), one has $d(x) = d(x') = \prod_{k=1}^{r} d(x'_k)$. Also, one has

$$\langle u, v\rangle = \sum_{k=1}^{r} \varepsilon_k t_k\Big(\xi_k + \frac{1}{2}\langle u_k, x'_k\rangle + \frac{1}{4}P_{k-1}(u)[x'_k]\Big)\ .$$

Following the method in [6], we define $Q_k = Q_k(u, \delta, \varepsilon_k) \in \mathrm{End}(V_k)$ with $\delta > 0$ by

$$Q_k = \delta 1 - \frac{\sqrt{-1}}{4}\varepsilon_k \rho_{k-1}(P_{k-1}(u)) \quad (2 \leqq k \leqq r)\ .$$

Then one has

$$\begin{aligned}\langle u, v\rangle &= \frac{\sqrt{-1}}{2}\lim_{\delta\to 0}\Big\{\delta\sum_{k=1}^{r} t_k(1 + \langle x'_k, x'_k\rangle) - 2\sqrt{-1}\,\langle u, v\rangle\Big\} \\ &= \frac{\sqrt{-1}}{2}\lim_{\delta\to 0}\sum_{k=1}^{r} t_k(\delta - 2\sqrt{-1}\varepsilon_k\xi_k + Q_k[x'_k] - \sqrt{-1}\varepsilon_k\langle u_k, x'_k\rangle)\ ,\end{aligned}$$

where we make a convention that $Q_1 = 1$, $u_1 = x'_1 = 0$. Therefore, one obtains

$$\begin{aligned}\Psi'_\varepsilon\Big(\hat{f}, \sigma - \frac{n}{r}\Big) &= 2^{r-n}\lim_{\delta\to 0}\int f(u)e\Big(\frac{\sqrt{-1}}{2}\sum_{k=1}^{r} t_k(\delta - 2\sqrt{-1}\varepsilon_k\xi_k + Q_k[x'_k] \\ &\qquad - \sqrt{-1}\,\varepsilon_k\langle u_k, x'_k\rangle)\Big)\prod_{k=1}^{r}(t_k^{\sigma_k+(k-1)d-n/r}dt_k)d(x)d(u) \\ &= 2^{r-n}\int_V f(u)\Big(\lim_{\delta\to 0}\prod_{k=1}^{r}\int_0^\infty F_k(t_k, u, \delta)\,dt_k\Big)d(u)\ ,\end{aligned}$$

where one puts

$$\begin{aligned}F_k(t_k, u, \delta) &= e\Big(\frac{\sqrt{-1}}{2}t_k\Big(\delta - 2\sqrt{-1}\varepsilon_k\xi_k + \frac{1}{4}Q_k^{-1}[u_k]\Big)\Big) \\ &\quad\times t_k^{\sigma_k+(k-1)d-n/r}\int_{V_k} e\Big(\frac{\sqrt{-1}}{2}t_k Q_k\Big[x'_k - \frac{\sqrt{-1}}{2}\varepsilon_k Q_k^{-1}u_k\Big]\Big)d(x'_k)\ .\end{aligned}$$

Since the last integral over V_k is equal to $\det(t_k Q_k)^{-1/2}$, one has

$$\begin{aligned}\int_0^\infty F_k(t_k, u, \delta)\,dt_k &= \det(Q_k)^{-1/2}\Gamma\Big(\sigma_k - \frac{d}{2}(r-k)\Big) \\ &\qquad\times\Big(\pi\Big(\delta - 2\sqrt{-1}\,\varepsilon_k\xi_k + \frac{1}{4}Q_k^{-1}[u_k]\Big)\Big)^{-\sigma_k+\frac{d}{2}(r-k)}\ .\end{aligned}$$

Now assume that $u \in \Omega_\eta$, $\eta = (\eta_k)$. Then, when δ tends to zero, one has by Lemma 2 and (14), (15).

$$\det(Q_k)^{-1/2} = N^{(k)}\Big(\delta 1^{(k)} - \frac{\sqrt{-1}}{4}\varepsilon_k P_{k-1}(u)\Big)^{-d/2}$$
$$= \prod_{l=1}^{k-1}\Big(\delta - \frac{\sqrt{-1}}{4}\varepsilon_k\eta_l|\chi_l(u)|\Big)^{-d/2}$$
$$\to 4^{\frac{d}{2}(k-1)}\Big(\prod_{l=1}^{k-1} e\Big(\frac{d}{8}\varepsilon_k\eta_l\Big)\Big)|N^{(k-1)}(P_{k-1}(u))|^{-d/2}$$

and

$$\Big(\delta - 2\sqrt{-1}\varepsilon_k\xi_k + \frac{1}{4}Q_k^{-1}[u_k]\Big)^{-\sigma_k+\frac{d}{2}(r-k)}$$
$$\to 2^{-\sigma_k+\frac{d}{2}(r-k)} e\Big(\frac{1}{4}\varepsilon_k\eta_k\Big(\sigma_k - \frac{d}{2}(r-k)\Big)|\chi_k(u)|\Big)^{-\sigma_k+\frac{d}{2}(r-k)} .$$

