Tôhoku Math. Journ.
36 (1984), 407-426.

TOWARDS AN ALGEBRO-GEOMETRIC INTERPRETATION OF THE NEUMANN SYSTEM

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(Received September 7, 1983)

Abstract. Lax equations and constants of motion for C. Neumann's system of constrained harmonic oscillators are derived in a systematic way from the Burchnall-Chaundy-Krichever theory of 2nd-order differential operators $D^2+q(t)$. The approach is based on a geometric step: to map the algebraic curve and linebundle associated with $D^2+q(t)$ to a larger projective space by means of a suitable linear system. The image of $D^2+q(t)$ is, roughly speaking, just the Lax operator for the Neumann system.

1. Introduction. In the 1920's, J. L. Burchnall, T. W. Chaundy, and H. F. Baker studied differential operators $L = D^n + q_2(t)D^{n-2} + \cdots + q_n(t)$ $(D = (d/dt))$ that commute with at least one other differential operator B of some order m relatively prime to n [2 − 5]. They realized that the commutant

$$\bar{L} = \{\text{diff. ops. } A \,|\, [L, A] = 0\}$$

of such an L has the structure of the affine coordinate ring of an algebraic curve $\mathscr{R}$,

$$\bar{L} \cong C[x_1, \cdots, x_r]/\text{ideal}\ ,$$

and, especially in Baker's paper [2], that the determination of the operator L from the curve $\mathscr{R}$ requires some additional data: most important among those, when $\mathscr{R}$ is a smooth affine curve, is a linebundle with a one-dimensional space of holomorphic sections. From a suitable section, one constructs an eigenfunction ψ of L, and then recovers L itself.

These remarkable results were largely forgotten, and eventually rediscovered, with improvements, by Krichever [13] in the late 1970's. His motivation was the recently established connection between commuting differential operators and soliton solutions of integrable partial differential equations, such as the Korteweg-de Vries equation [9, 11, 13, 14, 17]. In the last few years, the relation between linebundles over algebraic curves and special differential operators has led to the discovery of many amazing "coincidences". In this paper, I will explain one such "coincidence", the occurrence of integrable constrained oscillator systems (particularly, the classical one studied by C. Neumann) in the theory of commuting differential operators, by a simple and fairly natural geometric construction.

There is more background to be covered before the results can be made precise. I *will consider only second-order operators* $L = D^2 + q(t)$ *that commute with an operator* B *of odd order*. B will be taken to be of minimal order amongst all the odd-order operators commuting with L. Then there is an irreducible polynomial equation in $\boldsymbol{C}[L, B]$,

$$B^2 - P(L) = 0$$

satisfied by L and B. The explicit construction of all the L's and B's subject to this particular relation involves a common eigenfunction $\psi(t, (x, y))$ of

$$L\psi = x\psi$$
$$B\psi = y\psi .$$

Such a ψ exists precisely because L and B commute, and the eigenvalues x, y are related by

$$y^2 - P(x) = 0 .$$

Put differently, ψ is parametrized by the points $p = (x, y)$ of the hyperelliptic curve $\mathscr{R}$: $y^2 - P(x) = 0$, *which will be assumed to be smooth.* In Section 2, further details of this theory, which goes back to Baker [2], will be reviewed.

This is one ingredient of the present paper. The other ingredient, an observation apparently due to J. Moser and E. Trubowitz, is a surprising connection between these operators $D^2 + q(t)$ and a certain mechanical problem proposed and solved in 1853 by C. Neumann, probably to illustrate the power of the (then) new ultra-elliptic integrals (= hyperelliptic, of genus 2) [18]. Here is the "less famous Neumann problem". Take n independent linear oscillators,

$$\ddot{\xi}_j = x_j\xi_j$$

and impose an external force q,

$$\ddot{\xi}_j + q\xi_j = x_j\xi_j$$

chosen so that the constraints

$$\sum_{i=1}^{n} \xi_i^2 = 1, \sum_{i=1}^{n} \xi_i\dot{\xi}_i = 0 \tag{1}$$

on the displacements ξ_j are satisfied for all time. It is easy to see that the constraining force q must be

$$q = \sum_{i=1}^{n} x_i \xi_i^2 + \dot{\xi}_i^2 \tag{2}$$

The resulting system,

$$\ddot{\xi}_j + \left(\sum_{i=1}^{n} x_i \xi_i^2 + \dot{\xi}_i^2\right)\xi_j = x_j \xi_j \quad (j = 1, \cdots, n)\,, \tag{3}$$

was solved via (ultra-elliptic) quadratures by Neumann for $n = 3$. Over 100 years later, it resurfaced in some geometric investigations ([25]—I have not seen this paper), as a counterexample to dynamical systems misconceptions [8], as one of a whole class of integrable Hamiltonian equations living on coadjoint orbits of various infinite-dimensional Lie algebras [1, 19, 21], and finally as prototype of a number of remarkable dynamical systems related to the geometry of quadric surfaces in projective space [12, 15]. Neumann's system is clearly not a run-of-the-mill o.d.e.

Its relation to $L = D^2 + q(t)$ is suggested by the oscillator equation $\ddot{\xi}_j + q\xi_j = x_j\xi_j$. One may choose [14] certain values of x in the eigenvalue problem $L\psi = x\psi$ – say, $x_1, \cdots, x_n$ – and corresponding eigenfunctions $\xi_1, \cdots, \xi_n$ (taken with specific normalizations) such that

$$\sum_{i=1}^{n} \xi_i^2 = 1 \quad \text{and} \quad q(t) = \sum_{i=1}^{n} x_i \xi_i(t)^2 + \dot{\xi}_i(t)^2\,. \tag{4}$$

When this representation of $q(t)$ is inserted into the eigenvalue equation

$$\ddot{\xi}_j + q\xi_j = x_j\xi_j\,,$$

the Neumann system (3) is recovered.

Now, while it is quite easy to *derive* these formulas, and so to get the Neumann system from the Baker-Burchnall-Chaundy-Krichever theory of $D^2 + q(t)$, the actual solution (by hyperelliptic quadratures) of the Neumann system required some inspired guesses [15, 16]. It ought to be possible to be possible to say: take all that is known about $D^2 + q(t)$, do such-and-such, and arrive at all that is known about the Neumann system. This paper takes a step towards that goal. It will be shown that the Neumann sytem provides, in a sense to be made precise, the minimal projective model of the $D^2 + q(t)$/curve $\mathscr{R}$/Baker linebundle-setup. Within a projective model, one might be able to see the geometry of quadrics [15] appearing, but that remains to be worked out.

The content of the rest of this paper is the following. In Section 2, I review the facts about commuting pairs of operators L, B, with

$L = D^2 + q(t)$. As already mentiond, there arises, in a natural way, a curve $\mathscr{R}: y^2 - P(x) = 0$; moreover, there is a one-parameter family of linebundles, $\{\mathscr{L}_t\}$, such that an appropriately normalized section of $\mathscr{L}_{t_0}$ produces the common eigenfunction ψ of

$$L\psi = x\psi$$
$$B\psi = y\psi$$

at $t = t_0$. The discussion is only a very concrete version of what can be found in [10, 17].

