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EINSTEIN-KAEHLER METRICS ON OPEN ALGEBRAIC SURFACES OF GENERAL TYPE

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1. Introduction. If S is a compact Riemann surface, the universal covering of S is biholomorphic to the unit disk if and only if the genus of S is greater than one. This is equivalent to saying that S admits a Kaehler metric with negative constant Gaussian curvature. The following result due to T. Aubin [1] and S. T. Yau [23] is a generalization of the above fact to higher dimensions from the differential geometric viewpoint.

FACT A ([1], [23]). If M is a compact complex manifold with negative first Chern class, then M admits an Einstein-Kaehler metric which is unique up to multiplication by positive numbers.

As an application of Fact A, Yau [22] obtained the following uniformization theorem.

FACT B ([22]). Let M be a complex n-dimensional compact complex manifold with negative first Chern class. Then the inequality $(-1)^n 2(n+1)c_1(M)^{n-2}c_2(M) \geqq (-1)^n n c_1(M)^n$ holds and the equality occurs if and only if the universal covering of M is biholomorphic to the open unit ball B^n in $\boldsymbol{C}^n$.

The above inequality measures the integrated deviation of the canonical Einstein-Kaehler metric in Fact A from the ball-metric.

In dimension two, a much stronger result is known. Miyaoka [14] obtained the inequality $3c_2 \geqq c_1^2$ for the class of compact complex surfaces of general type, which includes all surfaces with negative first Chern class. Recently Miyaoka [15] proved the following more general result: if M is a compact complex surface of general type, then the inequality $3c_2(M) - c_1(M)^2 \geqq k(M)$ holds, where $k(M)$ is a nonnegative rational number which is universally determined by the configuration of all (-2)-curves (i.e., rational curve with self-intersection number -2) and equal to zero if and only if there is no (-2)-curve on M (see p. 77).

Generalizing Yau's method in [23], the author [10] independently proved the inequality $3c_2(M) - c_1(M)^2 \geqq k(M) \geqq 0$. To state our previous result precisely, we need some definitions. A compact complex space

X is called a V-manifold if it has at worst isolated quotient singularities (cf. [19]). Let X be a V-manifold and G a Riemannian metric defined in the smooth part of X. G is called a V-metric if we obtain G by pushing down a smooth metric of local unifomizations. Let M be a compact complex surface of general type and Φ_{mK} the m-canonical map. Then $M' = \Phi_{mK}(M)$ is independent of m if m is sufficiently large (cf. [12]), and is called the canonical model of M. We obtain M' by contraction of all (-2)-curves on M (cf. [12]). M' is a V-manifold with only rational double points as its singularities. M has negative Chern class if there is no (-2)-curve on M. We proved in [10] the following:

FACT C ([10]). Let M be a compact complex minimal surface of general type and M' the canonical model of M. Then M' admits an Einstein-Kaehler V-metric with negative Ricci curvature, which is unique up to multiplication by positive numbers.

Computing the Chern forms in terms of the above canonical Einstein-Kaehler V-metric, we have the following uniformization theorem.

FACT D ([10]). Let M be as above. Then the inequality $3c_2(M) - c_1(M)^2 \geqq k(M)$ holds. The equality occurs if and only if the universal covering of M minus all (-2)-curves is biholomorphic to the open unit ball minus a discrete set of points. In other words, we obtain M' by dividing the open unit ball B^2 with respect to a discrete group Γ of automorphisms acting on B^2 properly discontinuously and with only isolated fixed points.

The following example is due to F. Hirzebruch. Consider a surface in $\boldsymbol{P}^4$ defined by $Z_0^5 + Z_1^5 + Z_2^5 + Z_3^5 + Z_4^5 = 0$ and $Z_0^{15} + Z_1^{15} + Z_2^{15} + Z_3^{15} + Z_4^{15} = 0$. This has 50 singularities each of which is resolved in a smooth curve of genus 6 with self-intersection number -5 and 1875 rational double points of type A_4. The resolution gives a smooth minimal surface of general type M. M satisfies the extremal equality $3c_2(M) - c_1(M)^2 = k(M) = 27000$. By Fact D, M is the minimal resolution of some $\Gamma\backslash B^2$.

On the other hand, if $p_1, \cdots, p_k$ are distinct points in $\boldsymbol{P}^1$, the universal covering of $\boldsymbol{P}^1 - \{p_1, \cdots, p_k\}$ is biholomorphic to the disk if and only if k is greater than two. This is equivariant to saying $\boldsymbol{P}^1 - \{p_1, \cdots, p_k\}$ admits a complete Kaehler metric with negative constant Gaussian curvature with finite volume. In this paper, we shall obtain a two-dimensional analogue of the above fact on punctured Riemann surfaces. To state our results, we fix some notations. Let $\bar{M}$ be a compact complex surface and D a reduced divisor with normal crossings. We assume $(\bar{M}, D)$

satisfies the following conditions:

(1.1) (i) Let $L = K_{\bar{M}} \otimes [D]$ then $L^2 > 0$ and $L \cdot C \geqq 0$ for all irreducible curves C on $\bar{M}$,

(ii) the divisor determined by curves C such that $C \subset D$ and $L \cdot C > 0$ has only simple normal crossings as its singularities.

(When $D = \varnothing$, (1.1) is equivalent to saying that M is a minimal surface of general type by the Kodaira classification.) Let Φ_{mL} be the logarithmic m-canonical map. Then $\Phi_{mL}(\bar{M} - D)$ is independent of m if m is large enough [18], and called the logarithmic canonical model of $\bar{M} - D$, denoted by M'. We obtain M' by contraction of all (-2)-curves contained in $\bar{M} - D$, [18]. M' is a V-manifold with only rational double points as its singularities. We shall prove the following existence and uniformization theorems.

THEOREM 1. *Let $(\bar{M}, D)$ be as above. Then the logarithmic canonical model M' of $\bar{M} - D$ admits a complete Einstein-Kaehler V-metric with negative Ricci curvature, which is unique up to multiplication by positive numbers. Moreover, the total volume is finite and equal to L^2 if the Ricci tensor is $-(2\pi)^{-1}$ times the metric.*

THEOREM 2. *Let $(\bar{M}, D)$ be as above. Write $\bar{c}_i$ for the i-th logarithmic Chern class of $(\bar{M}, D)$. Then the inequality*

$$3\bar{c}_2 - \bar{c}_1^2 \geqq k(\bar{M} - D) \geqq 0 \tag{1.2}$$

holds. The equality occurs if and only if the universal covering of $\bar{M} - D$ minus (-2)-curves is biholomorphic to the open unit ball minus a discrete set of points. In other words, we obtain M' by dividing B^2 with respect to a discrete group Γ of automorphisms acting on B^2 properly discontinuously and with only isolated fixed points. In this case the canonical metric in Theorem 1 is the ball-metric.

Here, the logarithmic Chern classes are defined as follows. Let $\bar{M}$ be a compact complex manifold of dimension n and D a reduced divisor with normal crossings. $\Omega^1(\log D)$ is the bundle on $\bar{M}$ whose section in a polydisk Δ^n in $\bar{M}$ with $\Delta^n \cap D = \cup_{i=1}^k$ (coordinate hyperplanes $z_i = 0$) are given by $\sum_{i=1}^k a_i(z)dz_i/z_i + \sum_{j=k+1}^n b_j(z)dz_j$ where $a_i(z)$'s and $b_j(z)$'s are holomorphic in z. $\bar{c}_i$ is defined to be $(-1)^i c_i(\Omega^1(\log D))$.

Let $\Gamma \backslash B^2$ be the noncompact quotient of B^2 with respect to Γ such that Γ is a discrete group of automorphisms acting freely on B^2 and the volume of $\Gamma \backslash B^2$ is finite. Let $\bar{M} = (\Gamma \backslash B^2) \cup D$ be the minimal smooth compactification of $\Gamma \backslash B^2$. It is shown that $(\bar{M}, D)$ satisfies (1.1). Mumford [16] has proved that $3\bar{c}_2 = \bar{c}_1^2$. As a direct consequence of Theorem 2, the converse is also true in the following sense:

COROLLARY. *Let $(\bar{M}, D)$ as in Theorem 2. The equality $3\bar{c}_2 = \bar{c}_1^2$ holds if and only if the universal covering of $\bar{M} - D$ is biholomorphic to B^2.*

Now we shall show that Theorem 2 is a generalization of Fact D. Let $\bar{M}$ be a compact complex surface of general type. Let $D = \cup D_i$ be a union of mutually disjoint nonsingular elliptic curves D_i on $\bar{M}$. It is easy to see that $(\bar{M}, D)$ satisfies (1.1). From (1.2), we obtain the inequality

$$3c_2(\bar{M}) - c_1(\bar{M}) \geqq k(\bar{M} - D) - \sum D_i^2 \,. \tag{1.3}$$

Since $D_i^2 < 0$, (1.3) is an estimate better than (1.2). The right hand side of (1.3) represents an obstruction for the universal covering of $\bar{M}$ from being the ball, since any compact quotient of the ball has negative first Chern class and admits no elliptic curves. The inequality (1.3) is sharp. In fact, Hirzebruch [8] constructed a sequence $X_n (n = 2, 3, \cdots)$ of minimal surfaces of general type with the following properties: $c_2(X_n) = n^7$, $3c_2(X_n) - c_1(X_n)^2 = 4n^5$. There are $4n^4$ smooth disjoint elliptic curves on X_n with self-intersection number $-n$. So, we have the equality sign in (1.3). By Theorem 2, the universal covering of X_n $-$ ($4n^2$ elliptic curves) is the ball.

This paper is organized as follows.

In Section 2, we shall construct a V-volume form (whose definition is similar to that of a V-metric) Ψ on M' whose Ricci form is the -1 times a complete Kaehler V-metric on M'. In Section 3, we shall show that $-\mathrm{Ric}\,\Psi$ is of quasi-bounded geometry (i.e., of bounded geometry in terms of quasi-coordinates) in the sense similar to [4]. In Section 4, we shall follow the arguments developed in [4] to solve the Monge-Ampère equation $(\omega + \sqrt{-1}\partial\bar{\partial}u)^2 = \exp(u)\Psi$, where $\omega = -\mathrm{Ric}\,\Psi$. Then $\omega + \sqrt{-1}\partial\bar{\partial}u$ is a desired Einstein-Kaehler V-metric in Theorem 1. In Section 5, Theorem 2 will be proved. Examples of Theorem 1 will be given in Section 6. Section 7 contains miscellaneous remarks.

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2. Singular Volume Form with Negative Ricci Curvature. In this section, let $(\bar{M}, D)$ be as in Theorem 1. Set $L = K_{\bar{M}} \otimes [D]$. Denote by $\mathscr{C}$ the union of all irreducible curves C with $L \cdot C = 0$. By the Hodge index theorem [6], we have $C^2 < 0$. By (i) in (1.1), there exist no (-1)-curves (exceptional curves of the first kind) in $\bar{M} - D$. If E is a (-1)-curve with $L \cdot E = 0$, then E is one of the following: (a) E is a component of D such that E intersects the other components of D in exactly two points, (b) E is not a component of D and E intersects D in exactly one point. It is easy to see that if $\mathscr{E}$ contains a (-1)-curve E, the pair $(\bar{M}', D')$ we obtain by blowing down E also satisfies the condition (1.1) and contains no (-1)-curves E' with $L'\cdot E' = 0$ passing through the point coming from E. So, we may and do assume that $\mathscr{E}$ has no (-1)-curves, i.e., there exist no (-1)-curves with (a) and (b). We begin with the classification of the curves C on $\bar{M}$ with $L\cdot C = 0$.

LEMMA 1. *Let $(\bar{M}, D)$ and $\mathscr{E}$ be as above. Write $\mathscr{E} = \sum \mathscr{E}_\nu$ for the decomposition of $\mathscr{E}$ into connected components. Then each $\mathscr{E}_\nu$ is of one of the following five types*:

If $\mathscr{E}_\nu$ coincides with a connected component of D, then it is of one of the followings:

(2.1) *a nonsingular ellipitic curve with negative self-intersection number,*

(2.2) *a cycle of nonsingular rational curves with self-intersection number $\leqq -2$ and some of them $\leqq -3$,*

(2.3) *a rational curve with a node with negative self-intersection number.*

If $\mathscr{E}_\nu$ is properly contained in some connected component of D, $\mathscr{E}_\nu$ is

(2.4) *a chain of nonsingular rational curves with selfintersection number $\leqq -2$.*

If $\mathscr{E}_\nu$ is contained in $\bar{M} - D$, $\mathscr{E}_\nu$ is one of the followings:

(2.5) A_n, $D_m (m \geqq 4)$, E_6, E_7, E_8, *i.e., the Dynkin diagrams consisting of (-2)-curves.*

PROOF. There are no irreducible curves C with $L\cdot C = 0$ which is not a component of D and intersects D. Indeed, if C is such a curve, we have $0 = L\cdot C = K\cdot C + D\cdot C > K\cdot C$. By the Hodge index theorem, we see $C^2 < 0$. By the adjunction formula ([11, p. 118]), we have $2\pi(C) - 2 = K\cdot C + C^2 \leqq -2$. Hence C is a (-1)-curve with the property (b), which has been excluded by our assumption. If C is an irreducible curve with $L\cdot C = 0$ and $C \subset \bar{M} - D$, then $K\cdot C = L\cdot C = 0$. Since $C^2 < 0$, we have by the adjunction formula that C is a (-2)-curve. By the Hodge index theorem and classification of Cartan matrices, each connected component of $\mathscr{E}_\nu$ disjoint from D forms a Dynkin diagram. Next, Suppose $\mathscr{E}_\nu$ coincides with a connected component of D. Write $\mathscr{E}_\nu = \sum C_i$ for the decomposition into irreducible components. Then we have $0 = L\cdot C_i = K\cdot C_i + C_i^2 + \sum_{j\neq i} C_i\cdot C_j = 2g(\tilde{C}_i) - 2 + \deg(\mathfrak{c}_i) + \sum_{j\neq i} C_i\cdot C_j$, where $\tilde{C}_i$ is a normalization of C_i and $\mathfrak{c}_i$ is the conducter of C_i which is an effective divisor on $\widetilde{C}_i$ of even degree. C_i is nonsingular if and only if $\mathfrak{c}_i$ is zero. So, the following three cases are possible:

(i) $g(C_i) = 1$, $\deg(\mathfrak{c}_i) = 0$, $\sum_{j\neq i} C_i\cdot C_j = 0$,

(ii) $g(C_i) = 0$, $\deg(\mathfrak{c}_i) = 0$, $\sum_{j\neq i} C_i\cdot C_j = 2$,

(iii) $g(C_i) = 0$, $\deg(\mathfrak{c}_i) = 0$, $\sum_{j\neq i} C_i\cdot C_j = 0$.

These curves with $L\cdot C_i = 0$ have negative self-intersection numbers and some $C_i\cdot C_i \leqq -3$ in the case of (ii), by the Hodge index theorem. Finally, let $\mathscr{E}_\nu$ be of type (2.4) and C_i an irreducible component of $\mathscr{E}_\nu$. By the same argument as above, we see that the possibility is $g(\widetilde{C}_i) = 0$, $\deg(\mathfrak{c}_i) = 0$ and $\sum_{j\neq i} C_i\cdot C_j = 2$, i.e., the case (2.4). q.e.d.

