Tôhoku Math. Journ.
36 (1984), 505-519.

# ON THE FUNCTIONS OF LITTLEWOOD-PALEY AND MARCINKIEWICZ

GEN-ICHIRÔ SUNOUCHI

(Received August 27, 1983)

**1. Introduction.** Let $f(x)$ be a locally integrable function on the real line $\boldsymbol{R}$. The Fourier integral analogue of Marcinkiewicz function [7] is

$$\mu(f)(x)=\left(\int_0^\infty |F(x+t)+F(x-t)-2F(x)|^2 t^{-3}dt\right)^{1/2}$$

where

$$F(x)=\int_0^x f(u)du\ .$$

We generalize this as follows: for $\alpha>0$

$$\mu_\alpha(f)(x)=\left\{\int_0^\infty\left|\frac{\alpha}{t}\int_0^\infty\left(1-\frac{u}{t}\right)^{\alpha-1}(f(x-u)-f(x+u))du\right|^2\frac{dt}{t}\right\}^{1/2}, \tag{1.1}$$

$\mu_1(f)(x)$ coincides with $\mu(f)(x)$. (1.1) is the one dimensional form of the more general Marcinkiewicz function

$$\mu(f)(x)=\left\{c\int_0^\infty\left|\frac{1}{t}\int_{|u|\le t} f(x-u)\frac{\Omega(u')}{|u|^{k-1}}du\right|^2\frac{dt}{t}\right\}^{1/2}$$

where $\Omega(u')/|u|^k$ is the Calderón-Zygmund kernel on $k$-dimensional space and $c$ is a constant depending on $k$ only, see Stein [8].

On the other hand we have generalized the Littlewood-Paley function as follows

$$g_\beta^*(\phi)(x)=\left\{\frac{1}{\pi}\int_0^\infty y^{2\beta}dy\int_{-\infty}^\infty\frac{|\phi'(t+iy)|^2}{|t-x-iy|^{2\beta}}dt\right\}^{1/2}, \tag{1.2}$$

where $\phi(z)=\phi(x+iy)$ is analytic in the upper half-plane and has boundary value $\phi(x)=\lim_{y\to 0}\phi(x+iy)$. The original Littlewood-Paley function $g^*(\phi)(x)$ in Fourier integral form corresponds to the case $\beta=1$ in (1.2).

Let $\sigma_\beta(R;x,\beta)$ the $R$-th $(C,\beta)$-mean of Fourier integral of complex valued function $\phi(x)$ and set

$$\tau_\beta(R;x,\phi)=R\frac{d}{dR}\sigma_\beta(R;x,\phi)=\beta\{\sigma_{\beta-1}(R;x,\phi)-\sigma_\beta(R;x,\phi)\} \tag{1.3}$$

and set

$$h_\beta(\phi)(x) = \left(\int_0^\infty \frac{|\tau_\beta(R; x, \phi)|^2}{R} dR\right)^{1/2} . \tag{1.4}$$

Then (1.2) is equivalent to (1.4), that is,

$$Ah_\beta(\phi)(x) \leqq g_\beta^*(\phi)(x) \leqq Bh_\beta(\phi)(x) ,$$

where $A$ and $B$ are constants independent of $\phi$ and $x$; see Sunouchi [11]. Here after $A$ and $B$ mean such constants.

Now we consider the functional $h_\beta$ for imaginary part of $\phi$. Let $\bar{\sigma}_\beta(R; x, f)$ the $(C, \beta)$-mean of the conjugate Fourier integral of any $f(x)$ and define $\bar{\tau}_\beta(R; x, f)$ and $\bar{h}_\beta(R; x, f)$ analogously to the formula (1.3) and (1.4). We denote by $S$ the Schwartz space on $\boldsymbol{R}$, that is, the space of rapidly decreasing $C^\infty$-functions. Then our main theorem is as follows.

THEOREM 1. *If* $\alpha + 1/2 = \beta$ $(\alpha > 0)$, *then*

$$A\bar{h}_\beta(f)(x) \leqq \mu_\alpha(f)(x) \leqq B\bar{h}_\beta(f)(x)$$

*for any function* $f(x) \in S$ *and* $x \in \boldsymbol{R}$.

One of the inequalities

$$\bar{h}_\beta(f)(x) \leqq A\mu_\alpha(f)(x)$$

is already given by Flett [3] for the functions on the unit circle.

For a variant of this, let $f_\alpha(x)$ be the Riesz potential of $f(x)$, that is, $f_\alpha(x) = \int_{-\infty}^\infty |\xi|^{-\alpha}\hat{f}(\xi)e^{ix\xi}d\xi$ and set

$$D_\alpha(f)(x) = \left(\int_0^\infty \frac{|f_\alpha(x - t) - f_\alpha(x + t)|^2}{t^{1+2\alpha}} dt\right)^{1/2} .$$

THEOREM 2. *If* $\alpha + 1/2 = \beta$ $(0 < \alpha < 1)$, *then*

$$A\bar{h}_\beta(f)(x) \leqq D_\alpha(f)(x) \leqq B\bar{h}_\beta(f)(x)$$

*for* $f \in S$ *and* $x \in \boldsymbol{R}$.

The fact that, for $\alpha + 1/2 > \beta$

$$D_\alpha(f)(x) \leqq B\{\bar{h}_\beta(f)(x) + h_\beta(f)(x)\} = Bh_\beta(\phi)(x) ,$$

is given by Stein [9] for functions of several variables.