In Section 3, I ask: how can the curve $\mathscr{R}$ and the family $\{\mathscr{L}_t\}$ be realized geometrically? It will be remembered that each $\mathscr{L}_t$ has a one-dimensional space of regular sections. This is not an easy geometric picture to visualize: the linear system corresponding to $\mathscr{L}_t$ is not cut out on $\mathscr{R}$ by a simple family of curves. One might therefore try to increase the dimension of the linear systems corresponding to $\{\mathscr{L}_t\}$, most obviously by forming the tensor products $\{\mathscr{L}_t \otimes \mathscr{L}(Z)\}$, with $\mathscr{L}(Z)$ the linebundle corresponding to some t-independent divisor Z, so that $\mathscr{L}_t \otimes \mathscr{L}(Z)$ has enough sections to allow a reasonable map into projective space. This can be done, but in very many ways, so I impose one more requirement: I ask that the family $\{\mathscr{L}_t \otimes \mathscr{L}(Z)\}$ correspond to a pair of commuting first-order *matrix* differential operators, just as $\{\mathscr{L}_t\}$ determined the scalar operators L, B. So, I look for a projective image of $\mathscr{R}$ that "respects" the family $\{\mathscr{L}_t\}$, and realizes the commutant of L in the commutant of a first-order matrix differential operator $\boldsymbol{L}$. An embedding cannot always be achieved, but in any case, the least-dimensional reasonable map turns out to produce the Neumann systm.

Exactly how, will be explained in Section 4. There I derive the explicit formulas — constants of motion and Lax pairs — that form the basis of [1, 15, 19, 20]. It turns out that there are two types of oscillator systems that can arise, depending on the divisor Z used in the extension of $\mathscr{L}_t$ by $\mathscr{L}(Z)$. If Z is supported at Weierstrass points of $\mathscr{R}$, one recovers the Neumann system (3); otherwise, one arrives at another mechanical system, studied in 1877 by E. Rosochatius [22].

Section 5 describes, very briefly, the directions of already accomplished [23, 24] and still hoped-for generalizations, and comments on the case of of a singular curve $\mathscr{R}$.

Acknowledgements. The explict formulas in Section 4 were obtained while I was supported by a fellowship at RIMS, Kyoto University, in 1980-81. The explanation given in Section 3 was invented only recently; I am indebted to *N.* Ercolani and *R.* Schilling for discussions that helped me to understand my own computations. I also acknowledge support from NSF Grant MCS-80-02748A01, and DOA Contract DAAG 29-82-K-0068, and AFOSR-83-0227.

2. Review of Baker functions and linebundles. Let $L = D^2 + q(t)$, and let $B = D^{2g+1} + u_2(t)D^{2g-1} + \cdots + u_{2g+1}(t)$ be the operator of minimal odd order commuting with L. It is shown in [17] that there is only one such B, up to constant scalar multiples. One can learn from [3, 10, 17], furthermore, that L and B satisfy an irreducible polynomial equation

$$B^2 = P(L) ,$$

where

$$P(x) = \prod_{k=1}^{2g+1} (x - a_k) .$$

If P has no repeated factors, the affine curve $\mathscr{R}: y^2 - P(x) = 0$ will be nonsingular; *this is assumed from now on.* (See Section 5 for some comments about singular curves). Let $\bar{\mathscr{R}}$ be the projective one-point completion $\mathscr{R} \cup \{x = y = \infty\}$. Krichever [13] proves that a unique common solution $\psi(t, p)$ of

$$L\psi = x\psi ,$$
$$B\psi = y\psi ,$$

with $p = (x, y) \in \mathscr{R}$, is determined by the following requirements:

(5)
- (i) For each $t \in \boldsymbol{C}$ sufficiently close to 0, $\psi(t, p)$ is a meromorphic function on $\mathscr{R}$;
- (ii) The poles of $p \to \psi(t, p)$ are in a nonspecial divisor $\delta = \delta_1 + \cdots + \delta_g$ independent of t;
- (iii) $\psi(t = 0, p) \equiv 1$ on $\bar{\mathscr{R}}$;
- (iv) Near ∞, $\psi(t, p)e^{-\sqrt{x}t} = 1 + 0(x^{(-1/2)})$ is holomorphic.

The key to this set of conditions is that, at $t = 0$, all solutions of $L\psi = x\psi$, $B\psi = y\psi$ will vanish for some pairs $p = (x, y) \in \mathscr{R}$. Requiring $\psi(0, p) \equiv 1$ therefore amounts to a normalization by dividing by 0 at those p, which introduces poles at those points of $\mathscr{R}$ for all t. $\psi(t, p)$ is called a *Baker function*, in honor of H. F. Baker's original insights [2].

A more geometric description of Krichever's conditions (5) involves holomorphic linebundles over $\bar{\mathscr{R}}$.

First, $\mathscr{L}(\delta)$ be the linebundle corresponding to the nonspecial divisor $\delta = \delta_1 + \cdots + \delta_g$ in (5ii). This means: let $\{U_\alpha\}$ cover $\bar{\mathscr{R}}$, and choose f_α in

U_α so that f_α has zeros only at $\delta_1, \cdots, \delta_g$. The transition functions $g_{\alpha\beta} = f_\beta/f_\alpha$ define $\mathscr{L}(\delta)$. For sake of definiteness, pick a neighborhood U_∞ of ∞ that excludes all of δ, and take $f_\infty \equiv 1$.

Next, define a holomorphic linebundle $\mathscr{L}_t'$ as follows: The transition functions $g'_{\alpha\beta}$ are $\equiv 1$ if neither α nor β are ∞, while

$$g'_{\alpha\infty}(p) = e^{-\sqrt{x}t} \quad \text{in} \quad U_\alpha \cap U_\infty \,. \tag{6}$$

One can check that $\mathscr{L}_t'$ has Chern class zero; it corresponds to the divisor

(7) zeros $-$ poles, on the affine curve $\mathscr{R}$, of *any* Baker function associated to $\mathscr{R}$ (i.e., *any* function determined by the properties (5), with possibly different δ).

Set $\mathscr{L}_t = \mathscr{L}(\delta) \otimes \mathscr{L}_t'$. Now I verify that a normalized holomorphic section σ of $\mathscr{L}_t$ defines a Baker function with the properties (5). The transition functions of $\mathscr{L}_t$ are

$$\begin{aligned} h_{\alpha\beta} &= f_\beta/f_\alpha,\ \alpha, \beta \neq \infty \\ h_{\alpha\infty} &= e^{-\sqrt{x}t} f_\alpha \,. \end{aligned} \tag{8}$$

The section σ satisfies, in local coordinates,

$$\sigma_\beta = h_{\alpha\beta}\sigma_\alpha \,.$$

Away from ∞,

$$\frac{\sigma_\beta}{f_\beta} = \frac{\sigma_\alpha}{f_\alpha} \underset{\text{def}}{=} F$$

is meromorphic with poles at most δ. At ∞,

$$\sigma_\infty = \frac{e^{-\sqrt{x}t}}{f_\alpha}\sigma_\alpha \,,$$

so $F = (\sigma_\alpha/f_\alpha) = \sigma_\infty e^{\sqrt{x}t}$: F continues as (holomorphic function) $\times e^{\sqrt{x}t}$ up to ∞.