LEMMA 2 (See [18]). *Let* $(\bar{M}, D)$ *be as in Theorem* 2. *Then there is a positive integer* $m_0(\bar{M}, D)$ *with the following properties: for any integer* $m \geqq m_0(\bar{M}, D)$, (i) *the complete linear system* $|mL|$ *has no base points,* (ii) *if* $N + 1 = \dim H^0(\bar{M}, \mathcal{O}(mL))$ *and* $\{\phi_0, \phi_1, \cdots, \phi_N\}$ *is a* $\boldsymbol{C}$*-basis for* $H^0(\bar{M}, \mathcal{O}(mL))$, *then the logarithmic m-canonical map* $\Phi_{mL}: \bar{M} \to \boldsymbol{P}^N$; $z \mapsto (\phi_0(z): \phi_1(z): \cdots \phi_N(z))$ *is a holomorphic map whose restriction to* $\bar{M} - \mathscr{E}$ *is biholomorphic onto its image and* $\Phi_{mL}^{-1}(\Phi_m(z)) = \begin{cases} z & \text{if } z \in \bar{M} - \mathscr{E} \\ \mathscr{E}_\nu & \text{if } z \in \mathscr{E}_\nu. \end{cases}$

PROOF. Sakai [18] proved this fact under the assumption that D is semi-stable with some minimality condition and $\kappa(\bar{M}, D) = 2$. These assumptions are satisfied under our condition (1.1) and the minimality condition. Indeed, if C is a nonsingular rational component of D, we have $0 \leqq L\cdot C = K\cdot C + C^2 + (D - C)\cdot C = -2 + (D - C)\cdot C$, hence C intersects the other components of D in more than one points, i.e., D is semi-stable. Sakai's minimality condition follows from ours. By the Riemann-Roch formula [11], $\sum_{i=0}^{2} (-1)^i \dim H^i(\bar{M}, \mathcal{O}(mL)) = (m^2L^2 - mL\cdot K)/2 + \chi(\bar{M}, \mathcal{O})$. By (i) in (1.1), we see $H^2(\bar{M}, \mathcal{O}(mL)) = 0$. Indeed, $H^2(\bar{M}, \mathcal{O}(mL)) \cong H^0(\bar{M}, \mathcal{O}(-(m-1)L - D))$ by the Serre duality. If $-(m-1)L - D$ contains an effective divisor A, we have $0 \leqq L\cdot A = -(m-1)L^2 - L\cdot D < 0$ which is absurd. Since L^2 is positive, we have $\lim_{m\uparrow\infty} m^{-2} \dim H^0(\bar{M}, \mathcal{O}(mL)) > 0$, i.e., $\kappa(\bar{M}, D) = 2$. q.e.d.

LEMMA 3. *Let* $(\bar{M}, D)$ *and* L *be as in Theorem* 1. *Then* $c_1(L)$ (*in the de Rham cohomology*) *is represented by a real closed* (1, 1) *form* γ *with the following properties*: (i) γ *is positive definite outside of* $\mathscr{E}$, (ii) *for any irreducible component* C *of* $\mathscr{E}$, $i_C^*\gamma$ *vanishes, where* i_C *is the inclusion* $C \to \bar{M}$.

PROOF. Pick m sufficiently large so that the map Φ_{mL} of $\bar{M}$ to $\boldsymbol{P}^N$ satisfies the properties in Lemma 2. The Fubini-Study metric form of $\boldsymbol{P}^N$ is given by $\sqrt{-1}(2\pi)^{-1}\partial\bar{\partial}\log(\sum_{k=0}^{N}|Z_k|^2)$ where $Z = (Z_k)$ is the homogeneous coordinate. This represents the first Chern class of the hyperplane bundle over $\boldsymbol{P}^N$. Let γ be defined by $m\gamma = \sqrt{-1}(2\pi)^{-1}\partial\bar{\partial}\log(\sum_{k=0}^{N}|\phi_k(z)|^2)$, which is the pull back of the Fubini-Study metric form by Φ_{mL}. Then γ represents $c_1(L)$. Clearly γ has the desired properties. q.e.d.

The rest of this section is devoted to the construction of a singular volume form Ψ on $\bar{M}$. First, we shall determine the complex structure around each connected component $\mathscr{E}_\nu$ of $\mathscr{E}$:

LEMMA 4. *Let $\mathscr{E} = \sum \mathscr{E}_\nu$ be the decomposition of $\mathscr{E}$ into connected components. Then there exist mutually disjoint open neighborhoods U_ν of $\mathscr{E}_\nu$ with the following properties: for each $\mathscr{E}_\nu$ there exists an open set D_ν in $\boldsymbol{C}^2$ and a discrete group Γ_ν of automorphisms of D_ν such that the deleted neighborhood $U_\nu - \mathscr{E}_\nu$ is biholomorphic to $\Gamma_\nu \backslash D_\nu$. More precisely, D_ν and Γ_ν are defined as follows:*

(i) *When $\mathscr{E}_\nu$ is of type (2.1), i.e., $\mathscr{E}_\nu = A$, a smooth elliptic curve with self-intersection number $-b(b > 0)$, there exist a positive number a, real numbers α, β, and a lattice $\mathscr{L}$ in $\boldsymbol{C}$ defined by $\{m + n\omega;\ m, n \in \boldsymbol{Z}, \operatorname{Im}(\omega) = a\}$ such that*

$$\Gamma_\nu = \left\{ \begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2h(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix};\ \begin{array}{l} \gamma \textit{ runs over } \mathscr{L}, \textit{ and for each } \gamma = m + \\ n\omega,\ h(\gamma) \textit{ runs over the class of } m\alpha + \\ n\beta - mna \textit{ modulo } (2a/b)\boldsymbol{Z} \end{array} \right\},$$

D_ν is defined by $\{(u, v) \in \boldsymbol{C}^2;\ \operatorname{Im}(u) - |v|^2 > k\}$ for some positive number k. Γ_ν acts on D_ν from the left by

$$\begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2h(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u + 2i\bar{\gamma}u + i|\gamma|^2 - 2h(\gamma) \\ v + \gamma \end{pmatrix}$$

(ii) *When $\mathscr{E}_\nu$ is of type (2.2), i.e., $\mathscr{E}_\nu = \sum_{i=0}^{r-1} B_i (r \geqq 2)$ a cycle of smooth rational curves with $-B_i \cdot B_i = b_i \geqq 2$ (some $b_j \geqq 3$), we introduce an irrational quadratic number w_k as the infinitely cyclic continued fraction $[[q_k, q_{k+1}, \cdots]] = \lim_{s\uparrow\infty}\{q_k - (q_{k+1} - \cdots - (q_{s-1} - q_s^{-1})^{-1}\cdots)^{-1})^{-1}\}$, where $\{q_k\}$ is a periodic sequence with period r defined by $q_k = b_k$ for $0 \leqq k \leqq r-1$. Let $\{R_k\}_{k\in\boldsymbol{Z}}$ be a sequence defined by $R_{-1} = w_0$, $R_0 = 1$, $q_kR_k = R_{k-1} + R_{k+1}$ for $k \in \boldsymbol{Z}$. Using these, let M be a free $\boldsymbol{Z}$-module of rank 2 generated by 1 and w_0, V the infinite cyclic multiplicative group generated by R_r. Then*

$$\Gamma_\nu = G(M, V) = \left\{ \begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix};\ \varepsilon \in V,\ \mu \in M \right\},$$

$$D_\nu = \{(z_1, z_2) \in \boldsymbol{C}^2;\ \operatorname{Im}(z_1) > 0,\ \operatorname{Im}(z_2) > 0,\ \operatorname{Im}(z_1)\cdot\operatorname{Im}(z_2) > k\}$$

for some positive number k, and Γ_ν acts on D_ν from the left by

$$\begin{pmatrix}\varepsilon & \mu\\ 0 & 1\end{pmatrix}(z_1, z_2) = (\varepsilon z_1 + \mu, \varepsilon' z_2 + \mu')\ ,$$

where $'$ means to take the conjugate over $\boldsymbol{Q}$.

(iii) *When $\mathscr{E}_\nu$ is of type* (2.4), *i.e., $\mathscr{E}_\nu$ is a rational curve with a node with self-intersection number $-b_0$ $(b_0 > 0)$, D_ν is the same as that in* (ii) *and Γ_ν is also the same as that in* (ii) *provided we let $q_k = b_0 + 2$ for all k and $r = 1$.*

(iv) *When $\mathscr{E}_\nu$ is of type* (2.4), *i.e., $\mathscr{E}_\nu = \sum_{k=1}^{r} E_k (r \geqq 1)$ a chain of smooth rational curves with $-E_k \cdot E_k = b_k \geqq 2$, we let n/q the irreducible fraction representing $b_1 - (b_2 - (\cdots - (b_{r-1} - b_r^{-1})^{-1} \cdots)^{-1})^{-1}$. Then we have*

$$\Gamma_\nu = \left\langle \begin{pmatrix}\xi & 0\\ 0 & \xi^q\end{pmatrix} \right\rangle_{gen.}\ ,$$

the group generated by $\begin{pmatrix}\xi & 0\\ 0 & \xi^q\end{pmatrix}$, where $\xi = \exp(2\pi i/n)$, and $D_\nu = B - \{0\}$, where B is a small ball centered at the origin in $\boldsymbol{C}^2$.

(v) *When $\mathscr{E}_\nu$ is of type* (2.5), *D_ν is $B - \{0\}$, where B is a small ball centered at the origin in $\boldsymbol{C}^2$ and*

(v-i) *if $\mathscr{E}_\nu = A_n$, then*

$$\Gamma_\nu = \left\langle \begin{pmatrix}\xi & 0\\ 0 & \xi^n\end{pmatrix} \right\rangle_{gen.}\ ,$$

where $\xi = \exp(2\pi i/n + 1)$,

(v-ii) *if $\mathscr{E}_\nu = D_m (m \geqq 4)$, then $\Gamma_\nu =$ the binary dihedral group $\tilde{\mathfrak{D}}_{2(m-2)}$,*
(v-iii) *if $\mathscr{E}_\nu = E_6$, then $\Gamma_\nu =$ the binary tetrahedral group $\tilde{\mathfrak{A}}_4$,*
(v-iv) *if $\mathscr{E}_\nu = E_7$, then $\Gamma_\nu =$ the binary octahedral group $\tilde{\mathfrak{S}}_4$, and*
(v-v) *if $\mathscr{E}_\nu = E_8$, then $\Gamma_\nu =$ the binary icosahedral group $\tilde{\mathfrak{A}}_5$.*

Secondly we shall construct a singular volume form Ψ on $\bar{M}$. If we contract all connected components of $\mathscr{E}$ lying outside of D, we get a V-manifold $\bar{M}'$.

DEFINITION. A continuous function h on $\bar{M}'$ is called *V-smooth* if (i) h is smooth in the smooth part of $\bar{M}'$, (ii) $\pi^* h$ is smooth where π denotes a local uniformization.

In the following lemma, we shall use the notations in Lemmas 2 and 4.

LEMMA 5. *Let $\mathscr{E}' = \sum' \mathscr{E}_\nu$ be the union of all the connected components of $\mathscr{E}$ lying in D. Write $D = \sum D_l$ for the decomposition into irreducible components. Then there exists a smooth volume form Ω on $\bar{M}$,* *Hermitian metrics* $\|\cdot\|$ *for* $[D_l]$, *mutually disjoint open neighborhoods* U_ν *of* $\mathscr{E}_\nu \subset \mathscr{E}'$, *functions* f_ν *which are V-smooth outside of* $\mathscr{E}_\nu$ *and a V-smooth function h on* $\bar{M}$ *such that* $\Psi = h\Omega/\prod \|s_l\|^2 \prod' f_\nu \prod''(\log(c\|s_l\|^{-2})^2$ *is a V-volume form on* $\bar{M} - D$ *with negative Ricci curvature, where* $\prod'$ *means to take the product over* ν *such that* $\mathscr{E}_\nu$ *is contained in* $\mathrm{supp}(D)$, $\prod''$ *to take the product over l such that* D_l *is disjoint from* $\mathscr{E}$ *and c a suitably chosen positive number. Moreover,* $\omega = -\mathrm{Ric}\,\Psi$ *has the following properties:* ω *is a complete Kaehler V-metric on* $\bar{M} - D$ *with finite total volume and the inequality* $C^{-1} < \Psi/(\omega^2) < C$ *holds in* $\bar{M} - D$ *for some positive number C. Here,* f_ν *is a function going to* ∞ *at* $\mathscr{E}_\nu$, *written as follows:*

Case (i) $\mathscr{E}_\nu = A$ *is of type* (1.1): $f_\nu = \{\log\|\sigma\|^{-2}\}^{-3}$ *for some Hermitian norm* $\|\cdot\|$ *for* $[A]$ *and* $\sigma \in \Gamma(\bar{M}, [A])$ *such that* $(\sigma = 0) = A$. *More precisely,* $f_\nu | U_\nu = [\log\{c(\exp(-|z|^2))^{b\pi/a}|w|^{-2}\}]^{-3}$ *for some positive constant* c, $f_\nu \equiv$ *constant in a neighborhood of* $\mathscr{E} - \mathscr{E}_\nu$, *where* $w = \exp(b\pi i u/2a)$, $z = v$.

Case (ii) *and* (iii) $\mathscr{E}_\nu = \sum_{i=0}^{r-1} B_r (r \geqq 1)$ *is of type* (1.2) *or* (1.3): *let* C_k^2 *be a copy of* C^2 *with natural coordinate* (u_k, v_k). *We put an identification* $u_k = u_{k-1}^{q_{k-1}} \cdot v_{k-1}$, $v_k = u_{k-1}^{-1}$ *on the disjoint union* $\coprod_{k\in Z} C_k^2$. *Then* (u_k, v_k) *is a local coordinate in a neighborhood of* $B_{k-1} \cap B_k$ *for* $0 \leqq k \leqq r-1$ *where* $B_{-1} = B_{r-1}$. f_ν *is written in terms of* (u_k, v_k) *as*

$$(R_{k-1}\log|u_k|^{-1} + R_k \log|v_k|^{-1})^{-2}(R'_{k-1}\log|u_k|^{-1} + R'_k\log|v_k|^{-1})^{-2}\,,$$

in a neighborhood of $B_{k-1} \cap B_k$, $f_\nu \equiv$ *constant in a neighborhood of* $\mathscr{E} - \mathscr{E}_\nu$.

Case (iv) $\mathscr{E}_\nu = \sum_{k=1}^{r} E_k$ *is of type* (1.4): *let* $\pi: D_\nu = B - \{0\} \to U_\nu - \mathscr{E}_\nu$ *be the projection and* (λ, μ) *the natural coordinate of B. Denote by* E_0 *and* E_{r+1} *the irreducible components of D intersecting* E_1 *and* E_r, *respectively. Then there are continuous functions* g_1, g_2 *defined in a neighborhood of* $\sum_{k=0}^{r-1} E_k$ *such that* (i) $\pi^* g_1 | (U_\nu - \mathscr{E}_\nu) = |\mu|^2$, $\pi^* g_2 | (U_\nu - \mathscr{E}_\nu) = |\lambda|^2$, (ii) *the zero-locus of* g_1 *(resp.* g_2*) is* $\mathscr{E}_\nu \cup E_0$ *(resp.* $\mathscr{E}_\nu \cup E_{r+1}$*),* (iii) $g_1 = |\sigma_0|^2$ *(resp.* $g_2 = |\sigma_{r+1}|^2$*) near* E_0 *(resp. near* E_{r+1}*), where* σ_0 *(resp.* σ_{r+1}*) is a holomorphic section of* $[E_0]$ *(resp.* $[E_{r+1}]$*) with* $(\sigma_0 = 0) = E_0$ *(resp.* $(\sigma_{r+1} = 0) = E_{r+1}$*) and* $\|\cdot\|$ *is a certain Hermitian metric for* $[E_0]$ *(resp.* $[E_{r+1}]$*),* (iv) $\log(g_1)$ *and* $\log(g_2)$ *is smooth outside of* $\sum_{k=0}^{r+1} E_k$. *Finally,* f_ν *has the following properties:* $f_\nu = (\log(c/g_1))^2(\log(c/g_2))^2$ *near* $\sum_{k=0}^{r+1} E_k$, *where c is a positive constant,* $f \equiv$ *constant in a neighborhood of* $\mathscr{E} - \mathscr{E}_\nu$.