Corresponding to $h_\beta(f)(x)$, we consider

$$\delta_\alpha(f)(x) = \left(\int_0^\infty t \left|\frac{d}{dt} M_\alpha(t; x, f)\right| dt\right)^{1/2}$$

where

$$M_\alpha(t; x, f) = \frac{\alpha}{t} \int_{|u| \leqq t} \left(1 - \frac{|u|}{t}\right)^{\alpha - 1} f(x - u) du .$$

THEOREM 3. *If* $\alpha - (1/2) = \beta$ $(\alpha > 0)$, *then*

$$Ah_\beta(f)(x) \leqq \delta_\alpha(f)(x) \leqq Bh_\beta(f)(x)$$

*for* $f \in S$ *and* $x \in \boldsymbol{R}$.

In the last section, an analogous relation to the Littlewood-Paley function $g(f)(x)$, is also established. In particular the relationship between

$$\delta_0(f)(x) = \left(\int_0^\infty t \left|\frac{d}{dt}\{f(x-t) + f(x+t)\}\right|^2 dt\right)^{1/2}$$

and $g(f)(x)$ is clarified. This question is proposed as problem 6(a) of Stein-Wainger [10, p. 1289] for several variables case.

I wish to express my appreciations to Professor M. Kaneko, whose valuable suggestions have led to a material improvement in the presenting of this paper.

2. **Notations.** We suppose throughout this paper $f(x)$ belongs to the class $S$. We write for a fixed $x_0$,

$$\phi(t) = \phi(t; x_0, f) = f(x_0 - t) + f(x_0 + t)$$

and

$$\psi(t) = \psi(t; x_0, f) = f(x_0 - t) - f(x_0 + t) .$$

For $\alpha > 0$ and $t \geqq 0$, set

$$\phi_\alpha(t) = \phi_\alpha(t; x_0, f) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} \phi(u)du , \tag{2.1}$$

and

$$\psi_\alpha(t) = \psi_\alpha(t; x_0, f) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} \psi(u)du . \tag{2.2}$$

The generalized Marcinkiewicz function and a variant are written as

$$\mu_\alpha(f)(x_0) = \left(\int_0^\infty \left|\psi_\alpha(t; x_0, f)\right|^2 \frac{dt}{t}\right)^{1/2} , \tag{2.3}$$

$$\delta_\alpha(f)(x_0) = \left(\int_0^\infty t \left|\frac{d}{dt}\phi_\alpha(t; x_0, f)\right|^2 dt\right)^{1/2} . \tag{2.4}$$

For the Cesàro-Riesz mean of $f(x)$, we introduce the well-known Young function. Let

$$\gamma_\alpha(x) + i\bar{\gamma}_\alpha(x) = \int_0^1 (1-t)^{\alpha-1} e^{ixt} dt$$

where $\alpha > 0$, $x \geqq 0$, then it is known [1],

$$\gamma_\alpha(x) \sim x^{-p} \quad \text{as} \quad x \to \infty , \qquad \text{where} \quad p = \text{Min}(2, \alpha) .$$

Then the $R$-th Cesàro-Riesz mean of of the $\beta$-th order for Fourier integral of $f(x)$ is

$$\sigma_\beta(R) = \sigma_\beta(R; x_0, f) = c\int_0^\infty \phi(u)R\gamma_{\beta+1}(Ru)du \tag{2.5}$$

and for the conjugate Fourier integral of $f(x)$ is

$$\bar\sigma_\beta(R) = \bar\sigma_\beta(R; x_0, f) = c'\int_0^\infty \psi(u)R\bar\gamma_{\beta+1}(Ru)du \tag{2.6}$$

where $c$ and $c'$ are constants. Then we have

$$h_\beta(f)(x_0) = \left(\int_0^\infty |\sigma_{\beta-1}(R) - \sigma_\beta(R)|^2\frac{1}{R}dR\right)^{1/2} \tag{2.7}$$

and

$$\bar h_\beta(f)(x_0) = \left(\int_0^\infty |\bar\sigma_{\beta-1}(R) - \bar\sigma_\beta(R)|^2\frac{1}{R}dR\right)^{1/2} . \tag{2.8}$$

3. **Proof of Theorem 1.** Let $f \in S$ and fix a point $x_0$ in $\boldsymbol{R}$. By the change of the variables $u = e^{-y}$ and $t = e^{-x}$ (2.3) becomes

$$\mu_\alpha(f)(x_0) = \left[\int_{-\infty}^\infty \left|\int_x^\infty \alpha e^{(x-y)}(1 - e^{(x-y)})^{\alpha-1}\psi(e^{-y})dy\right|^2 dx\right]^{1/2} . \tag{3.1}$$

If we rewrite

$$K_\alpha(x) = \begin{cases} \alpha e^x(1 - e^x)^{\alpha-1} , & x \leqq 0 \\ 0 , & x > 0 \end{cases} \tag{3.2}$$

and $\Psi(x) = \psi(e^{-x})$, then (3.1) becomes

$$\mu_\alpha(f)(x_0) = \left(\int_{-\infty}^\infty |(\Psi * K_\alpha)(x)|^2 dx\right)^{1/2} . \tag{3.3}$$