By (15) one has

$$\prod_{k=1}^{r}(|\chi_k(u)|^{r-k}|N^{(k-1)}(P_{k-1}(u))|^{-1}) = 1 .$$

Hence one has

$$\lim_{\delta\to 0}\prod_{k=1}^{r}\int_0^\infty F_k(t_k, u, \delta)dt_k$$
$$= (8\pi)^{\frac{d}{4}r(r-1)} u_{\varepsilon\eta}(\sigma)\prod_{k=1}^{r}\Big((2\pi)^{-\sigma_k} e\Big(\frac{1}{4}\sigma_k\Big)\Gamma\Big(\sigma_k - \frac{d}{2}(r-k)\Big)|\chi_k(u)|^{-\sigma_k}\Big) ,$$

where one puts

$$(20)\qquad u_{\varepsilon\eta}(\sigma) = e\Big(\frac{d}{8}\Big(\sum_{l<k}\varepsilon_k\eta_l - \sum_{k=1}^{r}\varepsilon_k\eta_k(r-k)\Big) + \frac{1}{4}\sum_{k=1}^{r}(\varepsilon_k\eta_k - 1)\sigma_k\Big) .$$

Thus one has by (16)

$$(21)\qquad \Psi'_\varepsilon\Big(\hat{f}, \sigma - \frac{n}{r}\Big) = (2\pi)^{-\Sigma\sigma_k} e\Big(\frac{1}{4}\sum_{k=1}^{r}\sigma_k\Big)\Gamma_{\Omega_0}(\sigma)\sum_{\eta\in\mathscr{E}} u_{\varepsilon\eta}(\sigma)\Psi_\eta(f, -\sigma) .$$

2.3. When $\sigma = s = (s, \cdots, s)$, it is clear that $\sum_{\varepsilon\in\mathscr{E}_i} u_{\varepsilon\eta}(s)$ depends only on the sign of η. Hence we set

$$(22)\qquad u_{ij}(s) = \sum_{\varepsilon\in\mathscr{E}_i} u_{\varepsilon\eta}(s) \quad \text{for} \quad \eta\in\mathscr{E}_j .$$

Then one has

$$\Phi_i\Big(\hat{f}, s - \frac{n}{r}\Big) = \sum_{\varepsilon\in\mathscr{E}_i}\Psi'_\varepsilon\Big(\hat{f}, s - \frac{n}{r}\Big)$$

$$= (2\pi)^{-rs} e\Big(\frac{r}{4}s\Big) \Gamma_{\Omega_0}(s) \sum_{j=0}^{r} \Big(u_{ij}(s) \sum_{\eta \in \mathscr{E}_j} \Psi_\eta(f, -s) \Big)$$

$$= (2\pi)^{-rs} e\Big(\frac{r}{4}s\Big) \Gamma_{\Omega_0}(s) \sum_{j=0}^{r} u_{ij}(s) \Phi_j(f, -s) .$$

Thus we have shown that, if one defines $u_{ij}(s)$ by (20) and (22), the formula (5) holds for any function f in $\mathscr{S}(V)$ whose support is compact and contained in $\bigcup \Omega_\varepsilon$. Hence, by Theorem 1 in [6], the same formula holds for all f in $\mathscr{S}(V)$.

Taking $\eta = (-1, \cdots, -1, 1, \cdots, 1)$ (-1 repeated j times), one has

$$(23) \qquad u_{ij}(s) = \sum_{\varepsilon \in \mathscr{E}_i} e\Big(\frac{d}{8}\Big(\sum_{k=1}^{j} \varepsilon_k(r-2k+1) - \sum_{k=j+1}^{r} \varepsilon_k(r-2k+2j+1)\Big) + \frac{1}{4}\Big(-\sum_{k=1}^{j} \varepsilon_k + \sum_{k=j+1}^{r} \varepsilon_k - r\Big)s\Big) .$$

Thus we have proved the following

THEOREM 1. *Let V be a formally real simple Jordan algebra. Then the tempered distribution $f \mapsto \Phi_i(f, s)$ defined by* (4) *satisfies a system of functional equations*

$$\Phi_i\Big(\hat{f}, s - \frac{n}{r}\Big) = (2\pi)^{-rs} e\Big(\frac{r}{4}s\Big) \Gamma_{\Omega_0}(s) \sum_{j=0}^{r} u_{ij}(s) \Phi_j(f, -s) \quad (0 \leqq i \leqq r) ,$$

where Γ_{Ω_0} and u_{ij} are given by (3) *and* (23).