PROOF OF LEMMAS 4 AND 5. Step 1. Substep 1-1: Let A be a non-singular elliptic curve with self-intersection number $-b$ ($b > 0$) on $\bar{M}$ which is an isolated component of $\mathscr{E}$. We analyse a small neighborhood of A. We use the following theorem due to Grauert [5]: "Let $\pi: (\tilde{X}, C) \to (X,$

x) be the minimal resolution of the normal two-dimensional singularity (X, x) such that $\pi^{-1}(x) = C$ is a nonsingular curve of genus g. Let N denote the normal bundle of C in X. If $C^2 < 4 - 4g$, there are a neighborhood U of C in $\tilde{X}$, a neighborhood V of the zero-section ($\cong C$) of N and a biholomorphic map $\sigma: U \to V$ such that $\sigma|_C$ is the identity of C''. Combining this theorem with our assumption, we may assume that there is a neighborhood V of A in $\bar{M}$ which is biholomorphic to a neighborhood of the zero-section of the normal bundle N of A in $\bar{M}$. Since $A^2 = -b$ $(b > 0)$, N is a negative line bundle of degree $-b$. We set $A = \boldsymbol{C}/\mathscr{L}$, $\mathscr{L} = \{\boldsymbol{Z} + \boldsymbol{Z}\omega; \operatorname{Im}\omega > 0\}$, and denote the projection $\boldsymbol{C} \to A$ by π. Let a be the area of the fundamental domain of $\mathscr{L}$ measured by the usual flat metric $|dz|^2$ of $\boldsymbol{C}$. There is a real closed $(1, 1)$-form η on A such that $\pi^*\eta = i(2a)^{-1}dz \wedge d\bar{z}$. By the definition of a, $[\eta]$ is a generator of $H^2(A; \boldsymbol{R})$. Let ρ be the Hermitian metric of the line bundle $N \to A$ whose Chern form is given by $-b\eta = -ib(2a)^{-1}dz \wedge d\bar{z}$. The curvature form of the induced Hermitian line bundle $\pi^*N \to \boldsymbol{C}$ is given by $-ib\pi a^{-1}\cdot dz \wedge d\bar{z}$. Since $H^1(\boldsymbol{C}, \mathcal{O}^*) = \{1\}$, there is an isomorphism $\boldsymbol{C}^2 \xrightarrow{\sim} \pi^*N$ between holomorphic line bundles over $\boldsymbol{C}$ where $\boldsymbol{C}^2$ is the trivial line bundle over $\boldsymbol{C}$. In particular, we may regard ρ as a positive function on $\boldsymbol{C}$. There is an entire holomorphic function $\zeta(z)$ such that $\pi^*\rho(z)$ is equal to $\{\exp(-|z|^2)|\exp\zeta(z)|^2\}^{b\pi/a}$. The biholomorphic map of $\boldsymbol{C}^2$ into itself defined by $(w, z) \mapsto (\exp\{-b\pi\zeta(z)/a\}\cdot w, z)$ is an isomorphism of trivial line bundles over $\boldsymbol{C}$ and the Hermitian metric $(\exp(-|z|^2))^{b\pi/a}$ is pulled back to the Hermitian metric $(\exp(-|z|^2)|\exp\zeta(z)|^2)^{b\pi/a}$. We may assume that $\pi^*N = \boldsymbol{C}^2$, $\pi^*\rho(z) = (\exp(-|z|^2))^{b\pi/a}$. Let U be an open subset of $\boldsymbol{C}$ such that $\bar{U}$ is contained in a fundamental domain of $\mathscr{L}$. Let γ be an arbitrarily fixed element of $\mathscr{L}$. Since $\boldsymbol{C}\times U$ and $\boldsymbol{C}\times(U + \gamma)$ are local trivializations of $N|_{\pi(U)}$, there exists a non-vanishing holomorphic function $g(z)$ defined in $(z \in)U$ such that $(w, z) \in \boldsymbol{C}\times U$ and $(w', z') \in \boldsymbol{C}\times(U + \gamma)$ represent the same point of $N|_{\pi(U)}$ if and only if $z' = z + \gamma$ and $w' = g(z)\cdot w$. $g(z)$ must satisfy the following equality:

$$\begin{aligned} |w|^2(\exp(-|z|^2))^{-b\pi/a} &= |w'|^2(\exp(-|z'|^2))^{-b\pi/a} \\ &= |g(z)|^2|w|^2(\exp(-|z + \gamma|^2))^{-\pi/a} \end{aligned}$$

for all $z \in U$ and $w \in \boldsymbol{C}$. Therefore $g(z)$ must be written as

$$g(z) = \exp\left\{-\frac{b\pi}{a}\left(z\bar{\gamma} + \frac{|\gamma|^2}{2} + i\theta(\gamma)\right)\right\}, \tag{2.6}$$

where $\theta(\gamma)$ is a real number determined by $\gamma \in \mathscr{L}$ modulo $(2a/b)\boldsymbol{Z}$. Now we look at $\theta(\gamma)$ more closely. If $z' = z + \gamma$ and $z'' = z' + \gamma'(\gamma, \gamma' \in \mathscr{L})$, then (w, z), (w', z') and (w'', z'') represent the same point if and only if the following three equalities hold:

$$w'' = \exp\left\{-\frac{b\pi}{a}\left(z\overline{(\gamma + \gamma')} + \frac{|\gamma + \gamma'|^2}{2} + i\theta(\gamma + \gamma')\right)\right\}\cdot w\,,$$

$$w'' = \exp\left\{-\frac{b\pi}{a}\left(z'\bar{\gamma}' + \frac{|\gamma'|^2}{2} + i\theta(\gamma')\right)\right\}\cdot w'\,,$$

$$w' = \exp\left\{-\frac{b\pi}{a}\left(z\bar{\gamma} + \frac{|\gamma|^2}{2} + i\theta(\gamma)\right)\right\}\cdot w\,.$$

Hence $\theta(\gamma + \gamma') = \theta(\gamma) + \theta(\gamma') - \operatorname{Im}(\bar{\gamma}\gamma')$ modulo $(2a/b)\boldsymbol{Z}$. Recall that $\mathscr{L} = \{\boldsymbol{Z} + \boldsymbol{Z}\omega;\ \operatorname{Im}\omega = a > 0\}$. It follows that $\theta(m + n\omega) = m\alpha + n\beta - mna$ modulo $(2a/b)\boldsymbol{Z}$, where α and β are fixed representatives of $\theta(1)$ and $\theta(\omega)$, respectively. Using this $\theta(\gamma)$ $(\gamma \in \mathscr{L})$, we define a group Γ of 3×3 matrices as follows:

$$\Gamma = \left\{\begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2h(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix};\ \begin{matrix} \gamma \in \mathscr{L},\ h(m + n\omega) \text{ is an element} \\ \text{of } \theta(m + n\omega) = m\alpha + n\beta - mna \\ \text{modulo } (2a/b)\boldsymbol{Z} \end{matrix}\right\}.$$

Let B be the unit ball in $\boldsymbol{C}^2$ and $\mathscr{S}$ the domain in $\boldsymbol{C}^2$ defined by $\{(u, v) \in \boldsymbol{C}^2;\ \operatorname{Im}(u) - |v|^2 > 0\}$. Then $z_1 = (u - i)(u + i)^{-1}$ and $z_2 = (2v)(u + i)^{-1}$ give a biholomorphic map of B to $\mathscr{S}$. For a positive integer k, we consider the subdomain W of $\mathscr{S}$ defined by $W = \{(u, v) \in \mathscr{S}: \operatorname{Im}(u) - |v|^2 > k\}$. W corresponds to the horoball at $(1, 0)$ of B with the Bergman metric. Γ is a discrete subgroup of the group of the analytic automorphisms of $\mathscr{S}$, acting on $\mathscr{S}$ properly discontinuously and without fixed points. W is invariant under the action of Γ. This action is described as follows:

$$\text{(2.7)}\quad \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 1 & 2i\bar{\gamma} & i|\gamma|^2 - 2h(\gamma) \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} u + 2i\bar{\gamma}v + i|\gamma|^2 - 2h(\gamma) \\ v + \gamma \end{pmatrix}.$$

The map $F: W \to \boldsymbol{C}^2$ defined by $(u, v) \to (\exp(b\pi iu/2a), v)$ maps W onto the set $V' = F(W) = \{(w, z) \in \boldsymbol{C}^2;\ 0 < |w|^2/(\exp(-|z|^2))^{b\pi k/a}\}$. If we define $V = \{w \in N;\ 0 < \rho(w, w) < \exp(-b\pi k/a)\}$, then $V' = \pi^{-1}(V)$. V is a deleted neighborhood of the zero-section of N and a punctured disk bundle over an elliptic curve A. Now we show that $\Gamma\backslash W$ is biholomorphic to V. By (2.6) and (2.7), $\pi\circ F(u, v) = \pi\circ F(u', v')$ if and only if ${}^t(u', v') = \gamma\, {}^t(u, v)$ for some $\gamma \in \Gamma$. So, there is a unique biholomorphic map $\tilde{F}: \Gamma\backslash W \to V$ with $\pi\circ F = \tilde{F}\circ p$, where p is the projection $W \to \Gamma\backslash W$. The Kaehler form of the Bergman metric of B is written in terms of the coordinate (u, v) of $\mathscr{S}$ as

$$\frac{dv \wedge d\bar{v}}{\operatorname{Im}(u) - |v|^2} + \frac{(-idu - 2\bar{v}dv) \wedge (id\bar{u} - 2vd\bar{v})}{4(\operatorname{Im}(u) - |v|^2)^2} .$$

This projects down to a Kaehler metric of V, whose Kaehler form is given by

$$\frac{dz \wedge d\bar{z}}{(a/b\pi)\log|w|^{-2} - |z|^2} + \frac{((a/b\pi)(dw/w) + \bar{z}dz) \wedge ((a/b\pi)(\overline{dw/w}) + zd\bar{z})}{((a/b\pi)\log|w|^{-2} - |z|^2)^2}$$

where $(w, z) = F(u, v) = (\exp(b\pi iu/2a), v)$, i.e., a local trivialization of $V \to A$. If we define a function f of V to $\boldsymbol{R}^+$ by $\exp(f(w)) = \rho(w, w)^{-1}$, then $(F \circ p)^*(-i\partial\bar{\partial}\log f)$ is equal to the restriction to W of the Kaehler metric coming from the Bergman metric of B.

REMARK 1. Let (X, A) be a pair of a complex surface X and a nonsingular elliptic curve A holomorphically embedded into X with $A^2 < 0$. Then $K_X \otimes [A]$ is analytically trivial near A.

Substep 1–2: Let B be a component of D of type (2.2) in Lemma 1, i.e., B is a cycle of $\boldsymbol{P}^1$'s with the decomposition $B = \sum_{i=0}^{r-1} B_i (r \geqq 2)$ into irreducible components. For $0 \leqq i \leqq r-1$, we define a positive number b_i by $-b_i = B_i \cdot B_i$. By our assumption, $b_i \geqq 2$ for all i, $b_i \geqq 3$ for some i. Let q_k for $k \in \boldsymbol{Z}$ be the periodic sequence of period r with $q_k = b_k$ with $0 \leqq k \leqq r-1$. From now on, we choose an open neighborhood $\tilde{X}$ of B in $\bar{M}$ and analyse the differential geometric structure of $(\tilde{X}, B)$. By [5], B can be contracted to a normal singular point x. We review Hirzebruch's theory (cf. [8]) of the construction of the minimal resolution $\pi: (\tilde{X}, B) \to (X, x)$ in a manner suitable to our purpose. Let $E \to \boldsymbol{P}^1$ be the q-th tensor power of the tautological line bundle. Then E is covered by $U_i \cong \boldsymbol{C}^2 (i = 1, 2)$ with conrdinates (u_i, v_i) with the transition rules $u_2 = u_1^{-1}$, $v_2 = u_1^q v_1$ on $U_1 \cap \{u_1 \neq 0\} = U_2 \cap \{u_2 \neq 0\}$. Combining Grauert's theorem [5] cited above with this fact, we can naturally construct a complex manifold containing a cycle of $\boldsymbol{P}^1$'s with the same intersection matrix as B. This is done as follows. For a integer k, let $\boldsymbol{C}_k^2$ be the copy of $\boldsymbol{C}^2$ with the coordinate (u_k, v_k). We put the identifications defined by

$$u_k = u_{k-1}^{qk-1} \cdot v_{k-1} , \quad v_k = u_{k-1}^{-1} \tag{2.8}$$

on the disjoint union $\coprod_{k \in \boldsymbol{Z}} \boldsymbol{C}_k^2$. Let the resulting manifold be denoted by Y'. It follows that the curve in Y' defined by $v_k = 0$ in $\boldsymbol{C}_k^2$ and $u_{k+1} = 0$ in $\boldsymbol{C}_{k+1}^2$ is a nonsingular rational curve with self-intersection number $-q_k$. We denote this by S_k. $\bigcup_{k \in \boldsymbol{Z}} S_k$ forms a chain of infinitely many $\boldsymbol{P}^1$'s.

The identification (2.8) is written as

$$(\log u_k \ \log v_k) = (\log u_{k-1} \ \log v_{k-1}) \begin{pmatrix} q_{k-1} & -1 \\ 1 & 0 \end{pmatrix}$$
$$= (\log u_0 \ \log v_0) \begin{pmatrix} q_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_{k-1} & -1 \\ 1 & 0 \end{pmatrix} .$$

Set

$$\begin{pmatrix} P_k & -P_{k-1} \\ Q_k & -Q_{k-1} \end{pmatrix} = \begin{pmatrix} q_0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} q_{k-1} & -1 \\ 1 & 0 \end{pmatrix} .$$

Then $q_k P_k = P_{k-1} + P_{k+1}$, $P_0 = 1$, $P_1 = q_0$ and $q_k Q_k = Q_{k-1} + Q_{k+1}$, $Q_0 = 0$, $Q_1 = 1$. $\{P_k\}_{k \geqq 1}$ and $\{Q_k\}_{k \geqq 1}$ are determined by the continued fractions $[[q_0, q_1, \cdots, q_k]] = q_0 - (q_1 - (q_2 - \cdots - (q_{k-2} - q_{k-1}^{-1})^{-1} \cdots)^{-1})^{-1} = P_k/Q_k$, where P_k and Q_k are mutually prime for positive integers ($k \geqq 1$). We have assumed that all $q_k \geqq 2$ and some $q_k \geqq 3$. The infinite periodic continued fraction $[[q_0, q_1, \cdots, q_s, \cdots]] > 1$ represents a real quadratic irrational number w_0. Let $w_s := [[q_s, q_{s+1}, \cdots]] > 1$. The sequences $\{P_k\}_{k \geqq 1}$ and $\{Q_k\}_{k \geqq 1}$ are strictly increasing and $\lim_{k \to \infty} P_k/Q_k = w_0$. Now we define a sequence $\{R_k\}_{k \geqq 1}$ by $R_k = P_k - Q_k w_0$ ($k \geqq 1$). Then R_k's satisfy $q_k R_k = R_{k-1} + R_{k+1}$. Using this, we can define R_k for any integer k. From the definitions, $R_0 = 1$, $R_1 = w_1^{-1}, \cdots, R_k = w_1^{-1} w_2^{-1} \cdots w_k^{-1}$, $R_{-1} = w_0$, $R_{-2} = w_0 w_{-1}$, $\cdots, R_{-k} = w_0 w_1 \cdots w_{-k+1}$. So, $\{P_k/Q_k\}_{k \geqq 1}$ approximates w_0 and $\lim_{k \to \infty} (P_k - Q_k w_0) = 0$. $M = \boldsymbol{Z} + \boldsymbol{Z} w_0$ is a free $\boldsymbol{Z}$-module of rank 2 and $\{R_{k-1}, R_k\}$, for any integer k, is a $\boldsymbol{Z}$-basis of M. Since $w_k = w_{k+r}$ for any integer k, $R_r R_k = R_{k+r}$ holds. So, $R_r M = M$. By the Hamilton-Cayley theorem, R_r and $R_{-r} = R_r^{-1}$ are both algebraic integers, in particular, $R_r' = R_r^{-1}$, where $'$ means to take the conjugate over $\boldsymbol{Q}$. Let $V = \{R_r^n\}_{n \in \boldsymbol{Z}} \cong \boldsymbol{Z}$ under the correspondence $R_r^n \leftrightarrow n$. Then $G(M, V) = \left\{\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix}; \varepsilon \in V, \mu \in M\right\}$ acts on $\boldsymbol{C}^2$ properly discontinuously and without fixed points as follows:

$$\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix} (z_1, z_2) := (\varepsilon z_1 + \mu, \varepsilon' z_2 + \mu') .$$