In (2.6) and (2.8), we set $u = e^{-y}$ and $R = e^x$, then (2.8) is

$$\begin{aligned}\bar h_\beta(f)(x_0) &= \left[c'\int_0^\infty \left|\int_0^\infty \psi(u)R\{\bar\gamma_\beta(Ru) - \bar\gamma_{\beta+1}(Ru)\}du\right|^2\frac{dR}{R}\right]^{1/2} \\ &= \left[c'\int_{-\infty}^\infty\left|\int_{-\infty}^\infty \psi(e^{-y})e^{x-y}\{\bar\gamma_\beta(e^{x-y}) - \bar\gamma_{\beta+1}(e^{x-y})\}dy\right|^2 dx\right]^{1/2} .\end{aligned}$$

If we rewrite

$$\bar K_\beta^*(x) = e^x(\bar\gamma_\beta(e^x) - \bar\gamma_{\beta+1}(e^x)) = e^x\gamma_\beta'(e^x) \tag{3.4}$$

and $\Psi(x) = \psi(e^{-x})$, then

$$\bar h_\beta(f)(x_0) = \left(\int_{-\infty}^\infty |(\Psi * \bar K_\beta^*)(x)|^2 dx\right)^{1/2} . \tag{3.5}$$

To compare (3.3) with (3.5), we apply Fourier transform method.

Since $\psi(u) = f(x_0 - u) - f(x_0 + u) \in S$, $\Psi(x) = \psi(e^{-x}) = 0(e^{-x})$ as $x \to \infty$, and $0(e^{-|x|})$ as $x \to -\infty$. Since $K_\alpha(x)$ is integrable on $(-\infty, \infty)$, $(\Psi * K_\alpha)(x)$ is an ordinary convolution. However, since

$$\bar{\gamma}_\beta(e^x) \sim e^{-\beta x} \quad \text{as} \quad x \to \infty \qquad (0 < \beta \leqq 1)$$

we have

$$\bar{K}_\beta^*(x) \sim e^x \cdot e^{-\beta x} = e^{(1-\beta)x} \quad \text{as} \quad x \to \infty .$$

But

$$\bar{K}_\beta^*(x) = 0(e^x) \quad \text{as} \quad x \to -\infty \qquad (0 < \beta \leqq 1) .$$

If $1/2 < \beta \leqq 1$, then $1 - \beta \geqq 0$ and $\bar{K}_\beta^*(x)$ is locally integrable, but not integrable on $(-\infty, \infty)$. In fact this is the most interesting case. Hence we have to consider a distributional Fourier transform. As we shall show at (3.11), the Fourier transform $\hat{\bar{K}}_\beta^*(\xi)$ belongs to the class $L^\infty$ and evidently $\hat{\Psi}(\xi) \in L \cap L^\infty$. Accordingly we can apply convolution rule to $(\Psi * \bar{K}_\beta^*)(x)$, see Katznelson [5, p. 151, Lemma].

Now we take the complex Fourier transform of kernels (3.2) and (3.4). Let $s = \zeta - i\xi$, where $\zeta$ is a complex number. Then

$$\begin{aligned}(3.6) \qquad \int_{-\infty}^{\infty} K_\alpha(x) e^{sx} dx &= \alpha \int_{-\infty}^{0} e^{(s+1)x}(1 - e^x)^{\alpha-1} dx = \alpha \int_0^1 (1 - t)^{\alpha-1} t^s dt \\ &= \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)} \qquad (\alpha > 0, \ \operatorname{Re} s > -1) .\end{aligned}$$

Let $\theta(x) \in S$, and consider Parseval's formula:

$$\langle \bar{K}_\beta^*(x) e^{\zeta x}, \theta(x) \rangle = 2\pi \Big\langle \int_{-\infty}^{\infty} \bar{K}_\beta^*(x) e^{\zeta x} \cdot e^{-i\xi x} dx, \hat{\theta}(\xi) \Big\rangle .$$

Then both sides are analytic functions of a complex variable $\zeta$ in an appropriate domain. Therefore we can calculate the distributional Fourier transform of $\bar{K}_\beta^*(x)$ by analytic continuation method, see [4, p. 171]. We have

$$\begin{aligned}(3.7) \qquad \int_{-\infty}^{\infty} \bar{K}_\beta^*(x) e^{sx} dx &= \int_{-\infty}^{\infty} e^x \gamma'_\beta(e^x) e^{sx} dx \\ &= c' \frac{\Gamma(\beta)s}{\Gamma(\beta - s + 1) \sin \pi s/2} \\ &\quad (\beta > 0, \beta + 1 > \operatorname{Re} s, 2 > \operatorname{Re} s > -2) .\end{aligned}$$

Accordingly we have, from (3.6) and (3.7),

$$(3.8) \qquad \hat{K}_\alpha(\xi) = \frac{\Gamma(\alpha + 1)\Gamma(1 - i\xi)}{\Gamma(\alpha + 1 - i\xi)} \quad (\alpha > 0)$$

and

$$\hat{\bar{K}}_\beta^*(\xi) = \frac{c'\Gamma(\beta)(i\xi)}{\Gamma(\beta + i\xi + 1)\sin \pi i\xi/2} \quad (\beta > 0)\,; \tag{3.9}$$