At $t = 0$, the meromorphic function F must be identically 1, because δ is nonspecial; in particular, $\sigma_\infty(\infty) \neq 0$ for small t, and so it can be normalized to be 1. Hence, F has poles δ, is $\equiv 1$ at $t = 0$, and goes like $e^{\sqrt{x}t}(1 + \cdots)$ at ∞; it must therefore coincide with the Baker function described by (5).

The linebundles $\mathscr{L}_t$ have a useful alternate description. From (7), one can write $\mathscr{L}_t' \cong \mathscr{L}(z(t) - \delta)$, where $z(t)$ is the zero divisor of ψ, which turns out to depend on t (whereas the pole divisor δ does not). Then $\mathscr{L}_t \cong \mathscr{L}(\delta) \otimes \mathscr{L}_t' \cong \mathscr{L}(\delta) \otimes \mathscr{L}(z(t) - \delta)$, whence

$$\mathscr{L}_t \cong \mathscr{L}(z(t)) \,. \tag{9}$$

$z(t)$ is often called the "tied" or "auxiliary" spectrum in the literature on Hill's equation [14].

REMARK. One can see quite easily that the dependence of q upon t amounts to linear motion in the Jacobian. Because $\mathscr{L}'_{t_1+t_2} \cong \mathscr{L}'_{t_1} \otimes \mathscr{L}'_{t_2}$ (the transition functions (6) of the two sides are equal), $t \to \mathscr{L}'_t$ is a local one-parameter subgroup of $\mathrm{Pic}_0(\bar{\mathscr{R}}) \cong \mathrm{Jac}(\bar{\mathscr{R}})$. $\mathscr{L}_t$ may then be thought of as a subgroup of $\mathrm{Jac}(\bar{\mathscr{R}})$ as well, via the isomorphism between $\mathrm{Pic}_0(\bar{\mathscr{R}})$ and $\mathrm{Pic}_g(\bar{\mathscr{R}})$ provided by tensoring with some fixed basepoint $\mathscr{L}_0$ of Chern class g.

3. Augmenting the Baker bundle. With these preliminaries disposed of, the relation between the Neumann system and the operator L can be investigated. Let $D^2 + q(t)$, the curve $\mathscr{R}$, and the bundle $\mathscr{L}_t \cong \mathscr{L}(\delta) \otimes \mathscr{L}'_t$ be given. The construction—whose details are postponed to the next section—is, in outline, as follows.

Taking a nonspecial divisor Z of degree g, one forms bundles $\tilde{\mathscr{L}}_t = \mathscr{L}(\delta) \otimes \mathscr{L}(Z) \otimes \mathscr{L}'_t$. From $g + 1$ appropriate holomorphic sections, one builds a vector Baker function,

$$\Psi(t, p) = \begin{bmatrix} \psi_1(t, p) \\ \vdots \\ \psi_{g+1}(t, p) \end{bmatrix} . \tag{10}$$

This Ψ will have poles in $\delta + Z$, and exponential behavior at $g + 1$ points of $\mathscr{R}$. As in the case of a single Baker function ψ discussed earlier, there are commuting differential operators $\boldsymbol{L}$ and $\boldsymbol{B}$, now $(g + 1) \times (g+1)$ *matrices*, of which Ψ is a common eigenfunction. The statement $[\boldsymbol{L}, \boldsymbol{B}] = 0$ amounts to the Neumann system; this representation is essentially the Kac-Moody Lax pair of [1, 19].

The key steps are the construction of the vector Baker function Ψ in (10) from the original ψ, and the calculation of the matrix operators $\boldsymbol{L}$ and $\boldsymbol{B}$. If taken in isolation, they appear to be quite unmotivated. This section is intended to make the whole procedure seem reasonable.

As just mentioned, one first augments the bundle $\mathscr{L}_t = \mathscr{L}(\delta) \otimes \mathscr{L}'_t$ by tensoring with the bundle of a divisor Z. There are two reasons for trying this even if the Neumann system were of no concern.

(1) The resulting $\tilde{\mathscr{L}}_t$'s have more independent holomorphic sections than the original $\mathscr{L}_t$; these can be used to map $\bar{\mathscr{R}}$ to a projective space. From a geometric point of view, it would be interesting to have concrete models for the whole setup—to have the important pole divisor δ cut out

by hyperplanes, for instance. When one remembers the (still mysterious) geometric setting of the Neumann system [12, 15], the idea of constructing a projective model of the Baker function begins to seem quite sensible.

(2) One might inquire whether the commutative algebra generated by the scalar operators L, B has a "representation" by a commutative algebra of matrix operators. This concept must first be defined, of course, but the construction adopted here is a reasonable possibility. In this paper, I admit only *first* order matrix operators. Higher order operators seem to lead to interesting geometry, but have not been explored in any detail.

In this paper, both topics, whatever their independent interest, are used only as motivation. The specific question posed, which combines (1) and (2), is this:

> Find the divisor(s) Z of smallest degree for which $\mathscr{L}(\delta)\otimes\mathscr{L}(Z)\otimes\mathscr{L}_t'$ defines a commuting pair of first-order matix differential operators.

It is necessary to summarize certain results on commuting matrix differential operators, mostly due to Krichever [13]. (Several proofs in [13] are wrong; the only correct argument I know of is given in [23]).

On a smooth curve $\bar{\mathscr{R}}$ of genus $g>0$, let there be given a function h, with distinct simple poles $\boldsymbol{b}=b_1+\cdots+b_l$, and a nonspecial divisor Δ of degree of $g+l-1$. Let $C_1,\cdots,C_l$ be distinct constants.

There exist unique functions $\psi_\alpha(t, p)$, such that

(i) ψ_α is meromorphic on $\bar{\mathscr{R}}-\boldsymbol{b}$ with poles at worst Δ, (11)

(ii) $\psi_\alpha(t,p)=e^{C_\beta h(p)t}\underset{\text{holomorphic}}{(\delta_{\alpha\beta}+\cdots)}$ near b_β.

The $\psi_\alpha(t,p)$ turn out to be analytic for small t away from $p\in\boldsymbol{b},\Delta$. There is then a unique first-order, $l\times l$ matrix differential operator $\boldsymbol{L}$, $\boldsymbol{L}=C^{-1}D+P$ ($D=(d/dt)$, $C=\operatorname{diag}(C_1,\cdots,C_l)$) such that

$$\boldsymbol{L}\Psi(t,p)=h(p)\Psi(t,p)\,. \tag{12}$$

Ψ is the vector $(\psi_1,\cdots,\psi_l)^t$.