In particular, the action of $V \cong \boldsymbol{Z}$ is given by $n \cdot (z_1, z_2) = (R_r^n z_1, R_r^{-n} z_2)$. Proper discontinuity follows from $\lim_{k \to \infty} R_k = 0$. The action of $G(M, V)$ on $\boldsymbol{C}^2$ restricts to H^2, where H is the upper half plane. Since both R_r^n and R_r^{-n} are algebraic integers, the function $\operatorname{Im}(z_1) \cdot \operatorname{Im}(z_2)$ is invariant under the action of $G(M, V)$. We shall show that there is a neighborhood Y_+ of $\bigcup_{k \in \boldsymbol{Z}} S_k$ in Y' such that $Y_+ - \bigcup_{k \in \boldsymbol{Z}} S_k$ is biholomorphic to H^2/M. The equation

$$(2.9)_k \qquad 2\pi i(z_1, z_2) = (\log u_k, \log v_k)\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix}$$

determines the class of (z_1, z_2) in $\boldsymbol{C}^2/M$. This is well-defined, since if $(2.9)_k$ holds, then so does $(2.9)_{k+1}$ by the definitions of (u_k, v_k) and R_k. Hence $\boldsymbol{C}^2/M \cong Y' - \bigcup_{k\in\boldsymbol{Z}} S_k$. Y_+ is by definition the closure of the image of H^2/M under this isomorphism, which is given by

$$\begin{aligned} Y_+ = \{(u_k, v_k) \in \boldsymbol{C}_k^2;\ \infty &\geqq R_{k-1} \log |u_k|^{-1} + R_k \log |v_k|^{-1} > 0 \\ \infty &\geqq R_{k-1} \log |u_k|^{-1} + R_k \log |v_k|^{-1} > 0\}\ . \end{aligned}$$

It follows that

$$(2.10) \qquad H^2/M \cong Y^+ - \bigcup_{k\in\boldsymbol{Z}} S_k\ .$$

To make a cycle B of $\boldsymbol{P}^1$'s from the chain of $\boldsymbol{P}^1$'s, we need to put a periodic identification on Y_+. We consider the following $\boldsymbol{Z}$-action on Y'. For $n \in \boldsymbol{Z}$ and (α, β) in the coordinate neighborhood $\boldsymbol{C}_k^2$, $n\cdot(\alpha, \beta)$ is defined by (α, β) in terms of the $(k + nr)$-th coordinate. This $\boldsymbol{Z}$-action restricts to the action on $Y_+ - \bigcup_{k\in\boldsymbol{Z}} S_k$ and is compatible with the $V(\cong\boldsymbol{Z})$-action on H^2/M via isomorphism (2.10). Indeed, the point of $Y' - \bigcup_{k\in\boldsymbol{Z}} S_k$ expressed as (α, β) in the $(k + nr)$-th coordinate is written as $(\alpha^a\beta^b, \alpha^{-c}\beta^{-d})$ in the k-th coordinate, where

$$\begin{pmatrix} a & -c \\ b & d \end{pmatrix} = \left\{\begin{pmatrix} q_{k-r} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q_{k-r+1} & -1 \\ 1 & 0 \end{pmatrix}\cdots\begin{pmatrix} q_{k-1} & -1 \\ 1 & 0 \end{pmatrix}\right\}^n .$$

So,

$$\begin{aligned} (\log \alpha^a\beta^b\ \log \alpha^{-c}\beta^{-d})\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix} &= (\log\alpha\ \log\beta)\begin{pmatrix} R_{k-nr-1} & R'_{k-nr-1} \\ R_{k-nr} & R'_{k-nr} \end{pmatrix} \\ &= (\log\alpha\ \log\beta)\begin{pmatrix} R_{k-1} & R'_{k-1} \\ R_k & R'_k \end{pmatrix}\begin{pmatrix} R_{-r}^n & 0 \\ 0 & R_{-r}^n \end{pmatrix} . \end{aligned}$$

Since $c > d > 0$, if $|\alpha| < \varepsilon$ and $|\beta| < 1/\varepsilon$ then the cardinality of n such that $|\alpha^{-c}\beta^{-d}| \leqq 1$ is finite. It follows that this $\boldsymbol{Z}$-action on Y_+ is properly discontinuous and without fixed points. We define $Y = Y_+/\boldsymbol{Z}$ by this action. The image of S_k's forms a cycle $B = \sum_{k=0}^{r-1} B_k$ of $\boldsymbol{P}^1$'s such that $B_k\cdot B_k = -q_k$. Summing up the above arguments, there is a canonical biholomorphic map $H^2/G(M, V) \cong Y - B$, and the correspondence is given by $(2.9)_k$ in the k-th coordinate of Y' and the Euclidean one of H^2. The open set W_L of H^2 defined by $\{(z_1, z_2) \in H^2; \operatorname{Im}(z_1)\times\operatorname{Im}(z_2) > L\}$ is invariant under the action of $G(M, V)$ and its image in $Y - B$ is a deleted neighborhood of B. By Laufer [13], $(q_0, q_1, \cdots, q_{r-1})$ determines the complex structure of the neighborhood of B. Hence there is neighborhood V_L of B in $\bar{M}$ such that $V_L - B$ is biholomorphic to $W_L/G(M, V)$ for some large L. $\{4\pi^2 \cdot \operatorname{Im}(z_1)\operatorname{Im}(z_2)\}^{-2}$ defined in H^2 projects down to a function f defined near B which is given by

$$f(u_k, v_k) = (R_{k-1}\log|u_k|^{-1} + R_k \log|v_k|^{-1})^{-2}(R'_{k-1}\log|u_k|^{-1} + R'_k\log|v_k|^{-1})^{-2}$$

in terms of the k-th coordinate. The Poincaré metric $|dz_1|^2/(y_1)^2 + |dz_2|^2/(y_2)^2$ projects down to a Kaehler metric $i\partial\bar{\partial}\log f$.

REMARK 2. If $Y \supset B$ is as above, then $K_Y \otimes [B]$ is analytically trivial in a neighborhood of B.

Substep 1-3. Let C be an isolated component of D of type (2.3) in Lemma 1, i.e., a rational curve with a node. If $C^2 = -b_0$ ($b_0 > 0$), then we can reproduce a complex surface Y containing a rational curve with a node of self-intersection number $-b_0$ by the construction in the preceeding Substep 1-2 provided we let $q_k = b_0 + 2$ for all k and $r = 1$. Hence there is the same function f as above which comes from $\operatorname{Im}(z_1)^{-2}\operatorname{Im}(z_2)^{-2}(2\pi)^{-2}$ and whose "$i\partial\bar{\partial}\log$" is the Kaehler form coming from the Poincaré metric of H^2.

Substep 1-4. Let E be a connected component of D containing connected components of $\mathscr{E}$. For brevity, we assume that the number of such components of $\mathscr{E}$ is one. The following arguments will be easily extended to the general case. Let $\mathscr{E}_1 = \sum_{k=1}^r E_k$ be the component of $\mathscr{E}$ contained in E. By Lemma 1, $\mathscr{E}_1$ is a chain of $\boldsymbol{P}^1$'s with $b_k = -E_k^2 \geqq 2$. Let E_0 and E_{r+1} be the irreducible components of $E - \mathscr{E}$ which meets E_0 and E_r, respectively. We cover the chain $\mathscr{E}_1$ of $\boldsymbol{P}^1$'s by $r+1$ coordinate neighborhoods $(U_k; u_k, v_k)$, $0 \leqq k \leqq r$, with the following transition rules:

$$\begin{aligned}
&u_1 = u_0^{-1},\ v_1 = u_0^{b_1}\cdot v_0 && \text{in} \quad U_0 \cap U_1 = \{u_0 \neq 0\}\ , \\
&u_2 = u_1 v_1^{b_2},\ v_2 = v_2^{-1} && \text{in} \quad U^1 \cap U^2 = \{v_1 \neq 0\}\ , \\
&u_3 = u_2^{-1},\ v_3 = u_2^{b_3}\cdot v_2 && \text{in} \quad U^2 \cap U^3 = \{u^2 \neq 0\}\ , \\
&\cdots\cdots
\end{aligned} \tag{2.11}$$

E_s is given by $\{v_{s-1} = v_s = 0\}$ if s is odd and by $\{u_{s-1} = u_s = 0\}$ if s is even. We may assume that E_0 is given by $\{u^0 = 0\}$ and E_{r+1} by $\{u_r = 0\}$, if r is odd or by $\{v_r = 0\}$ if r is even. Let n/q be the irreducible representation of the continued fraction $b_1 - (b_2 - (\cdots (b_{r-1} - b_r^{-1})^{-1}\cdots)^{-1})^{-1}$. Let Γ be the cyclic group of order n generated by $\xi = \exp(2\pi i/n)$. The $\boldsymbol{P}^1$-configuration $\mathscr{E}_1$ appears as the minimal resolution of the quotient singularity $\Gamma\backslash B$, where the action of Γ on B is defined by

$$\xi \cdot {}^t(\lambda, \mu) := \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix} \begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \xi\lambda \\ \xi^q \mu \end{pmatrix} .$$

The minimal resolution is described in terms of (λ, μ) and (u_k, v_k) as follows: Let us assume that r is odd. Then

$$\begin{aligned} (2.12) \quad \lambda^n &= v_0 = u_1^{\mu_2} \cdot v_1^{\mu_1} = \cdots = u_{2s}^{\mu_{2s}} \cdot v_{2s}^{\mu_{2s+1}} = u_{2s+1}^{\mu_{2s+2}} \cdot v_{2s+1}^{\mu_{2s+1}} = \cdots = u_r^n v_r^{q'} , \\ \lambda^{n-q}\mu &= u_0 v_0 = \cdots = u_{2s}^{\nu_{2s}} \cdot v_{2s}^{\nu_{2s+1}} = u_{2s+1}^{\nu_{2s+2}} \cdot v_{2s+1}^{\nu_{2s+1}} = \cdots = u_r^{n-q} v_r^{q'-\alpha} , \\ \mu^n &= u_0^n v_0^q = \cdots = u_{2s}^{\lambda_{2s}} \cdot v_{2s}^{\lambda_{2s+1}} = u_{2s+1}^{\lambda_{2s+2}} \cdot v_{2s+1}^{\lambda_{2s+1}} = \cdots = v_r , \end{aligned}$$

where λ_j, μ_j, and ν_j are determined by

$$\begin{gathered} \lambda_0 = n,\ \lambda_1 = q,\ b_j \lambda_j = \lambda_{j-1} + \lambda_{j+1} \\ \mu_0 = 0,\ \mu_1 = 1,\ b_j \mu_j = \mu_{j-1} + \mu_{j+1} \\ \nu_0 = 1,\ \nu_1 = 1,\ b_j \nu_j = \nu_{j-1} + \nu_{j+1} \end{gathered}$$

for $1 \leqq j \leqq r$. Here, we set $\mu_r = q'$, and $n\alpha = qq' - 1$, so, $\lambda_r = 1$, $\lambda_{r+1} = 0$, $\mu_{r+1} = n$, $\nu_r = q' - \alpha$, $\nu_{r+1} = n - q$. If r is even, we have only to interchange v_r and u_r. The curves E_0 $(u_0 = 0)$ and E_{r+1} $(u_r = 0)$ correspond to the coordinate lines $(\mu = 0)$ and $(\lambda = 0)$, respectively. With these in mind, we consider the function f' defined in neighborhood of $\mathscr{E}_1$ by

$$f'(u_{2s}, v_{2s}) = \{\log |u_{2s}|^{-2\mu_{2s}/n} |v_{2s}|^{-2\mu_{2s+1}/n}\}^2 \cdot \{\log |u_{2s}|^{-2\lambda_{2s}/n} |v_{2s}|^{-\lambda_{2s+1}/n}\}^2 ,$$

which is well-defined, since the transition rules of (u_k, v_k)'s are given by (2.12). Since the function $(\log |\lambda|^{-2})^2 (\log |\mu|^{-2})^2$ projects down to f', the Kaehler metric $|d\lambda|^2/|\lambda|^2 (\log |\lambda|^{-2})^2 + |d\mu|^2/|\mu|^2 (\log |\mu|^{-2})^2$ defined in $B - (\lambda = 0) \cup (\mu = 0)$ also projects down to an Einstein-Kaehler metric $i\partial\bar\partial \log f'$ in a neighborhood of $\mathscr{E}_1$. The functions $|u_0|^2 |v_0|^{2q/n}$ and $|u_r|^2 |v_r|^{2q'/n}$ are well-defined near $\mathscr{E}_1$. So, we extend these to nonnegative functions g_1 and g_2 defined near E with the following properties: (i) g_1 and g_2 restricted to a neighborhood of $\mathscr{E}_1$ are $|u_0|^2 |v_0|^{2q/n}$ and $|u_r|^2 |v_r|^{2q'/n}$, (ii) the restriction of g_1 (resp. g_2) near E_0 (resp. E_{r+1}) is written as $\|\sigma_0\|^2$ (resp. $\|\sigma_{r+1}\|^2$), where σ_0 (resp. σ_{r+1}) is a holomorphic section of $[E_0]$ (resp. $[E_{r+1}]$) and $\|\cdot\|$ denotes the norm with respect to a certain Hermitian metric of $[E_0]$ (resp. $[E_{r+1}]$), (iii) the zero-locus of g_1 (resp. g_2) is $\mathscr{E}_1 \cup E_0$ (resp. $\mathscr{E}_1 \cup E_{r+1}$). We set $f = (\log 1/g_1)^2 (\log 1/g_2)^2$. Finally we introduce a singular volume form

$$\Psi' = \Omega \Big/ \prod_{s=0}^{r+1} \|\sigma_s\|^2 \cdot f \cdot \prod_{\tau} (\|\tau\|^2 \cdot (\log 1/\|\tau\|^2)^2) ,$$

where Ω is a smooth volume form of $\bar{M}$, τ's are local equations of irreducible components of $D - (E_0 \cup \mathscr{E}_1 \cup E_{r+1})$, σ_s is a local equation of E_s, and

$\|\cdot\|$ is a certain Hermitian norm of these line bundles. The -1 times the Ricci form of Ψ' is written as

$$-\mathrm{Ric}\left\{\Omega\Big/\prod_{s=0}^{r+1}\|\sigma_s\|^2\prod_{\tau}\|\tau\|^2\right\}-i\partial\bar{\partial}\log f-i\sum_{\tau}\partial\bar{\partial}\log(\log\|\tau\|^{-2})^2\,,$$

where the first term represents $c_1(K_{\bar{M}}\otimes[D])\geqq 0$ near E, the second term is Einstein-Kaehler near $\mathscr{E}_1$ complete toward $\mathscr{E}_1\cup E_0\cup E_{r+1}$, the third term gives the "Poincaré metric" in the direction transversal to $D-(E_0\cup\mathscr{E}_1\cup E_{r+1})$. By an appropriate choice of $\|\cdot\|$ and by changing f by $(\log c/g_1)^2(\log c/g_2)^2$ for a small positive number c if necessary, we may assume that $-\mathrm{Ric}\,\Psi'$ is a Kaehler metric in a deleted neighborhood of E complete toward E.