Both $\hat{K}_\alpha(\xi)$ and $\hat{\bar{K}}_\beta^*(\xi)$ have no zero on $\xi \in (-\infty, \infty)$ and finite. By the asymptotic formula of the Gamma function

$$|\Gamma(a + i\xi)| \sim (2\pi)^{-1/2}e^{-\pi|\xi|/2}|\xi|^{a-(1/2)}\,, \qquad a \in (-\infty, \infty)$$

as $|\xi| \to \infty$, we have as $|\xi| \to \infty$,

$$|\hat{K}_\alpha(\xi)| \sim \frac{c\,|\xi|^{1-(1/2)}}{|\xi|^{\alpha+1-(1/2)}} \sim c\,|\xi|^{-\alpha} \tag{3.10}$$

and as $|\xi| \to \infty$,

$$|\hat{\bar{K}}_\beta^*(\xi)| \sim c'\frac{|\xi|}{|\xi|^{1+\beta-(1/2)}e^{-\pi|\xi|/2}e^{\pi|\xi|/2}} \sim c'\,|\xi|^{(1/2)-\beta}\,. \tag{3.11}$$

Hence $|\hat{K}_\alpha(\xi)/\hat{\bar{K}}_\beta^*(\xi)|$ is bounded if $\alpha + 1/2 = \beta$ $(\alpha > 0)$. Thus

$$\begin{aligned}
0 &\leqq \int_{-\infty}^{\infty} |(\Psi * K_\alpha)(x)|^2 dx = (2\pi)\int_{-\infty}^{\infty} |\hat{\Psi}(\xi)\hat{K}_\alpha(\xi)|^2 d\xi \\
&= (2\pi)\int_{-\infty}^{\infty} \left|\hat{\Psi}(\xi)\cdot \hat{\bar{K}}_\beta^*(\xi)\cdot \frac{\hat{K}_\alpha(\xi)}{\hat{\bar{K}}_\beta^*(\xi)}\right|^2 d\xi \\
&\leqq c\int_{-\infty}^{\infty} |\hat{\Psi}(\xi)\hat{\bar{K}}_\beta^*(\xi)|^2 d\xi \\
&= c'\int_{-\infty}^{\infty} |(\Psi * \bar{K}_\beta^*)(x)|^2 dx\,,
\end{aligned}$$

provided that the last term is finite. Since the proof of converse part is done similarly, Theorem is proved completely.

**4. Proof of Theorem 3.** From (2.1),

$$\phi_\alpha(t) = \frac{\alpha}{t}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1}\phi(u)du = \alpha\int_0^1 (1 - v)^{\alpha-1}\phi(tv)dv\,,$$

and since $f(x_0 - u) + f(x_0 + u) \in S$,

$$\begin{aligned}
\frac{d}{dt}\phi_\alpha(t) &= \alpha\int_0^1 (1 - v)^{\alpha-1}\cdot v\phi'(tv)dv \\
&= \alpha\cdot\frac{1}{t^2}\int_0^t \left(1 - \frac{u}{t}\right)^{\alpha-1} u\phi'(u)du
\end{aligned} \tag{4.1}$$

where $\phi'(u) = -\{f'(x_0 - u) - f'(x_0 + u)\}$.

We set as in §3 $u = e^{-y}$, $t = e^{-x}$, then, by (2.4), we have

$$(4.2) \qquad \{\delta_\alpha(f)(x_0)\}^2 = \int_0^\infty \frac{1}{t}\left|t\frac{d}{dt}\phi_\alpha(t)\right|^2 dt$$
$$= \int_{-\infty}^\infty \left|\alpha\int_x^\infty \phi'(e^{-y})\cdot e^{-y}\cdot e^{(x-y)}(1-e^{x-y})^{\alpha-1}dy\right|^2 dx .$$

We set $\chi(x) = e^{-x}\phi'(e^{-x})$ and

$$K_\alpha(x) = \begin{cases} \alpha e^x(1-e^x)^{\alpha-1}, & x \leqq 0 \\ 0 \quad , & x > 0 . \end{cases}$$

Then

$$\{\delta_\alpha(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K_\alpha)(x)|^2 dx .$$

On the other hand, by (2.5)

$$\sigma_\beta(R) = c\int_0^\infty \phi(u)R\gamma_{\beta+1}(Ru)du$$

and

$$(4.3) \qquad \sigma'_\beta(R) = c\int_0^\infty \phi'(u)u\gamma_{\beta+1}(Ru)du .$$

By the definition (2.7)

$$\{h_\beta(f)(x_0)\}^2 = c'\int_0^\infty \frac{1}{R}|R\sigma'_\beta(R)|^2 dR$$
$$= c'\int_0^\infty \frac{1}{R}\left|R\int_0^\infty \phi'(u)\cdot u\gamma_{\beta+1}(Ru)du\right|^2 dR .$$

We set $u = e^{-y}$, $R = e^x$, then

$$(4.4) \qquad \{h_\beta(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K_\beta^*)(x)|^2 dx$$

where $\chi(x) = e^{-x}\phi'(e^{-x})$ and $K_\beta^*(x) = e^x\gamma_{\beta+1}(e^x)$.

Since $\phi'(e^{-x}) = f'(x_0 - e^{-x}) - f'(x_0 + e^{-x})$, $\chi(x)$ behaves better than $\Psi(x)$ in §3. However since

$$K_\beta^*(x) \sim e^x\cdot e^{-(\beta+1)x} = e^{-\beta x} \quad \text{as} \quad x \to \infty ,$$

$K_\beta^*(x)$ behaves for $0 \geqq \beta > -1/2$, as if $\bar{K}_\beta^*(x)$ in (3.4). Hence all things go analogously as in §3.