As in Section 2, there is a linebundle description. Let $\mathscr{L}_t^{(\alpha)}$ be the bundle with all transition functions $=1$, except for $g_{\beta b_\alpha}(p)=e^{-C_\alpha h(p)t}$ in $U_\beta\cap U_{b_\alpha}$. The functions $\psi_1,\cdots,\psi_l$ define independent holomorphic sections of the bundle

$$\tilde{\mathscr{L}}_t=\mathscr{L}(\Delta)\otimes\mathscr{L}_t^{(1)}\otimes\cdots\otimes\mathscr{L}_t^{(l)}\,. \tag{13}$$

The dependence of $\tilde{\mathscr{L}}_t$ upon t is again linear in the Jacobian.

PROPOSITION 1. *Let $\mathscr{R}$ be a smooth affine hyperelliptic curve of genus $g > 0$, as at the beginning of Section 2. Let $\mathscr{L}_t = \mathscr{L}(\delta) \otimes \mathscr{L}_t'$* be *a Baker bundle on $\bar{\mathscr{R}}$, as in section 2. $\mathscr{L}_t \otimes \mathscr{L}(Z)$ is isomorphic to a bundle* (13) *of a vector Baker function, i.e.,*

$$(14) \qquad \mathscr{L}(\delta) \otimes \mathscr{L}_t' \otimes \mathscr{L}(Z) \cong \mathscr{L}(\delta + Z) \otimes \mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(l)} ,$$

for any divisor Z of degree g such that $\delta + Z$ is nonspecial, but for no Z of smaller degree.

PROOF. Let Z be a divisor of degree $l - 1$. The possibility of an isomorphism (14) will turn out to be determined solely by the degree of Z, which must be one less than the number of poles of h. There are two conditions that restrict h:

(i) The function h has l simple poles on $\mathscr{R}$; call them $b_1, \cdots, b_l$. (Note: it is not assumed that $b_1 + \cdots + b_l$ is nonspecial).

(ii) there is an isomorphism $\mathscr{L}_t' \cong \mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(l)}$, or

$$\mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(l)} \otimes (\mathscr{L}_t'^{-1}) \cong \text{trivial bundle} .$$

Suppose that this holds. Then the bundle on the left has a nonvanishing holomorphic section σ, which defines a nonvanishing function F meromorphic on $\bar{\mathscr{R}} - \{\infty, b_1, \cdots, b_l\}$ (∞ is the point $x = y = \infty$ from earlier), with the behavior

$$(15) \qquad \begin{aligned} F(p) &= e^{C_\alpha h(p) t} \sigma_{b_\alpha}(p) \quad \text{near} \quad b_\alpha , \\ &= e^{-\sqrt{x(p)}t} \sigma_\infty(p) \quad \text{near} \quad \infty . \end{aligned}$$

$G = \log F$ is well defined, meromorphic on $\bar{\mathscr{R}}$, with simple poles $\infty, b_1, \cdots, b_l$. Conversely, if such a G exists, (ii) is easily seen to be true. So (ii) can be replaced by (ii′): there is a function G with simple poles $\infty + b_1 + \cdots + b_l$. If $l \leqq g$, then $b_1 + \cdots + b_l$ is a special divisor, and so has a piece of the form $b_1' + \tau b_1' + b_2' + \tau b_2' + \cdots$, where τ is the hyperelliptic involution. One can then form the product $\Pi(x - b_k') \cdot G$, which has a pole of order $\leqq l + 1$ at ∞ and no other poles. But the function y has a pole of order $2g + 1$ at ∞, and that is the minimal odd-order pole there. Hence $2g + 1 \leqq l + 1 \leqq g + 1$, or $g \leqq 0$ — contradiction.

It follows that $l \geqq g + 1$, and in fact for $l = g + 1$, requirements (i), (ii′) can be met. Let $\boldsymbol{b} = b_1 + \cdots + b_{g+1}$ be a nonspecial divisor (not including ∞), and let h be a function with simple poles $\boldsymbol{b}$. There is a unique h with a zero $h(p) = (1/\sqrt{x(p)}) + \cdots$ at ∞. The function $G(p) = -x(p)h(p)$ then satisfies (ii), and $e^{G(p)t}$ satisfies (15). □

It should be understood that this proposition says only that there *are* commuting first-order matrix operators $\boldsymbol{L}, \boldsymbol{B}$ (of rank at least $g + 1$)

whose dependence on t leads to the same straight line in the Jacobian as does the dependence of $q(t)$ on t; the identification of $\mathrm{Pic}_{2g}(\bar{\mathscr{R}})$ and $\mathrm{Pic}_g(\bar{\mathscr{R}})$ with $\mathrm{Jac}(\bar{\mathscr{R}})$,

$$\mathrm{Pic}_{2g}(\bar{\mathscr{R}}) \xrightarrow{\otimes \mathscr{L}(Z)^{-1}} \mathrm{Pic}_g(\bar{\mathscr{R}}) \xrightarrow{\otimes \mathscr{L}_0^{-1}} \mathrm{Jac}(\bar{\mathscr{R}})$$

depends on $\boldsymbol{Z}$, but *not on h*, because any two of the bundles denoted above by $\mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(g+1)}$ are isomorphic. It is evident that there must be a close relation between $\boldsymbol{L}, \boldsymbol{B}$ and L, B, but *that the Neumann system* (3) *emerges* is seen only after specific choices of $\boldsymbol{Z}$ and of sections of the augmented bundle are made.

4. **Calculation of Neumann and Rosochatius systems.** This section finally gives the concrete formulas only alluded to so far:

(i) the expression (4) of $q(t)$ in terms of selected eigenfunctions;

(ii) the construction of the vector Baker function Ψ in terms of the scalar Baker function ψ;

(iii) the explicit form of the Lax pair $[\boldsymbol{L}, \boldsymbol{B}] = 0$ for the Neumann and Rosochatius systems.

A. TRACE FORMULAS. Fix a curve $\mathscr{R}$: $y^2 = \prod_{k=1}^{2g+1} (x - a_k)$, and fix a Baker function $\psi(t, p)$ with poles in a nonspecial divisor δ.

LEMMA 1 [6]. *There is a unique differential Ω with zeros δ and a double pole of the form $(-x + \text{holomorphic})dx^{-1/2}$ at ∞.*

PROOF. By the Riemann-Roch theorem,

$$\dim\{f \mid (f) + \delta - 2\infty \geqq 0\} - \dim\{\omega \mid (\omega) \geqq \delta - 2\infty\} = (g-2) - g + 1 .$$

Since δ is nonspecial, the dimension of the first space is zero, and so there is just one differential Ω – up to scalar multiples—with $(\Omega) \geqq \delta - 2\infty$. Ω must have poles, because δ is nonspecial; it cannot have a simple pole at ∞, because $\sum \operatorname{res} \Omega = 0$, so that Ω has a double pole at ∞. The desired normalization can of course be achieved. $\square$

DEFINITION 1. [6] Let δ' be the (necessarily nonspecial) divisor of degree g such that the $2g$ zeros of Ω are $\delta + \delta'$. Let $\phi(t, p)$ be the uniquely determined Baker function which

(i) is $\equiv 1$ at $t = 0$,

(ii) is meromorphic on $\mathscr{R}$ with poles δ',

(iii) has behavior $e^{-\sqrt{x}t}(1 + \underset{\text{holomorphic}}{\cdots})$ at ∞.