Substep 1–5. Let D' be a connected component of D which does not contain any curve of $\mathscr{E}$. Let D'_k denote an irreducible component of D'. Then as in [9], we consider the singular volume form

$$\Psi''=\Omega\Big/\prod_k\|s_k\|^2(\log 1/\|s_k\|^2)^2$$

near D'. We may assume that $-\mathrm{Ric}\,\Psi''$ is a Kaehler metric in a deleted neighborhood of D' complete toward D'.

Remark 3. In the preceding substeps, we constructed the canonical Kaehler metrics in small deleted neighborhoods of the connected components of D. These metrics are complete toward D. The restriction of each of these metrics to a disk transversal to D is equivalent to the Poincaré metric of the punctured disk near the intersection point with D.

Step 2. Let $\mathscr{E}_\nu$ be a connected component of $\mathscr{E}$ of type (2.5) in Lemma 1. By [10, §1], there is a smooth volume form Ω on $\bar{M}$ and a V-smooth function h such that $h\Omega$ is a V-volume form whose Ricci form is the -1 times of a Kaehler V-metric in a neighborhood of $\mathscr{E}_\nu$.

Step 3. Construction of the singular volume form: Let $D=\sum_i D_i$ be the decomposition into irreducible components. There are a smooth volume form Ω in $\bar{M}$ and a Hermitian metric $\|\cdot\|$ for each $[D_i]$ such that the real closed (1, 1)-form $\gamma:=-\mathrm{Ric}\{\Omega/\prod_i\|s_i\|^2\}$, representing $2\pi c_1(L)$, satisfies the conditions in Lemma 3, where s_i is a holomorphic section of $[D_i]$ with D_i as its zero-locus.

Substep 3–1. Let A be a nonsingular elliptic curve of type (2.1) in Lemma 1. As in Substep 1–1, there are a local coordinate (w, z) such that A is given by $w=0$ and the function

$$f(w, z)=\log\{(\exp(-|z|^2)^{b\pi/a}/|w|^2\}$$

whose $-\sqrt{-1}\partial\bar{\partial}\log$ is a Kaehler metric complete toward A. We extend f to a smooth positive function $\tilde{f}$ in $\bar{M}-A$ such that (i) $\tilde{f}$ coincides with f near A, (ii) $\tilde{f}$ is constant in a small neighborhood of other components of $\mathscr{E}$. It follows that there is a positive constant c such that

$$\gamma-\sqrt{-1}\partial\bar{\partial}\log(\tilde{f}+c)\gg 0 \quad \text{in} \quad \bar{M}-\mathscr{E}$$

holds.

Substep 3-2. Let B be a cycle of P^1's or a rational curve with a node in Lemma 1. The function $f\colon (u_k, v_k)\mapsto (R_{k-1}\log|u_k^{-1}|+R_k\log|v_k^{-1}|)^{-2}\times(R'_{k-1}\log|u_k^{-1}|+R'_k\log|v_k^{-1}|)^{-2}$ is well-defined and $\sqrt{-1}\partial\bar{\partial}\log f$ is the canonical metric coming from the Poincaré metric of H^2. Note that $\log f=-\infty$ at B. We extend the function $\log f$ to the whole $\bar{M}$ in the following manner. First, we consider the image in P^N of $\bar{M}$ under the logarithmic m-canonical map $\Phi=\Phi_{mL}$ for m large, where $L=K_{\bar{M}}\otimes[D]$. B is mapped to a singular point p. We pick a small neighborhood of p in P^N and introduce a local coordinate (z_i) around $p=(0,\cdots,0)$. We set $t(z)=\sum_i|z_i|^2$. Secondly, we pick a function $\tau\colon R^+\to R^+$ for a positive number μ as follows:

$$\begin{aligned}
&\tau(s)=0 \quad \text{if} \quad 0\leqq s\leqq \mu/2\\
&\tau(s)=\mu \quad \text{if} \quad \mu\leqq s\\
&0\leqq\tau'(s)\leqq 3 \quad \text{for all} \quad s\geqq 0\\
&-10\leqq\tau''(s)\leqq 10 \quad \text{for all} \quad s\geqq 0
\end{aligned}$$

and consider the function F given by $\log(F)=(1-\tau\circ t/\mu)\log(f)$. $\sqrt{-1}\partial\bar{\partial}\log(F)$ is positive definite in a neighborhood $|t|<\mu/2$ of p and vanishes if $|t|>\mu$. It may have negative direction in the domain $\mu/2\leqq|t|\leqq\mu$. But the following computation tells us that the order of the negativity is $(\mu\cdot\log(\mu^{-1}))^{-1}$ in $\mu/2\leqq|t|\leqq\mu$:

$$\begin{aligned}
\partial\bar{\partial}\log(F)=&-\frac{1}{\mu}(\tau''(t)\partial t\wedge\bar{\partial}t+\tau'(t)\partial\bar{\partial}t)\log(f)-\frac{\tau'(t)}{\mu}\partial t\wedge\bar{\partial}\log(f)\\
&+\Big(1-\frac{\tau\circ t}{\mu}\Big)\partial\bar{\partial}\log(f)\,,
\end{aligned}$$

$$t=\textstyle\sum_i\bar{z}_i dz_i\,,$$

$$\log(f)=2(R_{k-1}du_k/u_k+R_k dv_k/v_k)\cdot(R_{k-1}\log(u_k^{-1})+R_k\log(v_k^{-1}))^{-1}$$
$$+\ (\text{a similar term}).$$

On the other hand, we can choose a Hermitian metric for the hyperplane bundle over P^N whose curvature is nonnegative and arbitrarily concentrates near a hyperplane with respect to the standard metric. Indeed, if we pull back the curvature form $\sqrt{-1}\partial\bar{\partial}\log|Z|^2$ of the hyperplane boundle by the linear transformation $(Z_0: Z_1: \cdots : Z_N) \to (Z_0: \cdots Z_{N-1}: \lambda Z_N)$, $\lambda \in \boldsymbol{R}$, we have $\sqrt{-1}\partial\bar{\partial} \log (1 + |z_1|^2 + \cdots + |\lambda z_N|^2)((\partial/\partial z_N) \wedge (\partial/\partial \bar{z}_N)) = \lambda^2(1 + \sum_{i=1}^{N-1} |z_i|^2)\cdot(1 + \sum_{i=1}^{N-1} |z_i|^2 + \lambda^2|z_N|^2)^{-1}$, wherh $z_i = Z_i/Z_0$ $(1 \leqq i \leqq N)$. If $|z_N| = a\lambda^{-1}$, then it is equal to $\lambda^2(1 + \sum_{i=1}^{N-1}|z_i|^2)(1 + \sum_{i=1}^{N-1} |z_i|^2 + a^2)^{-1}$. Hence there is a positive smooth function α on $\boldsymbol{P}^N$ such that $\sqrt{-1}\partial\bar{\partial} \log\{(\Phi_{mL}^* \alpha)F\} + \gamma$ is positive definite in $\bar{M} - \mathscr{E}$.

Substep 3–3. Let E be a connected component of D containing a connected component of $\mathscr{E}$ properly. First, we extend the functions g_1 and g_2 to positive smooth functions on $\bar{M} - \sum_{k=0}^{N+1} E_k$ which are constant in a neighborhood of other components of $\mathscr{E}$. Secondly, we note that there is a positive number c such that $\gamma + \sqrt{-1}\partial\bar{\partial}\{\log(c/g_1) \log(c/g_2)\}^{-2}$ is positive definite in the whole $\bar{M} - \mathscr{E}$.

Substep 3–4. Let $\sum_i C_i$ be the union of irreducible curves of D with $L \cdot C_i > 0$, and s_i the local equations of C_i. Since $\sum_i C_i$ has only simple normal crossings, we see that there is a Hermitian metric $\|\cdot\|$ of each $[C_i]$ and a positive number c such that $\gamma + \sum_i \sqrt{-1}\partial\bar{\partial}(\log c\|s_i\|^{-2})^{-2}$ is positive definite in $\bar{M} - \mathscr{E}$. q.e.d.

3. Good Quasi-Coordinate System on $(M, -\mathrm{Ric}(\Psi))$. In this section, we write $\mathscr{E}_0$ for the union of components of $\mathscr{E}$ lying outside of D and $\mathscr{E}_0 = \sum \mathscr{E}_{0\nu}$ the decomposition into connected components.

DEFINITION. Let V be a domain in $\boldsymbol{C}^m$. Let X be an m-dimensional complex manifold and ϕ a holomorphic map of V into X. ϕ is called a *quasi-coordinate* map if ϕ is of maximal rank everywhere. In this case, (V, ϕ) is called a quasi-coordinate of X.

LEMMA 6. *Let $(\bar{M}, D)$ be as in Theorem 1. Then there exists a system of local quasi-coordinates $\mathscr{V} = \{(V_\alpha; v_\alpha^1, v_\alpha^2)\}$ of $\bar{M} - D$, a neighborhood U of D and a neighborhood U_0 of $\mathscr{E}_0$ such that*

(i) $\bigcup_\alpha (\text{Image of } V_\alpha) \cup U_0 = \bar{M} - D$,

(ii) $\bigcup_\alpha (\overline{\text{Image of } V_\alpha}) \cap \mathscr{E}_0 = \varnothing$,

(iii) *if the image of V_α does not intersect U then (v_α^1, v_α^2) is a local coordinate in the usual sence,*

(iv) *there is a positive number ε independent of V_α in $\mathscr{V}$ such that each $V_\alpha(\subset C^2)$ contains a ball of radius ε,*

(v) *there are positive constants c and $\mathscr{A}_k$ $(k = 0, 1, \cdots)$ such that*

$$c^{-1}(\delta_{ij}) < (g_{\alpha i\bar{j}}) < c(\delta_{ij})$$

and $|\partial^{|p|+|q|} g_{\alpha i\bar{j}}/\partial v_\alpha^p \partial \bar{v}_\alpha^q| < \mathscr{A}_{|p|+|q|}$ for all multi-indices p, q, where $-\mathrm{Ric}(\Psi) = w = \sqrt{-1}\sum_{i,j} g_{\alpha i\bar{j}} dv_\alpha^i \wedge d\bar{v}_\alpha^j$ in terms of (v_α^1, v_α^2),

(vi) *each connected* $U_{0\nu}$ *of* U_0 *is the minimal resolution of the image of a small ball under the quotient map* $\pi_{0\nu}$ *induced by the action of the finite subgroup of* $SU(2)$ *corresponding to* $\mathscr{E}_{0\nu}$ (cf. Lemma 4).

PROOF. This is proved using the method developed in [4]. Since $\mathscr{E}_0$ are treated in [10], it suffices to consider the neighborhood of D. First, let A be a nonsingular elliptic curve, a component of D of type (1.1) in Lemma 1. We define $\Phi_\eta: (u, v) \to (s, t)$ *by* $s = (1-\eta)u/(1+\eta)$ and $t = (1-\eta)^{1/2}v/(1+\eta)^{1/2}$ for a real number $\eta \in (0, 1)$. Let $B(R) \subset \mathscr{S}(s, t)$ ($\mathscr{S}$ is as in Substep 1-1 in the last section) be defined by $\{(s, t); |s-\sqrt{-1}|^2 + 4|t|^2 < R^2|s+\sqrt{-1}|^2\}$ for a fixed $R \in (0, 1)$. There is a positive constant e such that $1/e < \mathrm{Im}(s) < e$. Let $B_\eta(R) = \Phi_\eta^{-1}(B(R)) \subset \mathscr{S}(u, v)$. It follows that for any positive number K, there exists a real number $\eta \in (0, 1)$ such that $B_\eta(R) \subset W_K$. Indeed,

$$\mathrm{Im}(u) - |v|^2 > ((1+\eta)/(1-\eta))(\mathrm{Im}(s) + 4^{-1}(|s+\sqrt{-1}|^2 - R^2|s+\sqrt{-1}|^2)) .$$

If we set $G = \{(u, v) \in \mathscr{S}; -(2a\pi/b) \leqq \mathrm{Re}(u) \leqq 2a\pi/b, v = 0, \mathrm{Im}(u) > K\}$, then G is contained in $W_K \cap \bigcup_{0<\eta<1} B(R)$. In Substep 1-1 in the last section, we defined the map $F: W_K \to V'$ by $(u, v) \to (\exp(b\pi\sqrt{-1}u/a), v)$, which was of maximal rank everywhere. G is mapped by F onto $\{(w, 0) \in \boldsymbol{C}^2; 0 < |w| < \exp(-b\pi K/2a)\}$. Therefore, some neighborhood of A is covered by the images of quasi-coordinate maps caused by F and π. Local quasi-coordinates are given by $(B(R); s, t) \subset \mathscr{S}(s, t) \subset \boldsymbol{C}^2$, and local quasi-coordinate maps are given by $\pi \circ F \circ \Phi_\eta^{-1}$ for $(1-\eta)$ a small positive number. So, the condition (iii) is satisfied. As for (iv), we need to substitute $z = v$, $w = \exp(b\pi\sqrt{-1}u/2a)$; $u = (1+\eta)s/(1-\eta)$, $v = ((1+\eta)/(1-\eta))^{1/2}t$ into $\gamma = \sqrt{-1}(\alpha|w|^2|dz|^2 + \beta w dz \wedge d\bar{w} + \bar{\beta}\bar{w} dw \wedge d\bar{z} + \delta|dw|^2)$ (which represents $c_1(L)$) and $\sqrt{-1}\partial\bar{\partial}\log f_A$. The results are

$$\gamma = \exp(-b\pi(1+\eta)\,\mathrm{Im}(s)/a(1-\eta))\Big\{\alpha'\Big(\frac{1+\eta}{1-\eta}\Big)|dt|^2 + \beta'\Big(\frac{1+\eta}{1-\eta}\Big)^{3/2} dt \wedge d\bar{s} + \gamma'\Big(\frac{1+\eta}{1-\eta}\Big)^{3/2} ds \wedge d\bar{t} + \delta'\Big(\frac{1+\eta}{1-\eta}\Big)^2 |ds|^2\Big\} ,$$

and

$$\partial\bar{\partial}\log f_A = \frac{dt \wedge d\bar{t}}{\mathrm{Im}(s) - |t|^2 + (1-\eta)c/(1+\eta)} + \frac{|-\sqrt{-1}ds/2 - \bar{t}dt|^2}{(\mathrm{Im}(s) - |t|^2 + (1-\eta)c/(1+\eta))^2}$$

for some positive constant c, where α', β', γ', and δ' are C^∞ in (w, z). Since $(s, t) \in B(R)$, and $\lim_{t\to\infty} t^p e^{-t} = 0$ for any real p, the condition (iv) is satisfied. Second, let B be a cycle of $\boldsymbol{P}^1$'s or a rational curve with a node of type (2.2) or (2.3) of Lemma 1. Let $\tilde{F}$ be the composite map $H^2(z^1, z^2) \to H^2/M \to Y_+ - \bigcup_{k\in Z} S_k \to Y_+/Z - \bigcup_{k=0}^{r-1} S_k$, where the second map is $(2.9)_k$ and the third map is the quotient by Z. $\widetilde{F}$ is of maximal rank everywhere. Let $W \subset H^2$ be defined by $\{(z^1, z^2) \in H^2;\ \mathrm{Im}(z^1)\,\mathrm{Im}(z^2) > K\}$. $\widetilde{F}(W)$ is a deleted neighborhood of B. If we set $W_L = \{(z^1, z^2) \in H^2;\ \mathrm{Im}(z^1) > L, \mathrm{Im}(z^2) > K/L\}$, then $W = \bigcup_{L>0} W_L$. We define a biholomorphic map $\Phi_{\mu,\eta}$ of H^2 by $\Phi_{\mu,\eta}(z) = \eta(z-\mu) + \mu$ for $\mu \in R$ and $\eta \in (0, \infty)$. Let $B(a, r)$ denote the disk in C centered at a with radius r. Then