The complex Fourier transform of $K_\alpha(x)$ is

$$\alpha\int_{-\infty}^0 e^{sx}e^x(1-e^x)^{\alpha-1}dx = \frac{\Gamma(\alpha+1)\Gamma(s+1)}{\Gamma(\alpha+s+1)} \quad (\alpha > 0, \operatorname{Re} s > -1)$$

and that of $K_\beta^*(x)$ is

$$\int_{-\infty}^{\infty} e^{sx}e^x\gamma_{\beta+1}(e^x)dx = \int_0^\infty t^s\gamma_{\beta+1}(t)dt$$
$$= \frac{\pi}{2}\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-s)\cos\pi s/2}$$
$$(\beta+1>0,\ \beta+1>\operatorname{Re}s,\ -1<\operatorname{Re}s<1)\ .$$

Since $\hat{K}_\alpha(\xi) = \Gamma(\alpha+1)\Gamma(1-i\xi)/\Gamma(\alpha+1-i\xi)$, $|\hat{K}_\alpha(\xi)| \sim c|\xi|^{-\alpha}$ and since $\hat{K}_\beta^*(\xi) = \Gamma(\beta+1)/(\Gamma(\beta+1-i\xi)\cos\pi i\xi/2)$, $|\hat{K}_\beta^*(\xi)| \sim c'|\xi|^{-\beta-(1/2)}$.

If $\beta > -1/2$, then $\hat{K}_\beta^*(\xi)$ is bounded, and the theorem is proved for $\alpha = \beta + (1/2)$.

REMARK. If $\alpha > 1$, we may eliminate differentiability of $f(x)$ in (4.1) by a partial integration. However $0 < \alpha \leqq 1$ case, we define $\phi_\alpha'(t)$ by (4.1) and $\sigma_\beta'(R)$ by (4.3) respectively assuming differentiability of $f(x)$.

5. **Proof of Theorem 2.** For a proof of Theorem, we need two lemmas.

LEMMA 1. *For $f$ in $S$ we have*

$$f_\alpha(x-t) - f_\alpha(x+t) = c\int_0^\infty \xi^{-\alpha}\sin\xi t d\xi\int_0^\infty \psi(u;x,f)\sin u\xi du\ .$$

PROOF. By definition of Riesz potential, we have

$$f_\alpha(x-t) - f_\alpha(x+t) = c\int_{-\infty}^\infty |\xi|^{-\alpha}\hat{f}(\xi)e^{ix\xi}(e^{-it\xi} - e^{it\xi})d\xi$$
$$= -2ic\int_{-\infty}^\infty |\xi|^{-\alpha}\sin t\xi\cdot\hat{f}(\xi)e^{ix\xi}d\xi\ .$$

Since $|\xi|^{-\alpha}\sin t\xi$ is odd, we may take the odd part of

$$\hat{f}(\xi)e^{ix\xi} = \int_{-\infty}^\infty f(x+u)e^{iu\xi}du\ ,$$

which implies the lemma.

LEMMA 2. *If $\alpha + 1/2 = \beta$ $(0<\alpha<1)$, then for $f\in S$ we have*

$$\left(\int_0^\infty \frac{|\psi(u)|^2}{u^{2\alpha}}\frac{du}{u}\right)^{1/2} \sim \left(\int_0^\infty R\left|\frac{d}{dR}\bar{\sigma}_{\beta,\alpha}(R)\right|^2 dR\right)^{1/2}$$

*where*

$$\bar{\sigma}_{\beta,\alpha}(R) = \bar{\sigma}_{\beta,\alpha}(R;x_0,f)$$

*is the $(C,\beta)$-mean of the Fourier integral*

$$\int_0^\infty \xi^\alpha\sin\xi t d\xi\int_0^\infty\psi(u;x,f)\sin u\xi du\ .$$

PROOF. We set $u = e^{-y}$. Then

$$\int_0^\infty \frac{|\psi(u)|^2}{u^{2\alpha}} \frac{du}{u} = \int_{-\infty}^\infty |\Theta(y)|^2 dy \,, \tag{5.1}$$

where

$$\Theta(y) = \psi(e^{-y}) e^{\alpha y} \,.$$

On the other side

$$\bar{\sigma}_{\beta,\alpha}(R) = c\int_0^\infty \psi(u)\left\{R\int_0^1 (1-z)^\beta (Rz)^\alpha \sin(Rzu)dz\right\}du \,,$$

where

$$\int_0^1 (1-z)^\beta z^\alpha \sin(zu)dz \qquad (\beta > -1, \alpha > -1)$$

is Kummer's confluent hypergeometric function. Set $u = e^{-y}$, $R = e^x$, then

$$\begin{aligned} \int_0^\infty R \left|\frac{d}{dR}\bar{\sigma}_{\beta,\alpha}(R)\right|^2 dR &= \int_0^\infty \frac{1}{R}|\bar{\sigma}_{\beta-1,\alpha}(R) - \bar{\sigma}_{\beta,\alpha}(R)|^2 dR \\ &= c\int_{-\infty}^\infty |(\Theta * \bar{K}^*_{\beta,\alpha})(x)|^2 dx \end{aligned} \tag{5.2}$$

where

$$\bar{K}^*_{\beta,\alpha}(x) = e^{(\alpha+1)x}\int_0^1 (1-z)^{\beta-1} z^{\alpha+1} \sin(e^x z)dz$$