δ' and ϕ are said to be *dual* to δ, ψ, respectively.

REMARK 1. The lemma and definition work for arbitrary curves $\mathscr{R}$, not necessarily hyperelliptic [6, 23]. If $\mathscr{R}$ is hyperelliptic, and if τ is the hyperelliptic involution, then $\delta' = \tau\delta$ and $\phi(t, p) = \psi(t, \tau p)$.

REMARK 2. The differential Ω has a simple expression in terms of ψ and ϕ. When $\mathscr{R}$ is the hyperelliptic curve under discussion, one checks easily that

$$\Omega = \frac{dx}{W(\psi(t, p), \phi(t, p))} \tag{16}$$

where W is the Wronskian, $\dot{\psi}\phi - \psi\dot{\phi}$. For non-hyperelliptic $\mathscr{R}$, the Wronskian is replaced by the Lagrange bilinear form $(\psi; \phi)$ familiar from the theory of boundary value problems for ordinary differential operators (see [23, 24]).

Because the exponential factors cancel, the product $\psi\phi$ is meromorphic on $\bar{\mathscr{R}}$; in fact, it is rational in x. The differential $\psi\phi\Omega$ has no poles except ∞, because (by definition) Ω has zeros at the poles of ψ and ϕ. One now picks a function h with selected poles; the identity $\sum_{p \in} \operatorname{Res}_p h\psi\phi\Omega = 0$ will lead to identies for eigenfunctions analogous to the constraint $\sum \xi_j^2 = 1$ in equ. (1) of the Introduction.

NOTATION. Fix a nonspecial divisor $\boldsymbol{b}$ of degree $g + 1$, $\boldsymbol{b} = b_1 + \cdots + b_{g+1}$, with all b_j distinct, and different from δ and ∞. There exists a unique function h with poles in $\boldsymbol{b}$ and a zero at ∞, normalized to $h(p) = x(p)^{-1/2} + \cdots$ as $p \to \infty$, and regular at all other points of $\mathscr{R}$. h has g finite zeros; let $Z = Z_1 + \cdots + Z_g$ be that divisor. Assume that $Z \cap \delta = \phi$, and that $Z + \delta$ is nonspecial. (One could become embroiled in cataloging non-generic situations, which is not interesting—hence, these assumptions are convenient).

PROPOSITION 2. *Let*

$$\rho_j = \operatorname{Res}_{b_j} h\Omega,\ \xi_j(t) = \sqrt{\rho_j}\psi(t, b_j),\ \eta_j(t) = \sqrt{\rho_j}\phi(t, b_j),\ x_j = x(b_j)\ .$$

Then

$$\sum_{j=1}^{g+1} \xi_j\eta_j = 1 \tag{17}$$

$$\sum_{j=1}^{g+1} \dot{\xi}_j\eta_j + \xi_j\dot{\eta}_j = 0\ , \tag{18}$$

$$\sum_{j=1}^{g+1} x_j\xi_j\eta_j + \dot{\xi}_j\dot{\eta}_j = q(t)\ . \tag{19}$$

Furthermore, the ξ_j, η_j satisfy the coupled Neumann systems

(20)
$$\ddot{\xi}_j + (\sum x_i\xi_i\eta_i + \dot{\xi}_i\dot{\eta}_i)\xi_j = x_j\xi_j ,$$
$$\ddot{\eta}_j + (\sum x_i\xi_i\eta_i + \dot{\xi}_i\dot{\eta}_i)\eta_j = x_j\eta_j .$$

PROOF. Suppose that (17) is known to be true. (18) is just the t-derivative of (17). Differentiate (18) once more, and use (17) and the eigenvalue equations $\ddot{\xi}_j = (x_j - q)\xi_j$, $\ddot{\eta}_j = (x_j - q)\eta_j$: (19) drops out. To get (20), replace q by the expression (19) just derived in the eigenvalue equations for ξ_j, η_j.

To prove (17), consider the differential $h\psi\phi\Omega$. Because of the normalization at ∞ of the four quantities, there is a simple pole at ∞ with residue -1. The other poles are at the b_j, with residues

$$\operatorname*{Res}_{b_j} h\Omega\cdot\psi(t, b_j)\phi(t, b_j) \underset{\text{def}}{=} \xi_j(t)\eta_j(t) .$$

$\sum_{p\in\mathscr{R}} \operatorname{Res}_p h\psi\phi\Omega = 0$ now gives (17). □

REMARKS. (1) If $\tau b_j = b_j$ for some j (τ is the hyperelliptic involution), then $\phi(t, b_j)$, which is always $\psi(t, \tau b_j)$, coincides with $\psi(t, b_j)$, so that $\xi_j = \eta_j$. If $\tau b_j = b_j$ for all $j = 1, \cdots, g+1$, then (17)-(20) reduce to the original Neumann equations (1) $-$ (3). (2) $\psi(t, b) \equiv \phi(t, b)$ implies $\tau b = b$. Indeed, let $x(b) = x_0$, $y(b) = y_0$. Then ψ, ϕ satisfy $L\psi = x_0\psi$, $L\phi = x_0\phi$, resp. $B\psi = y_0\psi$, $B\phi = -y_0\phi$. If $\psi\equiv\phi$, the last two equations force $y_0 = -y_0$, or $y_0 = 0$, and thus $\tau b = b$. (3) Note that all equations, (17)-(20), are invariant under the substitutions

(21)
$$\xi_j \to \alpha_j\xi_j,\ \eta_j \to \alpha_j^{-1}\eta_j .$$

If $\tau b_j = b_j$, so that $\xi_j \equiv \eta_j$, then $\alpha_j = \alpha_j^{-1}$ in (21), or $\alpha_j = \pm 1$. Otherwise, any $\alpha_j \in \boldsymbol{C}^*$ is allowed. We return to these symmetry groups later on.

B. LAX PAIRS. The next topic to be taken up, which a priori has no relation to the Neumann system, is the explicit description of the augmented linebundle from Section 3.

PROPOSITION 3. *Let $W(f, g) = \dot{f}g - f\dot{g}$ be the Wronskian (of functions of t). The functions*

(22)
$$\chi_j(t, p) = \frac{\sqrt{\rho_j}}{h(p)}\frac{W(\psi(t, p), \phi(t, b_j))}{x(p) - x_j} , \quad j = 1, \cdots, g+1 ,$$

are holomorphic sections(*) *of a linebundle $\mathscr{L}(\delta)\otimes\mathscr{L}(Z)\otimes\mathscr{L}_t'$. (According to the notation set down at the beginning of this section, Z is the divisor of finite zeros of h).*

(*) Here and below, I use the expression "are holomorphic sections" as convenient, if inaccurate, abbreviation for "are obtained from holomorphic sections".