REMARK. It can be shown that

$$(24) \qquad \underline{V} \ni x \mapsto \exp(e_r \square x_r) \cdots \exp(e_2 \square x_2) \in N$$

is a bijection of $\underline{V}$ onto N (cf. [3], [9]). One can give an alternate proof of Theorem 1 by using this parametrization instead of Lemma 1, (ii) and by proceeding by induction on r.

3. Properties of the matrix $U^{(r)}(x)$. In what follows, we put $x = e(-s/2)$ and write $u_{ij}(x)$ for $u_{ij}(s)$. Then $u_{ij}(x)$ is a polynomial in x of degree at most r. We consider the matrix $U(x) = U^{(r)}(x) = (u_{ij}(x))$.

From (23) one has

$$u_{ij}(x) = \sum_{\varepsilon \in \mathscr{E}_i} e\Big(\frac{d}{4}\Big(\sum_{k=1}^{j} \frac{1+\varepsilon_k}{2}(r-2k+1) - \sum_{k=j+1}^{r} \frac{1+\varepsilon_k}{2}(r-2k+2j+1)\Big) - \frac{s}{2}\Big(\sum_{k=1}^{j} \frac{1+\varepsilon_k}{2} + \sum_{k=j+1}^{r} \frac{1-\varepsilon_k}{2}\Big)\Big)$$

$$= \sum_{\varepsilon \in \mathscr{E}_i} \Big(\prod_{k=1}^{j} ((-1)^{dk} \sqrt{-1}^{\,d(r+1)} x)^{\frac{1}{2}(1+\varepsilon_k)} \prod_{k=j+1}^{r} ((-1)^{d(k-j)} \sqrt{-1}^{\,-d(r+1)})^{\frac{1}{2}(1+\varepsilon_k)} x^{\frac{1}{2}(1-\varepsilon_k)} \Big) .$$

Hence, putting $\zeta = \sqrt{-1}^{d(r+1)}$, one has

$$(25) \qquad \sum_{i=0}^{r} y^i u_{ij}(x) = \prod_{k=1}^{j} ((-1)^{dk}\zeta x + y) \prod_{k=j+1}^{r} ((-1)^{d(k-j)}\zeta^{-1} + xy) ,$$

which can also be written as

$$(25') \qquad \sum_{i=0}^{r} y^i u_{ij}(x) = \zeta^{-(r-j)} P_j(\zeta x, y) P_{r-j}(1, \zeta xy) ,$$

where

$$P_j(x, y) = \prod_{k=1}^{j} ((-1)^{dk}x + y) = \begin{cases} (x+y)^j & \text{for } d \text{ even} , \\ (x+y)^{[\frac{j}{2}]}(y-x)^{j-[\frac{j}{2}]} & \text{for } d \text{ odd} . \end{cases}$$

First, we consider the case where d is even. We distinguish two cases:

Case (a): $d \equiv 0 \pmod 4$ or $d \equiv 2 \pmod 4$ and r odd,

Case (a′): $d \equiv 2 \pmod 4$ and r even.

Then one has

$$\zeta = \begin{cases} 1 & \text{in Case (a)} , \\ -1 & \text{in Case (a')} . \end{cases}$$

THEOREM 2. *Let ρ_r denote the symmetric tensor representation of GL_2 of degree $r+1$. Then, when d is even, one has*

$$(26) \qquad U^{(r)}(x) = \begin{cases} \rho_r\left(\begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}\right) & \textit{in Case (a)} , \\ \rho_r\left(\begin{pmatrix} 1 & x \\ -x & -1 \end{pmatrix}\right) & \textit{in Case (a')} . \end{cases}$$

PROOF. In Case (a), (25) can be written as

$$(1, y, \cdots, y^r)U^{(r)}(x) = ((1+xy)^r, (x+y)(1+xy)^{r-1}, \cdots, (x+y)^r) .$$

For $r = 1$, one has $U^{(1)}(x) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$. Hence one obtains (26). The proof for Case (a′) is similar. (Note that in this case r is even.) q.e.d.