The complex Fourier tranform of $\bar{K}^*_{\beta,\alpha}(x)$ is

$$\int_{-\infty}^\infty \bar{K}^*_{\beta,\alpha}(x)e^{sx}dx = \frac{\Gamma(\beta)\Gamma(1-s)\Gamma(1+\alpha+s)\cos\{(\alpha+s)\pi/2\}}{\Gamma(\beta+1-s)}$$
$$(0 < \alpha < 1, 1/2 < \beta < 3/2, 1-\alpha > \operatorname{Re} s > -(1+\alpha))$$

which is analytic in the strip near the line $\operatorname{Re} s = 0$ and has no zero on $\operatorname{Re} s = 0$. Furthermore we have

$$\begin{aligned} |\hat{\bar{K}}^*_{\beta,\alpha}(\xi)| &\sim c\frac{e^{-(\pi|\xi|/2)}|\xi|^{1-(1/2)}e^{-(\pi|\xi|/2)}|\xi|^{\alpha+1-(1/2)}e^{\pi|\xi|/2}}{e^{-(\pi|\xi|/2)}|\xi|^{\beta+1-(1/2)}} \\ &= c|\xi|^{-\beta+\alpha+(1/2)} \quad \text{as} \quad |\xi| \to \infty \,. \end{aligned}$$

Furthermore $\hat{\bar{K}}^*_{\beta,\alpha}(\xi)$ is bounded on $\xi \in (-\infty, \infty)$. Thus any necessary condition analogous to §3 are satisfied. Comparing (5.1) with (5.2) we get the lemma.

Theorem 2 is obvious from Lemmas 1 and 2.

**6.** We consider here the Abel summability analogue of the preceeding sections. Let $f(x) \in S$ and the Poisson and conjugate Poisson integral of $f(x)$ be

$$u(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{yf(u)}{(x-u)^2+y^2}du = \frac{1}{\pi}\int_0^{\infty} \frac{y\phi(u; x, f)}{u^2+y^2}du ,$$

$$\bar{u}(x, y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-u)f(u)}{(x-u)^2+y^2}du = \frac{1}{\pi}\int_0^{\infty} \frac{u\psi(u; x, f)}{u^2+y^2}du .$$

The Littlewood-Paley function $g(f)(x)$ is defined by

$$g(f)(x) = \left\{\int_0^{\infty} y\left(\left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2\right)dy\right\}^{1/2} .$$

But since

$$\left|\frac{\partial u(x, y)}{\partial x}\right|^2 = \left|\frac{\partial \bar{u}(x, y)}{\partial y}\right|^2 ,$$

$$g(f)(x) = \left\{\int_0^{\infty} y\left(\left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial \bar{u}}{\partial y}\right|^2\right)dy\right\}^{1/2} .$$

We separate the real and imaginary part and set

$$h(f)(x_0) = \left(\int_0^{\infty} y\left|\frac{\partial u(x_0, y)}{\partial y}\right|^2 dy\right)^{1/2} , \tag{6.1}$$

$$\bar{h}(f)(x_0) = \left(\int_0^{\infty} y\left|\frac{\partial \bar{u}(x_0, y)}{\partial y}\right|^2 dy\right)^{1/2} . \tag{6.2}$$

We use notations $\phi(u) = \phi(u; x_0, f)$ and $\psi(u) = \psi(u; x_0, f)$. We set $R = y^{-1}$, and write

$$a(R) = a(R; x_0, f) = \frac{1}{\pi}\int_0^{\infty} \frac{R\phi(u)}{1+(uR)^2}du \tag{6.3}$$

and

$$\bar{a}(R) = \bar{a}(R; x_0, f) = \frac{1}{\pi}\int_0^{\infty} \frac{R^2 \cdot u\psi(u)}{1+(uR)^2}du . \tag{6.4}$$

Then

$$h(f)(x_0) = \left(c\int_0^{\infty} R\left|\frac{da(R)}{dR}\right|^2 dR\right)^{1/2}$$

and

$$\bar{h}(f)(x_0) = \left(c'\int_0^{\infty} R\left|\frac{d\bar{a}(R)}{dR}\right|^2 dR\right)^{1/2} .$$

Now we consider $\bar{h}(f)(x_0)$. By definition

$$\{\bar{h}(f)(x_0)\}^2 = c\int_0^\infty \frac{1}{R}\left|\int_0^\infty \psi(u)\cdot\frac{2R^2u}{(1+R^2u^2)^2}du\right|^2 dR\,.$$

We set $u = e^{-y}$, $R = e^x$ and $\Psi(y) = \psi(e^{-y})$, $\bar{K}^*(x) = (e^{2x}/(1+e^{2x})^2)$. Then

$$\{\bar{h}(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\Psi * \bar{K}^*)(x)|^2 dx\,. \tag{6.5}$$

The convolution is obviously well-defined. The complex Fourier transform of $\bar{K}^*(x)$ is

$$\begin{aligned}\int_{-\infty}^\infty e^{sx}\bar{K}^*(x)dx &= \int_{-\infty}^\infty \frac{e^{(s+2)x}}{(1+e^{2x})^2}dx\\ &= \int_0^\infty \frac{t^{s+1}}{(1+t^2)^2}dt\\ &= \frac{s}{2}\cdot\frac{\pi}{\sin \pi s/2}\,.\end{aligned}$$