PROOF. All that needs to be checked is that each χ_j has poles independent of t in $\delta + Z$, and that $\chi_j(t, p) \sim e^{\sqrt{x(p)}t} \times$ holomorphic function near ∞. This is immediate: $W(\psi, \phi)$ has poles δ because ψ does, $(1/h)$ has poles Z. The two poles at $x = x_j$, i.e. at b_j, τb_j, are cancelled by the zero of $1/h$ at b_j and the zero of $W(\psi(t, p), \phi(t, b_j))$ at $p = \tau b_j$ $(\psi(t, \tau b_j) = \phi(t, b_j))$. The condition at ∞ is obviously satisfied. □

COROLLARY. *The functions*

$$\psi_j(t, p) = \chi_j(t, p)e^{-h(p)x(p)t} \tag{23}$$

are holomorphic sections of a linebundle

$$\mathscr{L}(\delta) \otimes \mathscr{L}(Z) \otimes \mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(g+1)} . \tag{24}$$

PROOF. The exponential singularity at ∞ is cancelled, and the required exponential growth at $b_1, \cdots, b_{g+1}$ is introduced. (Compare the proof of Proposition 1). □

It will be seen shortly that the functions (23) are $g + 1$ *independent* holomorphic sections of the bundle (24), which has precisely that many sections since $\delta + Z$ is nonspecial. Hence, one can appeal to Krichever's theory [13] of commuting matrix differential operators, and, as already mentioned, construct operators $\boldsymbol{L}, \boldsymbol{B}$ of which the vector function $\Psi = (\psi_1, \cdots, \psi_{g+1})^t$ is a common eigenfunction.

PROPOSITION 4.

$$\dot{\Psi} = (-Xh + \dot{\eta} \otimes \xi - \eta \otimes \dot{\xi})\Psi \underset{\text{def}}{=} L\Psi , \tag{25}$$

where $X = \operatorname{diag}(x_1, \cdots, x_{g+1})$, $(\dot{\eta} \otimes \xi)_{ij} = \dot{\eta}_i \xi_j$.

PROOF. The calculation is familiar from [13], and can be disposed of quickly. One shows that

$$\psi_i(t, p) = (c_i\delta_{ij} + \cdots)e^{-h(p)x(p)t} \tag{26}$$

($c_i =$ some nonzero constant) near $p = b_j$. It follows that the ψ_i are *independent* holomorphic sections of the bundle (24). One then shows that $\dot{\Psi} - L\Psi$ has behavior

$$0\left(\frac{1}{h(p)}\right)e^{-h(p)x(p)t} \tag{27}$$

near each b_j. Since it has all the right poles, each component of $\dot{\Psi} - L\Psi$ is a holomorphic section of (24), hence expressible in the basis $\{\Psi_i\}$: for the j^{th} component,

$$(\dot{\Psi} - L\Psi)_j = \sum_i \alpha_{ij}\psi_i \,.$$

In view of (26) and (27), all α_{ij} must vanish, which establishes (25). So, the only thing to calculate is the expansion of $\psi_i(t, p)$ near each b_j; this is easy but tedious. Only the results are recorded here:

Near b_j, $j \neq i$:

$$\text{(28a)} \qquad \psi_i(t, p) = \left[\frac{\sqrt{\rho_i}\,W(\psi(t, b_j), \phi(t, b_i))}{h(p)(x_j - x_i)} + 0\left(\frac{1}{h(p)}\right)\right] e^{-h(p)x_j t} e^{-h_{-1}^{(j)} t} \,;$$

Near b_i:

$$\text{(28b)} \qquad \psi_i(t, p) = \left[\frac{1}{\sqrt{\rho_i}} + 0\left(\frac{1}{h(p)}\right)\right] e^{-h(p)x_i t} e^{-h_{-1}^{(i)} t} \,.$$

Here $h_{-1}^{(i)} = \lim_{p \to b_i} h(p)(x(p) - x_i)$ (this is zero if $\tau b_i = b_i$).

Everything follows from these formulas; the relation (16) between Ω and $W(\psi, \phi)$ is essential here. □

PROPOSITION 5. *Ψ satisfies the equation*

$$\text{(29)} \qquad Q\Psi \underset{\text{def}}{=} (-Xh^2 + (\dot{\eta} \otimes \xi - \eta \otimes \dot{\xi})h - \eta \otimes \xi)\Psi = -xh^2\Psi$$

PROOF. In view of (25), the relation to be proved can be written

$$\text{(30)} \qquad h\dot{\Psi} - \eta \otimes \xi\Psi + xh^2\Psi = 0 \,.$$

As before, the idea is to show that each component of the left side of (30) is a holomorphic section of the bundle (24), vanishing at the b_j. So, one writes

$$\psi_i = \left(\frac{1}{\sqrt{\rho_i}}\delta_{ij} + A_{ij}h^{-1} + B_{ij}h^{-2} + \cdots\right) e^{-h(x_j + \alpha_j h^{-1} + \beta_j h^{-2} + \cdots)t}$$

near b_j, and substitutes this into the left side of (30). The result, which does not require the detailed expressions for the various coefficients, is (for the i-th component) $(\dot{A}_{ij} - (1/\sqrt{\rho_j})\eta_i\xi_j + 0(1/h))e^{-hxt}$, near b_j. To check that the 0(1) term vanishes, one needs to know $\dot{A}_{ij}$. Now A_{ij}, for $i \neq j$, is given in (28a), and one sees easily that $\dot{A}_{ij} = (1/\sqrt{\rho_j})\eta_i\xi_j$. To verify that $\dot{A}_{ii} = (1/\sqrt{\rho_i})\eta_i\xi_i$, one must carry the expansion in (28b) through $0(1/h)$, which is again straightforward, but long, so the details wil be omitted. □

COROLLARY. *The Neumann equations admit the Lax representation*

$$\text{(31)} \qquad \dot{Q} = [L, Q] \,.$$

PROOF. Direct computation, using the Neumann equation, (20). □

REMARKS. (1) Krichever's method [13] provides a 2^{nd}-order matrix differential operator $\boldsymbol{B}$ satisfying

$$\boldsymbol{B}\Psi = xh^2\Psi ,$$

and commuting with $\boldsymbol{L}$. Because $[\boldsymbol{L}, \boldsymbol{B}] = 0$, $\boldsymbol{B}$ maps $\ker(\boldsymbol{L} - h\boldsymbol{I})$ to itself. On $\ker(\boldsymbol{L} - h\boldsymbol{I})$, $\boldsymbol{B}$ is represented by a matrix depending on h: that matrix is precisely $-(\boldsymbol{Q})$.