$$W_L = \left\{\bigcup_{\mu_1 \in R}\ \bigcup_{\eta_1 \geqq \eta_1(L)} \Phi^{-1}_{\mu_1,\eta_1}(B(\mu_1 + \sqrt{-1}, 1/2))\right\}$$
$$\times \left\{\bigcup_{\mu_2 \in R}\ \bigcup_{\eta_2 \geqq \eta_2(L)} \Phi^{-1}_{\mu_2,\eta_2}(B(\mu_2 + \sqrt{-1}, 1/2))\right\}$$

for some positive number $\eta_1(L)$ and $\eta_2(L)$. By substituting $t_i = \eta_i(z_i - \mu_i) - \mu_i$ into $\widetilde{F}^*(\partial\bar\partial \log f_B)$, where f_B is one of the f_{B_j}'s in Lemma 4, we obtain $\sum_{i=1}^2 |dt_i/(\mathrm{Im}(t_i))|^2$. Here, z_i is in $\Phi^{-1}_{\mu_i,\eta_i}(B(\mu_i + \sqrt{-1}, 1/2))$ if and only if t_i is in $B(\mu_i + \sqrt{-1}, 1/2)$. On the other hand, the $(1, 1)$-form which is a pull back of a smooth form in P^N under the logarithmic pluricanonical map consists of terms $|u_k|^2|v_k|^2 dt_i \wedge d\bar t_j/\eta_i\eta_j$ multiplied by some C^∞-function in (u_k, v_k) and $|u_k|^2|v_k|^2 = \exp[2\pi\{(R'_{k-1} - R'_k)\eta_1^{-1}\mathrm{Im}(t_1) + (R_k - R_{k-1})\eta_2^{-1}\mathrm{Im}(t_2)\}/(R_{k-1}R'_k - R_kR'_{k-1})]$, where $R_k - R_{k-1} < 0$, $R'_{k-1} - R'_k < 0$, $R_{k-1}R'_k - R_kR'_{k-1} > 0$. Therefore, some neighborhood of B is covered by the images of quasi-coordinate maps given by the restriction of $\widetilde{F}\circ(\Phi^{-1}_{\mu_1,\eta_1} \times \Phi^{-1}_{\mu_2,\eta_2})$ to $B(\mu_1 + \sqrt{-1}, 1/2) \times B(\mu_2 + \sqrt{-1}, 1/2)$ for $\mu_i \in R$, $\eta_i \in (0, \infty)$, and the conditions (iii) and (iv) of Lemma 5 are satisfied by these quasi-coordinate maps. On the other hand, combining the arguments in Substeps 1-4 in the last section with those in [10, §2], we can show that there is a quasi-coordinate system with (iii) and (iv) around the other components of D. q.e.d.

4. Existence of the Canonical Einstein-Kaehler V-metric, Proof of Theorem 1. Following the arguments developed in [4] together with those of [10], we shall show the existence of a complete Einstein-Kaehler V-metric on M' with negative Ricci curvature.

We use the same notations as in the preceding sections.

DEFINITION. A continuous function u on M' is of class V-C^k, where k is a nonnegative integer possibly ∞, if u is of C^k in $M' - \mathscr{S}$ and each $\pi^*_{0\nu}u$ is C^k in a small ball centered at the origin.

DEFINITION. Let u be of class V-C^∞. For a nonnegative integer k and $\alpha \in (0, 1)$, the V-$C^{k,\alpha}(M)$ norm of u is defined by $\|u\|_{V,k,\alpha} = \max(A, B)$, with

$$A = \sup_{0\nu}\left\{\sup_{\zeta\in B}\sum_{|p|+|q|\leq k}|\partial^{|p|+|q|}\pi_{0\nu}^{*}u(\zeta)/\partial\zeta^{p}\partial\bar{\zeta}^{q}|\right.$$

$$\left.+\sup_{\substack{\zeta,\zeta'\in B\\ \zeta\neq\zeta'}}\sum_{|p|+|q|=k}|\zeta-\zeta'|^{-\alpha}|\partial^{k}\pi_{0\nu}^{*}u(\zeta)/\partial\zeta^{p}\partial\bar{\zeta}^{q}-\partial^{k}\pi_{0\nu}^{*}u(\zeta')/\partial\zeta^{p}\partial\bar{\zeta}^{q}|\right\},$$

$$B = \sup_{\alpha}\left\{\sup_{z\in V_\alpha}\sum_{|p|+|q|\leq k}|\partial^{|p|+|q|}u(z)/\partial v_{\alpha}^{p}\partial\bar{v}_{\alpha}^{q}|\right.$$

$$\left.+\sup_{\substack{z,z'\in V_\alpha\\ z\neq z'}}\sum_{|p|+|q|=k}|z-z'|^{-\alpha}|\partial^{k}u(z)/\partial v_{\alpha}^{p}\partial\bar{v}_{\alpha}^{q}-\partial^{k}u(z')/\partial v_{\alpha}^{p}\partial\bar{v}_{\alpha}^{q}|\right\},$$

where u is locally lifted to a function in $(V_\alpha;\ v_\alpha^1, v_\alpha^2)$, if necessary.

DEFINITION. For a nonnegative integer k and $\alpha\in(0,1)$ the function space $V\text{-}C^{k,\alpha}(M)$ is the Banach space obtained as the completion of $V\text{-}C^{\infty}(M)$ with respect to the norm $\|\cdot\|_{V,k,\alpha}$.

LEMMA 7. *Let Ψ be the singular volume form defined in Lemma 5. Then the function* $\log(\Psi/\omega^2)$ *is of class $V\text{-}C^{k,\alpha}(M)$ for any admissible k and α.*

PROOF. This comes from the arguments in §2. q.e.d.

By Lemma 5, $\omega=-\mathrm{Ric}\,\Psi$ is an $V\text{-}C^{\infty}$ Kaehler metric on $M=\bar{M}-D$ which is complete toward D. We consider the equation

$$\Delta_\omega u - b(x)u = f(x)\,, \tag{4.1}$$

where Δ_ω is the Laplacian with respect to ω and we assume that $b(x)$ and $f(x)$ are of class $V\text{-}C^{k,\alpha}(M)$.

LEMMA 8. *If $b(x)\geqq b$ for some positive number b, the equation* (4.1) *has a unique solution u belonging to $V\text{-}C^{k+2,\alpha}(M)$.*

PROOF. Let $M=\bigcup_i\Omega_i$ be the exhaustion of M by an increasing sequence of domains $\Omega_i\subset M$ with smooth boundary. We may assume that each Ω_i contains $\mathscr{E}_0$. Consider the Dirichlet problem

$$\begin{cases}\Delta_\omega u_i - b(x)u_i = f & \text{in}\quad \Omega_i\\ u_i = 0 & \text{on}\quad \Omega_i\,.\end{cases} \tag{4.2}$$

Although the metric ω is singular along $\mathscr{E}_0$, we can use the direct method in the calculus of variations to produce a weak solution of (4.2). Let us consider the Hilbert space $V\text{-}\mathring{H}_1^2(\Omega_i)$ which is the completion of the vector space $V\text{-}C_0^{\infty}(\Omega_i)$ of all $V\text{-}C^{\infty}$ functions with compact support in Ω_i, with respect to the norm

$$\|u\|^2_{V\text{-}\mathring{H}_1^2(\Omega_i)}=\int_{\Omega_i}|u|^2\omega^2+\int_{\Omega_i}\|du\|^2_\omega\omega^2\,.$$

We find a weak solution of (4.2) in this function space by minimizing the functional

$$J_i(\phi) = \int_{\Omega_i} (\|d\phi\|_\omega^2 + b(x)\phi^2 + 2f\phi)\omega^2 \quad \text{for} \quad \phi \in V\text{-}\mathring{H}_1^2(\Omega_i) .$$

Since $b(x) \geqq b$ in M, it is shown by the standard arguments that there is an element $\bar{\phi}$ of $V\text{-}\mathring{H}_1^2(\Omega_i)$ which minimizes J_i. $\bar{\phi}$ satisfies the Euler-Lagrange equation of J_i:

$$\int_{\Omega_i} \{\bar{\phi}\Delta_\omega \psi + \bar{\phi}\psi + f\psi\}\omega^2 = 0 , \quad \text{for any} \quad \psi \in V\text{-}C_0^\infty(\Omega_i) .$$

Hence $\bar{\phi}$ is a weak solution of (4.2) and the regularity theorem (cf. [2, p. 85]) tells us that $\bar{\phi}$ is of class $V\text{-}C^{k+2,\alpha}(\Omega_i)$ with zero boundary value. Here, the singularity of ω along $\mathscr{E}_0$ causes no trouble, since $\pi_{0\nu}^*\bar{\phi}|_{U_{0\nu}}$ is a weak solution of the equation $\Delta_{\pi_{0\nu}{}^*\omega} u + (\pi_{0\nu}{}^*b|_{U_{0\nu}})u = \pi_{0\nu}{}^*f|_{U_{0\nu}}$ in B (i.e., $\pi_{0\nu}^*\bar{\phi}|_{U_{0\nu}} = \bar{\phi}_{0\nu}$ satisfies $\int_B ((d\bar{\phi}_{0\nu}, d\psi)_{\pi_{0\nu}{}^*\omega} + \bar{\phi}_{0\nu}\psi + f\psi)(\pi_{0\nu}{}^*\omega)^2 = 0$, for any $\psi \in C_0^\infty(B)$) and $\pi_{0\nu}{}^*\omega$ is a smooth metric of $\bar{B}$. By the maximum principle, $|u_i| \leqq \max_M |f|/b$, for the unique solution u_i of (4.2). Hence u_i's are bounded above by a constant independent of i. Applying the interior Schauder estimate to u_i's in $(V_\alpha; v_\alpha^1, v_\alpha^2)$, we can show that a subsequence of $\{u_i\}_{i=1}^\infty$ converges to a function u of class $V\text{-}C^{k+2,\alpha}(M)$, and u is a solution of (4.2). Uniqueness comes from the inequality $\|u_i\|_{V,k+2,\alpha} \leqq C(\|u_i\|_0 + \|f\|_{V,k,\alpha})$ for any solution u_i of (4.2), where C depends only on M. q.e.d.

Now we are ready to prove Theorem 1 by following the arguments developed in [4]. By Lemma 5, there is a $V\text{-}C^\infty$ Kaehler metric ω on M which is complete toward D. We claim that for all f of class $V\text{-}C^{k+2,\alpha}(M)$ ($k \geqq 5$, $\alpha \in (0, 1)$), there is a unique solution u of

$$(\omega + \sqrt{-1}\partial\bar{\partial}u)^2 = \exp(u + f)\omega^2 \tag{4.3}$$

belonging to $\mathscr{U} = \{u \in V\text{-}C^{k,\alpha}(M);\ c^{-1}\omega < \omega + \sqrt{-1}\partial\bar{\partial}u < c\omega,$ for some positive constant $c\}$. This is proved by the method of "bounded geometry" (cf. Lemma 5) provided we lift everything up to V_α's by quasi-coordinate maps near D, lift everything up to B by $\pi_{0\nu}$; $B \to U_{0\nu}$ near each $\mathscr{E}_{0\nu}$ and represent everything in terms of coordinates (v_α^1, v_α^2) of V_α or (ζ^1, ζ^2) of B, respectively. To prove the claim, we consider the map $\Phi: V\text{-}C^{k,\alpha}(M) \to V\text{-}C^{k-2,\alpha}(M)$, $u \mapsto e^{-u}(\omega + \sqrt{-1}\partial\bar{\partial}u)/\omega^2$. It suffices to show that $C = \{t \in [0, 1];$ there is a solution u of $\Phi(u) = e^{tf}$ belonging to $\mathscr{U}\} \ni 0$ is open and closed. For example, openness follows from the fact that the Fréchet derivative $\Phi'(u): V\text{-}C^{k,\alpha}(M) — V\text{-}C^{k-2,\alpha}(M)$ of Φ at $u \in \mathscr{U}$ which is given by $\Phi'(u)h = \Delta_{\tilde{\omega}}h - h$ ($h \in V\text{-}C^{k,\alpha}(M)$), is an isomorphism by Lemma

7. Uniqueness of the V-C^∞ Einstein-Kaehler metric complete toward D follows from Yau's generalized maximum principle [20]. q.e.d.

REMARK. One of the main tools in the proof of Lemma 7 and Theorem 2 is Yau's maximum principle. We remark here that the singularity of of ω along $\mathscr{E}_0$ causes no trouble. Indeed, let u be a function bounded above on (M, ω). If u has a maximum value in some relatively compact domain in M, there is no trouble provided we lift everything up to B by $\pi_{0\nu}: B \to U_{0\nu}$. If $\sup_M(u) > \max_\Omega(u)$ for all compact subdomains Ω of M, we can use Yau's maximum principle provided we modify the Kaehler metric ω near $\mathscr{E}_0$ to a smooth complete metric of M.

5. Logarithmic Miyaoka-Yau Inequality. In this section we prove Theorem 2. Let us begin with general remarks. Let (E, h) be a Hermitian vector bundle of rank r over a complex manifold N and $e_U = (e_1, e_2, \cdots, e_r)$ a local holomorphic frame valid in U. Let $h = (h_{\bar{i}j}), h_{\bar{i}j} = h(e_j, e_i)$ be the Hermitian metric of E. The connection form and the curvature form of (E, h) are given by $\theta_U = h^{-1}\partial h$ and $\Theta_U = \bar\partial\theta_U = \bar\partial(h^{-1}\partial h)$, respectively. If $e_U = e_V g_{VU}$ is the transition rule between local holomorphic frames, Θ_U and Θ_V are related by $\Theta_U = g_{VU}^{-1}\Theta_V g_{VU}$.

Let $(\bar{M}, D)$ be as in Theorem 1 and $\omega = -\mathrm{Ric}\,\Psi$ and $\tilde\omega = \omega + \sqrt{-1}\partial\bar\partial u$ the canonical Einstein-Kaehler metric in Theorem 1.