For an Abel analogue of Marcinkiewicz function, we set

$$\psi_a(t) = \frac{1}{t}\int_0^\infty \left(\frac{u}{t}\right)^{1/2} e^{-u/t}\psi(u)du\,, \tag{6.6}$$

(see Levinson [6]) and

$$\mu_a(f)(x_0) = \left(\int_0^\infty |\psi_a(t)|^2\frac{dt}{t}\right)^{1/2}. \tag{6.7}$$

Set $t = e^{-x}$, $u = e^{-y}$, then

$$\{\mu_a(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\Psi * K)(x)|^2 dx \tag{6.8}$$

where

$$K(x) = e^{(1+1/2)x}\exp(-e^x)$$

The complex Fourier transform of $K(x)$ is

$$\begin{aligned}\int_{-\infty}^\infty e^{sx}K(x)dx &= \int_{-\infty}^\infty e^{(s+3/2)x}\exp(-e^x)dx\\ &= \int_0^\infty t^{(s+1/2)}e^{-t}dt = \Gamma\left(\frac{3}{2}+s\right).\end{aligned}$$

Therefore

$$\hat{\bar{K}}^*(\xi) = \frac{-i\xi}{2}\cdot\frac{\pi}{\sin\pi(-i\xi)/2}\,; \qquad \hat{\bar{K}}^*(\xi) \sim \frac{c|\xi|}{e^{\pi|\xi|/2}} \quad \text{as} \quad |\xi|\to\infty$$

and

$$\hat{K}(\xi) = \Gamma(3/2 - i\xi) ; \qquad \hat{K}(\xi) \sim c' \frac{|\xi|}{e^{\pi|\xi|/2}} \quad \text{as} \quad |\xi| \to \infty .$$

Thus we get the following theorem.

THEOREM 4. *For $f(x) \in S$, and $x_0 \in \boldsymbol{R}$,*

$$A\bar{h}(f)(x_0) \leqq \mu_a(f)(x_0) \leqq B\bar{h}(f)(x_0) .$$

If we set

$$\psi_a^*(t) = t \int_0^\infty \left(\frac{t}{u}\right)^{1/2} e^{-t/u} \psi(u) \frac{du}{u^2}$$

and change $t = e^{-x}$, $u = e^{-y}$, then the corresponding kernel is

$$K_a^*(x) = e^{-(1+1/2)x} \exp(-e^{-x}) .$$

Since

$$\int_{-\infty}^\infty e^{sx} K_a^*(x) = \Gamma\left(\frac{3}{2} - s\right), \qquad \hat{K}_a^*(\xi) = \Gamma\left(\frac{3}{2} + i\xi\right)$$

equals asymptotically to that of $\psi_a(t)$ as $|\xi| \to \infty$. Therefore we have

THEOREM 4′. *For $f(x) \in S$ and $x_0 \in \boldsymbol{R}$,*

$$A\bar{h}(f)(x_0) \leqq \mu_a^*(f)(x_0) \leqq B\bar{h}(f)(x_0) ,$$

*where*

$$\mu_a^*(f)(x_0) = \left\{\int_0^\infty \frac{1}{t} \left| t \int_0^\infty \left(\frac{t}{u}\right)^{1/2} e^{-t/u} \psi(u) \frac{du}{u^2} \right|^2 dt \right\}^{1/2} .$$

For the real part function $h(f)(x_0)$, we consider the following function. Let

$$\phi_a(t) = \frac{1}{t} \int_0^\infty \left(\frac{u}{t}\right)^{-1/2} e^{-u/t} \phi(u) du \tag{6.9}$$

and

$$\delta_a(f)(x_0) = \left(\int_0^\infty t \left| \frac{d}{dt} \phi_a(t) \right|^2 dt \right)^{1/2} . \tag{6.10}$$

Moreover put

$$\phi_a^*(t) = t \int_0^\infty \left(\frac{t}{u}\right)^{-1/2} e^{-t/u} \phi(u) \frac{du}{u^2}$$

and

$$\delta_a^*(f)(x_0) = \left(\int_0^\infty t\left|\frac{d}{dt}\phi_a^*(t)\right|^2 dt\right)^{1/2} .$$

Then we have

THEOREM 5. *For $f(x) \in S$ and $x_0 \in \boldsymbol{R}$,*

$$Ah(f)(x_0) \leqq \delta_a(f)(x_0) \leqq Bh(f)(x_0)$$

*and*

$$Ah(f)(x_0) \leqq \delta_a^*(f)(x_0) \leqq Bh(f)(x_0)$$

PROOF. By definition

$$\{h(f)(x_0)\}^2 = c\int_0^\infty \frac{1}{R}\left|R\frac{d}{dR}a(R)\right|^2 dR .$$

By (6.3) we have $(d/dR)a(R) = c\int_0^\infty \{u\phi'(u)/(1+(Ru)^2)\}du$. Now set $u = e^{-y}$, $R = e^x$, $\chi(x) = e^{-x}\phi'(e^{-x})$ and $K^*(x) = e^x/(1+e^{2x})$, then

$$\{h(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K^*)(x)|^2 dx . \tag{6.11}$$

On the other hand, since

$$\phi_a(t) = \frac{1}{t}\int_0^\infty \phi(u)\left(\frac{u}{t}\right)^{-1/2} e^{-u/t}du = \int_0^\infty \phi(tv)v^{-1/2}e^{-v}dv ,$$

$$\phi_a'(t) = \int_0^\infty \phi'(tv)v^{1-(1/2)}e^{-v}dv = \int_0^\infty \phi'(u)\left(\frac{u}{t}\right)^{1/2} e^{-u/t}du .$$