It is easier to *discover* $\boldsymbol{Q}$ than $\boldsymbol{B}$, once the basic eigenvalue equation $\dot{\Psi} = \boldsymbol{L}\Psi$ is given; fewer terms in the expansion of Ψ are required. Therefore, I did not follow the commuting-matrix-differential-operator route, but it is straightforward to deduce those formulas from Proposition 5.

(2) Equation (31) was given a Lie-algebraic interpretation in [1, 19, 21], for the case where $\xi_j \equiv \eta_j$ for all j. So far, I have not been able to fit Lie algebras into my setup in any natural way; on the other hand, the connection with the operator $D^2 + q$ is entirely foreign to the discussion in [1, 19, 21].

Finally, so that the discussion in Section 3 is not left completely up in the air, I should comment on the "projective model" mentioned there.

The map $p \to [\psi_1(t, p): \cdots : \psi_{g+1}(t, p)]$ is, for fixed t, a well defined map from $\mathscr{R}$ to $\boldsymbol{P}^g$, because the linear system associated with $\mathscr{L}(\delta) \otimes \mathscr{L}(\boldsymbol{Z}) \otimes \mathscr{L}_t^{(1)} \otimes \cdots \otimes \mathscr{L}_t^{(g+1)}$, of degree $2g$, has no base points. When $g = 2$, the image of $\mathscr{R}$ is a plane quartic of genus 2, so it must have one double point (this was explained to me by Nick Ercolani). Furthermore, it seems to lie in a very nice way on Kummer's quadric surface—but that is only a tantalizing observation (Ercolani's) so far. If $g > 2$, the map is an embedding, i.e., it separates points and directions, off a 2-dimensional set of divisors $\boldsymbol{b} = b_1 + \cdots + b_{g+1}$.

Once the curve is so embedded in $\boldsymbol{P}^g$, the zero divisor $z(t)$ of ψ defining the bundle $\mathscr{L}_t \cong \mathscr{L}(z(t))$ (see Sec. 3) can be related to hyperplanes. This follows from the formula [23]

$$\psi(t, p) = \sum_1^{g+1} \eta_j(t)\chi_j(t, p) ;$$

the image of $\mathscr{R}$ is intersected by the plane $\sum \eta_j \chi_j = 0$ precisely in $Z + z(t)$.

I hope that these ideas will soon lead to a truly geometric version of the Burchnall-Baker-Chaundy-Krichever theory. At the moment, they are just a reassuring background to help one keep other calculations in some perspective.

C. INTEGRABILITY. From a Lax equation like (31), $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$, one tries to deduce complete integrability of the system of o.d.e.'s it represents; the procedure is reasonably standard by now [1]. Only the main features, and their relation to the underlying curve $\mathscr{R}$, need be outlined.

It is natural and convenient to view the coupled Neumann system as a Hamiltonian system, and to this end $\dot{\xi}_j$ and $\dot{\eta}_j$ are replaced by new variables, the momenta u_j and v_j. So, the system to be considered is now

$$
\begin{aligned}
(32)\qquad &\dot{\xi}_j = u_j\,, &&\dot{\eta}_j = v_j\\
&\dot{u}_j = x_j\xi_j - (\textstyle\sum x_i\xi_i\eta_i + u_iv_i)\xi_j\,, &&\dot{v}_j = x_j\eta_j - (\textstyle\sum x_i\xi_i\eta_i + u_iv_i)\eta_j
\end{aligned}
$$

subject to the constraints

$$
(33)\qquad \sum \xi_i\eta_i = 1\,, \quad \sum \xi_iv_i + \eta_iu_i = 0\,.
$$

For details about Hamiltonian mechanics with constraints, see, e.g., [7, 16]. Likewise, $\dot{\xi}_j$ and $\dot{\eta}_j$ are replaced by u_j and v_j in the definitions (25) and (29) of $\boldsymbol{L}$ and $\boldsymbol{Q}$.

It was seen above that (32) and (33) imply (31), $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$. Conversely, $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$, with the constraint (33), implies (32) [19] *provided* that all b_j are Weierstrass points, so that $\xi_j = \eta_j$, $u_j = v_j$ for all j (the original Neumann system). If some b_j are not Weierstrass points, one can scale $\xi_j \to \xi_j e^{-f(t)}$, $\eta_j \to \eta_j e^{f(t)}$, $u_j \to u_j e^{-f(t)}$, $v_j \to v_j e^{f(t)}$, where $f(t)$ is an arbitrary function of t. It is clear that $\boldsymbol{L}, \boldsymbol{Q}$ are unaffected, since the $e^{\pm f}$ cancel each other in all terms, but equation (32) not invariant under this scaling. Specifically, (31) does not imply the two equations $\dot{\xi}_j = u_j$, $\dot{\eta}_j = v_j$.

One can get rid of this scaling freedom by a sort of elimination of angular momentum. Set

$$
\xi_j = r_je^{\theta_j},\ \eta_j = r_je^{-\theta_j},\ u_j = s_je^{\theta_j} + w_jr_je^{\theta_j}\,,\qquad v_j = s_je^{-\theta_j} - w_jr_je^{-\theta_j}\,,
$$

and define

$$
2c_j = 2w_jr_j^2 = u_j\eta_j - v_j\xi_j\,.
$$

This last expression is, up to a factor ρ_j, just the Wronskian $W(\psi(t, b_j), \phi(t, b_j))$, which is independent of t, and is zero iff $\tau b_j = b_j$. (32) and (33) transform to

$$
(34)\qquad \dot{r}_j = s_j,\ \dot{s}_j = x_jr_j - \left(\sum x_ir_i^2 + s_i^2 - \frac{c_i^2}{r_i^2}\right)r_j - \frac{c_j^2}{r_j^3}\,,
$$

subject to

$$
(35)\qquad \sum r_i^2 = 1\,, \quad \sum r_is_i = 0\,.
$$

(34), (35) is a mechanical system studied by Rosochatius [22]. See the

Appendix in [15] for details; the reduction that converts (32), (33) into (34), (35) is there credited to P. Deift.

Correspondingly, $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$ turns into $\dot{\tilde{\boldsymbol{Q}}} = [\tilde{\boldsymbol{L}}, \tilde{\boldsymbol{Q}}]$, where $\tilde{\boldsymbol{L}} = -\boldsymbol{X}h + s \otimes r - r \otimes s - (c/r) \otimes r - r \otimes (c/r) + (c/r^2)$, $\tilde{\boldsymbol{Q}} = -h^2 + h(s \otimes r - r \otimes s - (c/r) \otimes r - r \otimes (c/r)) - r \otimes r$, and this Lax representation, together with the constranints (35), *does* imply (34). It was given a Lie-algebraic interpretation in [20].

Once one knows that a Lax equation $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$ (or $\dot{\tilde{\boldsymbol{Q}}} = [\tilde{\boldsymbol{L}}, \tilde{\boldsymbol{Q}}]$) is equivalent to the Neumann system (or to the Rosochatius system), one has a tested procedure to prove integrability in the sense of Liouville.