LEMMA 9. *$\tilde\gamma_1$ and $\tilde\gamma_2$ be the Chern forms of the Hermitian vector bundle $(TM, \tilde\omega)$, where TM denotes the holomorphic tangent bundle of M. Then $\tilde\gamma_1^2$ and $\tilde\gamma_2$ are summable over M with their values $\int_M \tilde\gamma_1^2 = c_1(\bar{M}, D)^2$, $\int_M \tilde\gamma_2 = c_2(\bar{M}, D) - \delta$. Here, $c_i(\bar{M}, D) = \bar{c}_i$ denotes the i-th logarithmic Chern class of $(\bar{M}, D)$, δ is a rational number determined by $\mathscr{E}_0$, which is non-positive and zero if and only if $\mathscr{E}_0 = \varnothing$.*

PROOF. Let γ_i denote the i-th Chern form of ω. Since u belongs to $\mathscr{U}$ for any admissible k and α, the following equalities clearly hold: $\int_M \tilde\gamma_1^2 = \int_M \gamma_1^2$, $\int_M \tilde\gamma_2 = \int_M \gamma_2$. Let h be a smooth Hermitian metric of $E = \Omega^1_{\bar{M}}(\log D)^*$. We think of ω as a Hermitian metric h' of E with singularity along D. The Chern forms with respect to the Hermitian connection of the Hermitian metric h' are γ_1 and γ_2 in the complement of D, since any local holomorphic frame for E is a local holomorphic frame for TM in the complement of D. By [6, pp. 400-406], $\gamma_k = \gamma_k(E, h') = \gamma_k(E, h) + d\{P_k(\theta(h) - \theta(h'), \Theta(h), \Theta(h'))\}$, where $\cdot(h)$ and $\cdot(h')$ are the connection or the curvature form of h and h', respectively and P_k is a universal polynomial. If $(\varDelta; u, v)$ is a coordinate polydisk in $\bar{M}$ such that $\varDelta \cap D$ is given by the coordinate lines, the local expression with respect to

(u, v) of the connection form and the curvature form of h' are of order at D described in the following table:

equation of $\Delta \cap D$	equation of $\Delta \cap D \cap \mathscr{E}$	order of the connection form	order of the curvature form
$v = 0$	$v = 0$	$(\lvert dv/v \rvert + \lvert du \rvert)/\log \lvert v \rvert^{-1}$	$\lvert dv \rvert^2/\lvert v \rvert^2(\log \lvert v \rvert^{-1})^2 + \lvert du \rvert^2/\log \lvert v \rvert^{-1}$
$v = 0$	—	$\lvert dv \rvert/\lvert v \rvert \log \lvert v \rvert^{-1} + \lvert du \rvert$	$\lvert dv \rvert^2/\lvert v \rvert^2(\log \lvert v \rvert^{-1})^2 + \lvert du \rvert^2$
$uv = 0$	$uv = 0$	$(\lvert du/u \rvert + \lvert dv/v \rvert)(\log \lvert u \rvert^{-1} + \log \lvert v \rvert^{-1})^{-1}$	$(\lvert du/u \rvert^2 + \lvert dv/v \rvert^2)(\log \lvert u \rvert^{-1} + \log \lvert v \rvert^{-1})^{-2}$
$uv = 0$	$v = 0$	$\lvert du \rvert/\lvert u \rvert(\log \lvert u \rvert^{-1} + \log \lvert v \rvert^{-1}) + \lvert dv \rvert/\lvert v \rvert \log \lvert v \rvert^{-1}$	$(\lvert du \rvert/\lvert u \rvert(\log \lvert u \rvert^{-1} + \log \lvert v \rvert^{-1}))^2 + \lvert dv \rvert^2/\lvert v \rvert^2(\log \lvert v \rvert^{-1})^2$
$uv = 0$	—	$\lvert du \rvert/\lvert u \rvert \log \lvert u \rvert^{-1} + \lvert dv \rvert/\lvert v \rvert \log \lvert v \rvert^{-1}$	$\lvert du \rvert^2/\lvert u \rvert^2(\log \lvert u \rvert^{-1})^2 + \lvert dv \rvert^2/\lvert v \rvert^2(\log \lvert v \rvert^{-1})^2$

Let $\{\Omega_n\}$ be an exhaustion of M by relatively compact domains Ω_n with smooth boundary. Then $\lim_n \int_{\partial\Omega_n} (\gamma_1(E, h) \wedge P_1) = 0$, $\lim_n \int_{\partial\Omega_n} P_1 \wedge dP_1 = 0$, and $\lim_n \int_{\partial\Omega_n} P_2 = 0$, by the estimates in the above table. Here we have used the fact that $\int_{0<|z|<c} |dz|^2/|z|^2(\log|z|^{-1})^2$ for $0 < c < 1$ is finite. Let $U_{0\nu}$ be a small neighborhood of $\mathscr{E}_{0\nu}$. Then by [10, Proposition 4], the following equalities hold (see p. 77, Added in Proof):

$$\int_M \gamma_1(E, h')^2 - \int_M \gamma_1(E, h)^2 = 0 ,$$

$\int_M \gamma_2(E, h') - \int_M \gamma_2(E, h) = -\lim \sum_\nu \int_{\partial U_{0\nu}} P_2 = -[\sum_n \{n(n+2)/(n+1)\} \times {}^\#(\text{type } A_n) + \sum_m \{(4n^2 - 4n - 9)/4(n-2)\} \times {}^\#(\text{type } D_m,\ m \geqq 4) + \{167/24\} \times {}^\#(\text{type } E_6) + \{383/48\} \times {}^\#(\text{type } E_7) + \{1079/120\} \times {}^\#(\text{type } E_8)] = -\delta$, where "lim" means to go to the limit as $U_{0\nu}$ tends smaller and smaller. Then we have the following two equalities:

$$\begin{aligned} \int_M \tilde{\gamma}_1^2 &= \int_M \gamma_1^2 = \int_M \gamma_1(E, h')^2 = \int_M \gamma_1(E, h)^2 + 2 \lim_n \int_{\partial\Omega_n} \gamma_1(E, h) \wedge P_1 \\ &\quad + \lim_n \int_{\partial\Omega_n} P_1 \wedge dP_1 = \int_M \gamma_1(E, h)^2 = \bar{c}_1^2 , \\ \int_M \tilde{\gamma}_2 &= \int_M \gamma_2 = \int_M \gamma_2(E, h') = \int_M \gamma_2(E, h) - \delta + \lim_n \int_{\partial\Omega_n} P_2 \\ &= \int_M \gamma_2(E, h) - \delta = \bar{c}_2 - \delta . \end{aligned}$$

q.e.d.

PROOF OF THEOREM 2. We begin with the following general remark due to Chen and Ogiue [3]. Let (X, ds^2) be an arbitrary Einstein-Kaehler surface. Write s and $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ for the scalar curvature and the Riemann curvature tensor. We introduce the tensor $T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ which measures the pointwise deviation of ds^2 from the complex space form metric. This is given by $T_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} - s(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})/6$. By a direct tensor calculation, we see $0 \leqq \sum |T_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2 = \sum |R_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2 - s^2/3$. On the other hand, $3\gamma_2 - \gamma_1^2 = (1/16\pi^2)(3\sum |R_{\alpha\bar{\beta}\gamma\bar{\delta}}|^2 - s^2)$, where γ_i denotes the i-th Chern form formally computed from ds^2. We apply this to our case. Let $(M', \hat{\omega})$ be the complete Einstein-Kaehler V-manifold in Theorem 1. Then we have $3\tilde{\gamma}_2 - \tilde{\gamma}_1^2 \geqq 0$. We thus have the required inequality by integrating this pointwise inequality on M' and applying Lemma 9. Next, we consider the case where the equality holds. In general, let (Y, g) be a V-manifold with a complete V-metric q. The notion of geodesics on a V-manifold was defined as follows. Let γ' be a curve in Y passing through a singular point p. We say that γ' is a geodesic through p if γ' is a push down of a geodesic in a local uniformization, i.e., if $\pi: B \rightarrow B'$ is a local uniformization near p, there is a geodesic γ in B such that $\gamma' = \pi\gamma$. In our case, $Y = M'$ and ds^2 is the canonical complete Einstein-Kaehler V-metric. The equality sign holds in our inequality if and only if $T_{\alpha\bar{\beta}\gamma\bar{\delta}}$ vanishes identically on M', i.e., $\tilde{\omega}$ is of negative constant holomorphic sectional curvature and hence the ball-metric. There are domains U in B^2, U' in M' and an isometry $\phi: U \rightarrow U'$. Choose a point o in U. We define a continuous map Φ of B^2 to M', which is an extention of ϕ, as follows:

$$\Phi(\exp_o(v)) = \exp_{\phi(o)}(\phi_*(v)) , \quad \text{for} \quad v \in T_o B^2 .$$

Since the Hopf-Rinow theorem is true for $(M', \tilde{\omega})$, Φ is a continuous surjective map. Let Q be the set of all singular points of M'. Q is a discrete set in M'. $\Phi^{-1}(Q)$ is also discrete in M'. Note that $B^2 - \Phi^{-1}(Q)$ is simply-connected. In the following, we shall prove that $\Phi' = \Phi|(B^2 - \Phi^{-1}(Q))$ is a locally biholomorphic universal covering map of $M' - Q$. Let $\gamma: [0, l] \rightarrow B^2$ be a broken geodesic outside of $\Phi^{-1}(Q)$ starting from o, with the break points $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = l$. Write ${}_i\gamma$ for $\gamma|[0, t_i]$ and x_i for the tangent vector to $\gamma|[t_i, t_{i+1}]$ at t_i. We define a broken geodesic γ' in M' starting from $\phi(o)$ as follows. Define ${}_1\gamma': [0, t_1] \rightarrow M'$ by ${}_1\gamma'(t) = \exp_{\phi(o)}(t\phi_*(x_0))$. If ${}_i\gamma': [0, t_i] \rightarrow M'$ is defined, we define ${}_{i+1}\gamma': [0, t_{i+1}] \rightarrow M'$ by

$${}_{i+1}\gamma'(t) = \begin{cases} {}_i\gamma'(t) & \text{if} \quad t \in [0, t_i] \\ \exp_{{}_i\gamma'(t_i)}((t - t_i)(P_{{}_i\gamma'} \circ \phi_* \circ P_{{}_i\gamma}^{-1}(x_i))) & \text{if} \quad t \in [t_i, t_{i+1}] , \end{cases}$$

where P_γ stands for the parallel transportation along a piecewise smooth curve γ. Using broken geodesics γ and γ', we define a map $\Psi: B^2 - \Phi^{-1}(Q) \rightarrow$

$M' - Q$ as follows. For a broken geodesic $\gamma\colon [0, l] \to B^2$ outside of $\Phi^{-1}(Q)$, we define $\Psi(\gamma(l)) = \gamma'(l)$. This is well defined as a map to M', since $\Phi^{-1}(Q)$ is discrete in B^2. Suppose $\Psi(\gamma(t)) = \gamma'(t)$ is in Q, for some γ not passing through $\Phi^{-1}(Q)$. We consider the geodesic γ_0 in B^2 from o to $\gamma(t)$ and approximate γ_0 by broken geodesics from o to $\gamma(t)$ disjoint from $\Phi^{-1}(Q)$. By the continuity argument, we see $\Phi(\gamma(t)) = \gamma'(t)$ is in Q, which is absurd. We claim that Ψ is a local isometry. Let R and $\bar{R}$ be the Riemann curvature tensor of B^2 and $(M', \tilde{\omega})$. If we set $\phi_\gamma = P'_\gamma \phi_* P_\gamma^{-1}$, then we have $\phi_\gamma(R(x, y)z) = \bar{R}(\phi_\gamma(x), \phi_\gamma(y))\phi_\gamma(z)$, for any broken geodesic γ in $B^2 - \Phi^{-1}(Q)$ starting from o, since the curvature tensor is parallel for the ball-metric. Hence the claim follows. Since any isometry between germs of real-analytic Riemannian manifolds is real-analytic, Ψ is real-analytic on $B^2 - \Phi^{-1}(Q)$. By the continuity argument, Ψ coincides with Φ'. So, Φ' is real-analytic. Since Φ' is holomorphic in U, Φ' is holomorphic in $B^2 - \Phi^{-1}(Q)$ by analytic continuation. Since Ψ is of maximal rank everywhere on $B^2 - \Phi^{-1}(Q)$, so is Φ' everywhere. Therefore $\Phi'\colon B^2 - \Phi^{-1}(Q) \to M' - Q$ is locally biholomorphic universal covering map. The deck transformation group consists of biholomorphic automorphisms of $B^2 - \Phi^{-1}(Q)$. By the Hartogs theorem, these are extended to biholomorphic automorphisms of B^2. Thus we obtain a group Γ acting properly discontinuously on B^2 with only isolated fixed points such that $\Gamma \backslash B^2 \cong M'$. q.e.d.

6. Examples.

EXAMPLE 1. Let D' consist of n lines in general position in $\boldsymbol{P}^2$. We blow up $n(n-3)/2$ intersection points so that there remain two points not blown up on each line. Let $\bar{M}$ be the resulting manifold and D the proper transform of D'. Then D is a cycle of $\boldsymbol{P}^1$'s. If $n \geqq 7$, then $(\bar{M}, D)$ is an example of Theorem 1. In this case, $\bar{c}_1^2 = (n^2 - 9n + 18)/2$, $\bar{c}_2 = n^2 - 4n + 3$.

EXAMPLE 2. Let D' consist of three nonsingular curves of degree 3 in general position in $\boldsymbol{P}^2$. Blow up all 27 intersections. Let $\bar{M}$ be the resulting manifold and D the proper transform of D'. Then D consists of three nonsingular elliptic curves of negative self-intersection number. $(\bar{M}, D)$ is an example of Theorem 1. In this case, $\bar{c}_1^2 = 9$, $\bar{c}_2 = 57$.

EXAMPLE 3. Let B^2 be the open unit ball in $\boldsymbol{C}^2$, Γ a discrete group of automorphisms acting on B without fixed points. Assume that the volume of $\Gamma \backslash B$ is finite. Then we can compactify $\Gamma \backslash B$ by adding nonsingular elliptic curves to a nonsingular projective surface $\bar{M}$. This is verified by representing B^2 as the Siegel domain of the second kind and determining the form of the parabolic automorphisms fixing the boundary point $u = \infty$. By a simple matrix computation, we can show that they are of the form $(n, v) \mapsto (u + 2i\bar{\gamma}v + i|\gamma|^2 + \text{(real number)},\ v + \gamma)$. The resulting pair $(\bar{M}, D)$, where D is a divisor at infinity consisting of mutually disjoint elliptic curves, is an example of Theorem 1.

EXAMPLE 4. In [17, p. 134], U. Persson gives an example of the degenerations of Godeaux surfaces $\pi\colon M \to \Delta$, where M is a smooth threefold, Δ an open disk, and π a proper surjective holomorphic map. A Godeaux surface is a compact complex surface whose universal cover is a nonsingular quintic surface in $\boldsymbol{P}^3$ and whose fundamental group is $\boldsymbol{Z}_5$. $M_t = \pi^{-1}(t)$, $t \neq 0$, is a nonsingular Godeaux surface, and $M_0 = \pi^{-1}(0)$ is a singular surface. We obtain M_0 by contraction of a multisection of a certain elliptic ruled surface as follows. There is a blown up elliptic ruled surface $\bar{M}$ with the following structure. There is an elliptic curve C which is a five-section in $\bar{M}$ (i.e., the restriction of the projection $\bar{M} \to B$ to C is a five fold covering over the elliptic curve B) with the numerical conditions $(K_{\bar{M}} + [C])^2 = 1$, $C^2 = -3$. There are two singular fibers in $\bar{M}$ both of which consist of two (-1)-curves meeting transversely at one point. C cuts these (-1)-curves as in Figure 1. E_i and E_i' are (-1)-curves in this figure. We shall show that $(\bar{M}, C)$ satisfies the condition $(*)$ in Theorem 2. In general, let $\bar{X} \to B$ be a blown up elliptic ruled surface, C an elliptic curve which is an n-section. Let $\pi\colon \bar{X} \to \bar{X}_0$ be the blowing up from a minimal model $\bar{X}_0$, N the number of blow ups and E_i $(i = 1, \cdots, N)$ the exceptional divisors. Assume that C cuts each E_i at ν_i points. Then $C = \pi^* C_0 - \sum_{i=1}^N \nu_i E_i$, $K = \pi^* K_0 + \sum_{i=1}^N E_i$, where $K = K_{\bar{X}}$, $K_0 = K_{\bar{X}_0}$. $H_2(X_0, \boldsymbol{Z})$ is generated by the homology classes x of a fiber and y of a section such that $y^2 = -e$ $(-1 \leqq e)$. Since $K_0 = -2y - ex$ and

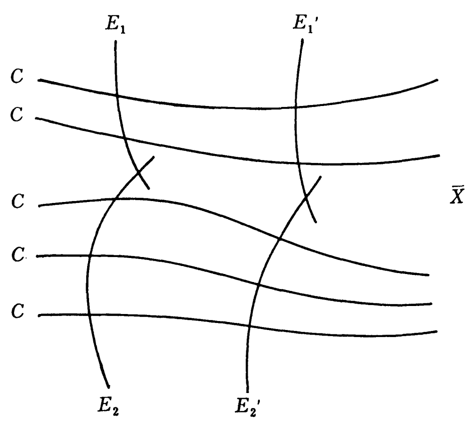

FIGURE 1

$(K_0 + C_0)C_0 = \sum_{i=1}^N \nu_i(\nu_i - 1)$ (the Plücker relation), C_0 can be written as $C_0 = ny + \alpha x$ with $\alpha = \sum_{i=1}^N \nu_i(\nu_i - 1)/2(n-1) + ne/2$. Since $(K + C)C = 0$, we get

$$\begin{aligned}(K + C)^2 = (K + C)K &= (\pi^*(K_0 + C_0) - \sum_{i=1}^N (\nu_i - 1)E_i)\Big(\pi^*K_0 + \sum_{i=1}^N E_i\Big)\\ &= (K_0 + C_0)K_0 + \sum_{i=1}^N (\nu_i - 1)\\ &= ((n-2)y + (\alpha - e)x)(-2y - ex) + \sum_{i=1}^N (\nu_i - 1)\\ &= 2e(n-2) - 2(\alpha - e) - e(n-2) + \sum_{i=1}^N (\nu_i - 1)\\ &= -N - \sum_{i=1}^N \nu_i(\nu_i - 1)/(n-1) + \sum_{i=1}^N \nu_i\\ &= -N + \sum_{i=1}^N \nu_i(n - \nu_i)/(n-1)\ .\end{aligned}$$