Set $t = e^{-x}$, $u = e^{-y}$, $\chi(x) = e^{-x}\phi'(e^{-x})$ and

$$K(x) = e^{x/2}\exp(-e^x)$$

then

$$\{\delta_a(f)(x_0)\}^2 = \int_{-\infty}^\infty |(\chi * K)(x)|^2 dx . \tag{6.12}$$

The Fourier transforms of the kernels (6.11) and (6.12) are

$$\hat{K}(\xi) = c\Gamma\left(-i\xi + \frac{1}{2}\right) \quad \text{and} \quad \hat{K}^*(\xi) = \frac{\pi}{2\cos\pi(-i\xi)/2} ,$$

respectively. Hence we get the first part of Theorem. By the same method we can prove the another part.

**7.** Here we give some corollaries of the above theorems. Fefferman [2] proves that $\bar{h}_\beta(f)(x)$ and $h_\beta(f)(x)$ is of weak type $(p, p)$ for $1 < p < 2$ and $\beta = (1/p)$. We assume this results in the sequel. In fact he proved the theorem in several variables form.

COROLLARY 1. *For* $\alpha = (1/p) - (1/2)$ *and* $1 < p < 2$, *the operator* $\mu_\alpha(f)(x)$ *is of weak type* $(p, p)$.

This is given from Theorem 1. $\alpha = (1/p) - (1/2)$, so if $\alpha = (1/2)$ then $\mu_{1/2}(f)(x)$ is of strong type $(p, p)$ for any $p$ $(1<p<2)$. Zygmund [12] proved that $\mu(f)(x) = \mu_1(f)(x)$ is of strong type $(p, p)$ for any $p > 1$.

COROLLARY 2. *For* $\alpha = (1/p) - (1/2)$ and $1 < p < 2$, *the operator* $D_\alpha(f)(x)$ *has weak type* $(p, p)$.

This comes from Theorem 3. Fefferman [2] remarks that this corollary is established by the same method to proof of $g_\beta^*(f)(x)$.

COROLLARY 3. *For* $\alpha = (1/p) + (1/2)$ and $1 < p < 2$, *the operator* $\delta_\alpha(f)(x)$ *has weak type* $(p, p)$.

Since, $h_\beta(f)(x)$ is of weak type $(p, p)$ for $1 < p < 2$, the corollary comes from Theorem 2.

COROLLARY 4. *Let*

$$\delta_0(f)(x) = \left(\int_0^\infty t\left|\frac{d}{dt}\{f(x-t) + f(x+t)\}\right|^2 dt\right)^{1/2} .$$

*Then, for* $\alpha_1 > \alpha_2 > 0$

$$h(f)(x) \sim \delta_\alpha(f)(x) \prec \delta_{\alpha_1}(f)(x) \prec \delta_{\alpha_2}(f)(x) \prec \delta_0(f)(x)$$

*and for* $\beta_1 > \beta_2 > -1/2$

$$h(f)(x) \prec h_{\beta_1}(f)(x) \prec h_{\beta_2}(f)(x) \sim \delta_{\beta_2+1/2}(f)(x) \prec \delta_0(f)(x) ,$$

*where* $\prec$ *means that if the right side is finite then the left side is finite.*

A comparison each other of Fourier transform of corresponding kernels and Theorems 3 and 5 yield the corollory.

This is an answer of Problem 6 (a) of Stein-Wainger [11, p. 1289] in one dimensional form.

REMARK. Several variables analogues in spherical sense of the above theorems will appear in the forthcoming paper.

## LITERATURES

[1] L. S. BOSANQUET, On the summability of Fourier series, Proc. London Math. Soc., 31 (1930), 144-164.

[2] C. FEFFERMAN, Inequalities for strongly singular convolution operators, Acta, Math., 124 (1970), 9-36.

[3] T. M. FLETT, On the absolute summability of a Fourier series and its conjugate series,

Proc. London Math. Soc., 8 (1958), 258–311.
[4] I. M. GELFAND AND G. E. SHIROV, Generalized functions I, Academic Press, 1968.
[5] Y. KATZNELSON, An introduction to harmonic analysis, John Wiley, 1968.
[6] N. LEVINSON, On the Poisson summability of Fourier series. Duke Math. Journ., 2 (1936), 138–146.
[7] J. MARCINKIEWICZ, Sur quelques integrales de type de Dini, Annales de la Soc. Polonaise, 17 (1928), 42–50.
[8] E. M. STEIN, On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz, Trans. Amer. Math. Soc., 88 (1958), 430–466.
[9] E. M. STEIN, The characterization of functions arising as potentials, Bull. Amer. Math. Soc., 67 (1961), 102–104.
[10] E. M. STEIN AND S. WAINGER, Problems in harmonic analysis related to curvature, Bull. Amer. Soc., 84 (1978), 1239–1295.
[11] G. SUNOUCHI, On functions regular in a half-plane, Tôhoku Math. Journ., 9 (1957), 37–44.
[12] A. ZYGMUND, On certain integrals, Trans. Amer. Math. Soc., 58 (1944), 170–204.

DEPARTMENT OF TECHNOLOGY
TAMAGAWA UNIVERSITY
MACHIDA, TOKYO 194
JAPAN