(i) $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$ implies that the characteristic equation of $\boldsymbol{Q}$, det $(\boldsymbol{Q} - \lambda \boldsymbol{I}) = 0$, is independent of t. This equation is a polynomial in h and λ, and the coefficients of the various terms $h^\alpha \lambda^\beta$ are integrals of the Neumann system.

(ii) One proves that there are enough integrals in involution.

(iii) One linearizes the Neumann equations on the Jacobian of the algebraic curve $\det(\boldsymbol{Q} - \lambda \boldsymbol{I}) = 0$.

This program is carried out in full in [1]. In the present context, some of it is automatic, and some of it cannot be done without introduction of new techniques. Property (ii) falls into the second category. My starting point was a curve and a scalar Baker function, and the end-result was one particular solution of the Neumann system. The Hamiltonian structure, however, involves the whole (ξ, η, u, v) phase-space, and there seems to be no way to study it one curve $\mathscr{R}$ at a time, so to say. Property (iii), on the other hand, is essentially built in, because the linearization of the scalar equation $[L, B] = 0$ for $q(t)$ was assumed from the beginning (cf. Section 3).

Property (i) does fit naturally and nontrivially into the framework of this paper. The fact that $\det(\boldsymbol{Q} - \lambda \boldsymbol{I}) = 0$ is independent of t can be understood from Proposition 5, which precedes the derivation of $\dot{\boldsymbol{Q}} = [\boldsymbol{L}, \boldsymbol{Q}]$. It said that $\boldsymbol{Q}\boldsymbol{\Psi} = -xh^2\boldsymbol{\Psi}$; this implies $\det(\boldsymbol{Q} + xh^2\boldsymbol{I}) = 0$, which is a polynomial equation satisfied by the meromorphic functions x and h on $\mathscr{R}$, and should not depend on t.

The explicit integrals of the coupled Neumann system (20) can be derived very neatly by the following intrinsic method. For $j = 1, \cdots, g + 1$, define the meromorphic differentials

$$\omega_j = \frac{\rho_j W(\psi(t, p), \phi(t, b_j)) W(\phi(t, p), \psi(t, b_j))}{x(p) - x_j} h(p) \Omega .$$

$\sum_{p \in \bar{\mathscr{R}}} \mathrm{Res}_p\, \omega_j = 0$ leads to the formula

$$G_j \underset{\text{def}}{=} \xi_j\eta_j + \sum_{k(\neq j)} \frac{(\xi_j v_j - \eta_j u_k)(\eta_k u_j - \xi_j v_k)}{x_k - x_j} = \overset{0}{G}_j\,, \tag{36}$$

where $\overset{0}{G}_j = -\mathrm{Res}_{b_j}\,\omega_j$; $\xi_j\eta_j$ is the residue at ∞, and the k^{th} term in the sum is obviously the residue at b_k. With some calculation, one finds that $G_j^0 = h_{-1}^{(j)2}$ if $\tau b_j = b_j$, resp. $h_{-1}^{(j)}h_0^{(j)}$ if $\tau b_j \neq b_j$; the $h_\alpha^{(j)}$ are the coefficients in the Laurent series of h at b_j, $h(p) = h_{-1}^{(j)}\eta^{-1} + h_0^{(j)} + \cdots$, $\eta = \sqrt{x(p) - x_j}$, resp., $\eta = x(p) - x_j$. As in [15], $\sum G_j = \sum \xi_j\eta_j = 1$, and generically, g of the $g+1$ expressions (36) are independent. It was observed earlier that the quantities

$$u_j\eta_j - \xi_j v_j = \rho_j W(\psi(t, b_j), \phi(t, b_j)) \tag{37}$$

are also independent of t. The value of (37) is $(1/2)h_{-1}^{(j)}$ if $\tau b_j \neq b_j$; if $\tau b_j = b_j$, then $\xi_j = \eta_j$ and $u_j = v_j$, so (37) vanishes.

Altogether, then, there are g independent integrals (36), and as many integrals (37) as there are non-Weierstrass points among the b_j. Observe that the values of these integrals depend only on the choice of h on the curve $\mathscr{R}$. All the integrals are in involution, but I have only a computational, not a function theoretic, proof of this fact, copied from [7, 15].

5. Conclusion. This study is the result of an attempt to generalize the geometry of quadrics, as shown by Moser [15] to be related to the Neumann system, to nonhyperelliptic situations. The obvious idea is to replace $D^2 + q$ by—say $-D^3 + qD + p$ (related to the Boussinesq equation), to find the corresponding "Neumann" system, its $\dot{Q} = [L, Q]$ representation, and then to develop the geometry. It turned out to be difficult even to get started, because the understanding of the Neumann-$D^2 + q$ connections was at the level of observations and inspired guesses. So, as a first step, I tried to derive the basic facts about the Neumann system in a systematic way from the Baker function theory of $D^2 + q$. This was the topic of the present paper. There are now two ways to go.

First, the analytical treatment in Section 4 can be generalized to other L, B pairs. This turns out to be nontrivial; the hyperelliptic case is in many ways misleading. R. Schilling [23, 24] obtained Neumann systems, and their representations by *two* types of Lax equations $\dot{Q} = [L, Q]$, for very general pairs L, B of commuting matrix or scalar differential operators of arbitrary order. In the process, most calculations in Section 4 are seen in a more natural setting.

Second, the geometric picture hinted at in Section 3 should be developed, first to explain the geometry of quadrics in [12, 15] and then to produce analogous, and presumably new, geometric structures associated with Schilling's general Neumann systems. That work has barely begun.

In this whole development, singular curves $\mathscr{R}$ have not been admitted, even though the coefficients $q(t)$ associated with some of these curves are among the most interesting in soliton theory: multi-soliton and rational potentials. To some extent, the theory carries over unchanged: as long as the function h is regular at singular points, all results remain valid. The new wrinkle comes about as follows. The differential Ω will have poles at singular points; if h is regular at a singular point, then $\sum_{\text{branches at } p} \operatorname{Res} h\psi\phi\Omega = 0$. If, however, h is not regular at p, even if it has no pole there, the singular point will contribute a term to $\sum \operatorname{Res} h\psi\phi\Omega$. In other words, one can trade off poles of h for irregular behavior at singular points. In the most interesting example—the multi-soliton case—, the curve is $\mathscr{R}$: $y^2 = x\prod_1^g (x - a_j)^2$, and $h = (\prod_1^g (x - a_j)/y)$. This h has only one pole, at $x = y = 0$, but is not regular at the double points. The Neumann variables ξ_j are the values of the Baker function ψ at these double points and at $x = y = 0$, and the representation ($a_0 = 0$)

$$q = \sum_{j=0}^{g} a_j\xi_j^2 + \dot{\xi}_j^2$$

is the most natural one, because the eigenvalues and eigenfunctions occurring there are the ones encountered in inverse scattering theory (compare [11], formula (3.13)).

Krichever's theory as used in Proposition 4 does not apply in such singular situations, however, and since there was no readily available substitute, I excluded singular curves. It may be interesting, once the results of this paper have been made more geometric, to study singular curves as well.

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