COROLLARY 1. *When d is even, the matrix $U(x)$ is diagonalizable.*

In fact, one has

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1+x & 0 \\ 0 & 1-x \end{pmatrix} ,$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & x \\ -x & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1-x \\ 1+x & 0 \end{pmatrix} .$$

Hence, putting $A^{(r)} = \rho_r\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right)$, one has

$$
(27)\qquad A^{(r)}U^{(r)}(x)A^{(r)-1} = \rho_r\left(\begin{pmatrix} 1+x & 0 \\ 0 & 1-x \end{pmatrix}\right) \quad \text{or} \quad \rho_r\left(\begin{pmatrix} 0 & 1-x \\ 1+x & 0 \end{pmatrix}\right)
$$

$$
= \begin{cases} \begin{pmatrix} (1+x)^r & & & 0 \\ & (1+x)^{r-1}(1-x) & & \\ & & \ddots & \\ 0 & & & (1-x)^r \end{pmatrix} & \text{in Case (a)}, \\ \begin{pmatrix} 0 & & & (1-x)^r \\ & & (1+x)(1-x)^{r-1} & \\ & \iddots & & \\ (1+x)^r & & & 0 \end{pmatrix} & \text{in Case (a}'\text{)}. \end{cases}
$$

This proves our assertion. In Case (a′), the eigen values of $U^{(r)}(x)$ are $(1-x^2)^{r/2}$ repeated $r/2+1$ times and $-(1-x^2)^{r/2}$ repeated $r/2$ times.

We note that the matrix $A^{(r)} = (a_{ij})$ is determined by the identity

$$
(28)\qquad (1+y+z-yz)^r = \sum_{i,j=0}^{r} \binom{r}{j} a_{ij} y^i z^j .
$$

COROLLARY 2. *When d is even, $U^{(r)}(x)$ satisfies the functional equation*

$$
(29)\qquad U^{(r)}(x)U^{(r)}\left(-\frac{\zeta}{x}\right) = \left(x - \frac{1}{x}\right)^r J^{(r)} ,
$$

where

$$
J^{(r)} = \rho_r\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}.
$$

This follows from the relation

$$
\begin{pmatrix} 1 & x \\ \zeta x & \zeta \end{pmatrix}\begin{pmatrix} 1 & -\zeta x^{-1} \\ -x^{-1} & \zeta \end{pmatrix} = \zeta(x - x^{-1})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .
$$

((29) follows also from (5).)

The formulas (5) and (27) imply that, if one puts

$$\Phi'_i(f, s) = \sum_{j=0}^{r} a_{ij}\Phi_j(f, s) \quad (0 \leqq i \leqq r) ,$$

then one has

$$(30) \qquad \Phi'_i\Big(\hat{f}, s - \frac{n}{r}\Big)$$
$$= (2\pi)^{-rs}\Gamma_{\Omega_0}(s)\Big(e\Big(\frac{r}{2}s\Big) + e\Big(-\frac{r}{2}s\Big)\Big)^{r-i}\Big(\Big(\frac{r}{2}s\Big) - e\Big(-\frac{r}{2}s\Big)\Big)^{i}$$
$$\times \begin{cases} \Phi'_i(f, -s) & \text{in Case (a)} , \\ \Phi'_{r-i}(f, -s) & \text{in Case (a}') . \end{cases}$$

Next, we consider the case where d is odd. According to the classification theory, we have the following two possibilities:

Case (b): $r = 2$ and d odd ($n = 2 + d$),

Case (c): r arbitrary and $d = 1$ $\Big(n = \frac{1}{2}r(r + 1)\Big)$.

In Case (b), one has by (25)

$$\sum y^i u_{ij}(x) = \begin{cases} (-\zeta^{-1} + xy)(\zeta^{-1} + xy) = 1 + x^2y^2 \quad (j = 0) , \\ (-\zeta x + y)(-\zeta^{-1} + xy) = x + \zeta(1 - x^2)y + xy^2 \quad (j = 1) , \\ (-\zeta x + y)(\zeta x + y) = x^2 + y^2 \quad (j = 2) , \end{cases}$$

where $\zeta = \sqrt{-1}^{\,n}$. Hence $U^{(2)}(x)$ is given by

$$(31) \qquad U^{(2)}(x) = \begin{pmatrix} 1 & x & x^2 \\ 0 & \sqrt{-1}^{\,n}(1 - x^2) & 0 \\ x^2 & x & 1 \end{pmatrix}.$$

Thus one see that $U^{(2)}(x)$ is again diagonalizable with simple eigen values $1 + x^2$, $1 - x^2$, $\sqrt{-1}^{\,n}(1 - x^2)$. This case was treated in [8].

The Case (c) is the one treated in [7]. The case $r = 2$ is contained in Case (b), while the case $r = 1$ may be included in Case (a), because for $r = 1$ the number d is actually undetermined. Hence $U^{(r)}(x)$ is diagonalizable for $r = 1, 2$. But, in general, it is not known whether $U^{(r)}(x)$ is diagonalizable or not.

It can be shown by (5) that, when d is odd, $U^{(r)}(x)$ satisfies the following functional equation

$$(32) \qquad U^{(r)}(x)U^{(r)}(\zeta^{-1}x^{-1}) = (x + x^{-1})^{[\frac{r}{2}]}(x - x^{-1})^{r-[\frac{r}{2}]}J^{(r)} .$$

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