By [17, Propositions 1.2 and 1.3], we may assume that each $\nu_i \leqq n/2$. Then $\nu_i(n - \nu_i) - (n-1) = -(\nu_i - n/2)^2 + (n^2/4) - (n-1) \geqq 0$, where the equality occurs if and only if every ν_i equals one. Hence if $(K + C)^2 > 0$, then $N > 0$ and some $\nu_i \geqq 2$, in particular $n \geqq 2\nu_i \geqq 4$. In Persson's example, $(K + C)^2 = 1$, $C^2 = -3$, $N = 2$ and $\nu_1 = \nu_2 = 2$. Namely, one obtains $\bar{X}_0$ by blowing down E_1 and E_1' in Figure 1, and C_0 has ordinary double points at the images of E_1 and E_1'. Now we try to find necessary additional conditions for $K + C$ to be ample outside of C (i.e., $K + C$ satisfies the numerical conditions $(K + C)^2 > 0$, $(K + C)Z \geqq 0$ for all irreducible curves Z in $\bar{X}$ and $(K + C)Z = 0$ if and only if $Z = C$). If Z is a curve contained in E_i, then $(K + C)Z = (\pi^*(K_0 + C_0) - \sum_{i=1}^N (\nu_i - 1)E_i)Z = (\nu_i - 1)(E_iZ)$. So, if $(K + C)Z > 0$, then every $\nu_i \geqq 2$. If Z is a general fiber, then $(K + C)Z = (K_0 + C_0)x = n - 2$. If Z is an irreducible curve which is not a fiber and cuts each E_i μ_i-times, then Z is written as $Z = qy + px$, where $p \geqq (eq/2) + \{\sum_{i=1}^N \nu_i\mu_i - \sum_{i=1}^N \nu_i(\nu_i - 1)q/2(n-1)\}/n$, because $C_0Z_0 = (ny + \alpha x)(qy + px) \geqq \sum_{i=1}^N \nu_i\mu_i$. Therefore,

$$\begin{aligned}(K + C)Z &= (\pi^*(K_0 + C_0) - \sum_{i=1}^N (\nu_i - 1)E_i)\Big(\pi^*Z_0 - \sum_{i=1}^N \mu_iE_i\Big)\\ &= (K_0 + C_0)Z_0 - \sum_{i=1}^N (\nu_i - 1)\mu_i = -qe(n-2)\\ &\quad + q\Big\{\sum_{i=1}^N \nu_i(\nu_i - 1)/2(n-1) + (n-2)e/2\Big\} + p(n-2)\\ &\geqq q\sum_{i=t}^N \nu_i(\nu_i - 1)/n(n-1) + \frac{n-2}{n}\sum_{i=1}^N \nu_i\mu_i - \sum_{i=1}^N (\nu_i - 1)\mu_i\ .\end{aligned}$$

Here, $\nu_i(\nu_i - 1)/n(n-1) + (n-2)\nu_i/n - (\nu_i - 1) = 1 - (2n - \nu_i - 1)/n(n-1) > 0$, since $2 \leqq \nu_i \leqq n/2$ for every i. From the above arguments, we conclude the following: If $2 \leqq \nu_i \leqq n/2$ and $N > 0$, then the pair satisfies the condition (*) of Theorem 1. In this case, $K_{\bar{X}} \otimes [C]$ is ample outside of C.

The logarithmic Chern numbers are given by

$$\bar{c}_1^2 = -N + \sum_{i=1}^{N} \nu_i(n - \nu_i)/(n-1) = -N - C^2, \bar{c}_2 = N .$$

The inequality $3\bar{c}_2 \geqq \bar{c}_1^2$ is rewritten as $4N \geqq -C^2 = \sum_{i=1}^{N} \nu_i(n - \nu_i)/(n-1)$. $X = \bar{X} - C$ admits a unique complete Einstein-Kaehler metric with negative Ricci curvature up to constant multiple. We do not know the convergence of the canonical Einstein-Kaehler metric of M_t to the canonical Einstein-Kaehler metric of $X = M_0 -$ (singular point).

EXAMPLE 5. Let D' be a sum of three curves C_1: $z_0^n + z_1^n + z_2^n = 0$, C_2: $z_0 = 0$, and C_3: $z_1 = 0$, where $n \geqq 2$, in $\boldsymbol{P}^2$. Let $\varepsilon = \exp(2\pi i/n)$, and $g: \boldsymbol{P}^2 \to \boldsymbol{P}^2$ be the automorphism defined by $g(z_0: z_1: z_2) = (\varepsilon z_0: \varepsilon^k z_1: z_2)$, with $(n, k) = 1$. The group G generated by g is $\boldsymbol{Z}/n\boldsymbol{Z}$. Each element of G has a unique fixed point $(0: 0: 1)$. By Fact E in the next section, there is a unique Einstein-Kaehler metric in $\boldsymbol{P}^2 - D'$ with negative Ricci curvature. It follows that $(\boldsymbol{P}^2 - D')/G$ admits a unique Einstein-Kaehler metric which comes from that of $\boldsymbol{P}^2 - D'$. On the other hand, the quotient variety $M' = \boldsymbol{P}^2/G$ has one singular point corresponding to the fixed point $(0: 0: 1)$. The exceptional set E of the minimal resolution $\bar{M}$ of M' is given by the chain of $\boldsymbol{P}^1$'s with self-intersection numbers $(b_1, b_2, \cdots, b_r)$, where b_i's are determined by the continued fraction $n/k = b_1 - (b_2 - (\cdots (b_{r-1} - b_r^{-1})^{-1} \cdots)^{-1})^{-1}$.

Let D be the sum of E and the proper transform of D'. Then $(\bar{M}, D)$ satisfies the condition (1.1) of Theorem 1. $K_{\bar{M}} \otimes [D]$ is not ample because $(K_{\bar{M}} + D)Z = 0$ for any curve Z in $E \subset D$ but it is not trivial near D, because $(K_{\bar{M}} + D)Z > 0$ for any curve in $D - E$.

7. Miscellaneous Results. I would like to take this opportunity to state some consequenses of the following:

FACT E (cf. [9]). Let $\bar{M}$ be a compact complex manifold and D a divisor with only simple normal crossings. Suppose that the first Chern class of $K_{\bar{M}} \otimes [D]$ is positive. Then there is a complete Einstein-Kaehler metric on $M = \bar{M} - D$ with negative Ricci curvature which is unique up to multiplication of positive numbers.

We shall prove the following:

THEOREM 3. *There exists a simply-connected noncompact complex manifold which admits a complete Einstein-Kaehler metric of finite volume with negative Ricci curvature but which admits no Riemannian metric with nonpositive sectional curvature.*

The following well-known result can also be proved using Fact E.

PROPOSITION 1. *Let* $(\bar{M}, D)$ *be as in Fact* E. *Then the group of biholomorphic automorphisms of* $M = \bar{M} - D$ *is a finite group.*

In the proof of Proposition 1, we shall use the following:

PROPOSITION 2. *Let* $(\bar{M}, D)$ *be as in Fact* E. *Then any biholomorphic automorphism* $f\colon \bar{M} - D \to \bar{M} - D$ *extends to that of* $\bar{M}$ *preserving* D.

We can prove Proposition 2 by induction using the following: (1) There exists a complete Einstein-Kaehler metric on $\bar{M} - D$ with negative Ricci curvature (Fact E). (2) The general Schwarz lemma of Yau [21] tells us that any biholomorphic map between complete Einstein-Kaehler manifolds of the same negative scalar curvature is an isometry. (3) If Δ^n is a coordinate polydisk of $\bar{M}$ such that $(\bar{M} - D) \cap \Delta^n = (\Delta^*)^k \times \Delta^{n-k}$, then the canonical Einstein-Kaehler metric of $\bar{M} - D$ is equivalent to $(\Delta^*$, the Poincaré metric$)^k \times (\Delta$, the flat metric$)^{n-k}$ in a small deleted neighborhood of D in Δ^n. (4) Write $C_i = \sum_j D_i \cap D_j$ for the intersection of D_i with the other D_j's, which is a redused divisor with only simple normal crossings. Then $K_{\bar{M}} \otimes [D] \,|\, D_i = K_{D_i} \otimes [C_i]$ and $c_1(K_{D_i} \otimes [C_i]) > 0$.

The rest of this section contains proofs of Theorem 3 and Proposition 1.

PROOF OF THEOREM 3. Let S be a nonsingular hypersurface in $\boldsymbol{P}^n$ $(n \geqq 2)$ of degree $d \geqq n + 2$. Since $c_1(K_{\boldsymbol{P}^n}) = -(n + 1)h$ and $c_1([S]) = dh$, where h is the Poincaré dual of a hyperplane, we see $c_1(K_{\boldsymbol{P}^n} \otimes [S]) > 0$. By Fact E, there exists a complete Einstein-Kaehler metric with negative Ricci curvature on $\boldsymbol{P}^n - S$. So, it suffices to prove that the universal covering of $\boldsymbol{P}^n - S$ does not admit any Riemannian metrics of nonpositive sectional curvature. We give a proof of a classical result $\pi_1(\boldsymbol{P}^n - S) = \boldsymbol{Z}_d$.

(1) Case $n \geqq 3$: Since $\boldsymbol{P}^n$ and S are simply connected and S is nonsingular, $\pi_1(\boldsymbol{P}^n - S, p)$ is generated by a single element g represented by a loop starting from p and linking S simply. Let l be a line through p intersecting S transversely at $p_1, \cdots, p_d$. Let g_i be a loop which starts from p and goes around p_i alone. Then each g_i is homotopic to g in $\boldsymbol{P}^n - S$. Since l is topologically a 2-sphere, it is clear that $g_1 g_2 \cdots g_d = 1$ in $\pi_1(l - p_1 - \cdots - p_d)$ and hence $g^d = 1$ in $\pi_1(\boldsymbol{P}^n - S)$. Moreover no lower

power of g is equal to the identity. Otherwise, there are only d_1-fold coverings of $P^n - S$ where $d_1 < d$. On the other hand, if S is given by the zero locus of a homogeneous polynomial $f(x_0, x_1, \cdots, x_d)$ of degree d, then the zero locus of $x_{n+1}^d - f(x_0, x_1, \cdots, x_d)$ in P^{n+1} is a d-fold branched covering of P^n branched over S, a contradiction. Hence $\pi_1(P^n - S)$ is the cyclic group of order d.

(2) Case $n = 2$: Note $\pi_1(S) \neq 0$. To begin with, we note that any loop γ in $P^2 - S$ can be deformed in $P^2 - S$ to a loop lying in a generic line of P^2. Let l be a line intersecting S transversely. Then $P^2 - l = C^2$. γ can be deformed in $P^2 - S$ to a polygon $\tilde{\gamma}$ with a finite number of edges. Let K be the union of all lines determined by two vertices of $\tilde{\gamma}$. Pick a point x from $P^2 - S - l - K$. Regard $P^2 - x$ as the total space of the hyperplane bundle over P^1. Then each fiber contains at most a finite number of points of $\tilde{\gamma}$. After rotating $\tilde{\gamma}$ by the $U(1)$-action on $P^2 - x$ if necessary, we can deform $\tilde{\gamma}$ to lie in l. Let p be in $l \setminus S$, $l \cap S = \{p_1, \cdots, p_d\}$, and g_i a loop determined by l, p, p_i as above. Then $\pi_1(P^2 - S)$ is generated by $g_1, \cdots, g_d$. In the homotopy exact sequence $0 \to \pi_1(S^1) \to \pi_1(\partial N) \to \pi_1(S) \to 0$, of the S^1-bundle $N(S) \to S$, every g_i comes from S^1. We may assume $g_1 = g_2 = \cdots = g_d$ under suitable orientation of g_i's. The rest goes exactly the same as in the case of $n \geqq 3$.

From the above arguments, the universal cover of $P^n - S$ is the nonsingular hypersurface $x_{n+1}^d - f(x_0, \cdots, x_n) = 0$ minus S. Let it be denoted by M and the compactification of M in P^{n+1} by $\bar{M} = M \cup S$. There is a complete Einstein-Kaehler metric on M with negative Ricci curvature. Now we show that M admits no Riemannian metric of nonpositive curvature. Suppose there is such a metric on M. The Cartan-Hadamard theorem tells us that $M = \bar{M} - S$ is diffeomorphic to R^{2n} and therefore there is a small tubular neighborhood $N(S)$ such that $X = \bar{M} - N(S)$ is homotopic to $\bar{M} - S \sim R^{2n}$. It follows that $\partial X = N(S)$ is a homology $(2n-1)$-sphere by the exact sequence

$$\cdots \to H_q(X) \to \underset{\substack{\wr\| \\ H^{2n-q}(X)}}{H_q(X, \partial X)} \to H_{q-1}(\partial X) \to H_{q-1}(X) \to \cdots .$$

Applying the Gysin exact sequence to the S^1-bundle $\partial X \to S$, we obtain the following exact sequence.

$$\cdots \to H_q(\partial X) \to H_q(S) \to H_{q-2}(S) \to H_{q-1}(\partial X) \to \cdots .$$

Since ∂X is a homology $(2n-1)$-sphere, S has the same Betti numbers as P^{n-1}. On the other hand, the total Chern class of S is given by $c(S) = (1 + h)^{n+1}(1 + dh)^{-1}$, where $h \in H^2(S)$ is the restriction to S of the

Poincaré dual of a hyperplane in $\boldsymbol{P}^n$. Hence

$$(-1)^{n-1}c_{n-1}(S) = \{(d-1)^{n+1} - (-1)^n(n+1)d + (-1)^n\}h^{n-1}/d^2 ,$$

and

$$(-1)^{n-1}e(S) = \int_S (-1)^{n-1}c_{n-1}(S) > nd/2 , \qquad n = e(\boldsymbol{P}^{n-1}) .$$

A contradiction. q.e.d.

PROOF OF PROPOSITION 1. There exists a canonical complete Einstein-Kaehler metric on $\bar{M}-D$ by Fact E. Any biholomorphic map of $\bar{M}-D$ is an isometry with respect to the canonical metric. Since the volume of $\bar{M}-D$ measured by the canonical metric is finite, the group of isometries is compact with respect to the compact-open topology. Let f_t $(-1 \leqq t \leqq 1)$ be a smooth one-parameter family of biholomorphic maps of $\bar{M}-D$. By Proposition 1, we can and do extend this family to a family of holomorphic maps f_t of $\bar{M}$. The vector field defined by $X(z) = (d/dt)_{t=0}f_t(z)$ in $\bar{M}$ is a Killing vector field in $\bar{M}-D$ with respect to the canonical metric. By the same arguments as in Proposition 2, we may assume that D is a nonsingular hypersurface. The extended $\tilde{f}_t$ induces a smooth one-parameter family g_t of biholomorphic maps of D. Since $c_1(K_D) > 0$, the group of biholomorphic maps of D is a finite group. Thus $g_t \equiv g_0$. It follows that $X(z) = 0$ on D. Hence the length of $X(z)$ measured by the canonical metric goes to zero at infinity D. Indeed, if D is given locally by $z_1 = 0$, the length of X is of order $O(|z_1|)$, and the canonical metric is of order $|dz_1|^2/|z_1|^2(\log|z_1|^{-2}) + |dz_2|^2 + \cdots + |dz_n|^2$. For any Killing vector field Y,

$$\Delta(\|Y\|^2/2) = \sum_{i=1}^{n} \|\nabla_{V_i}Y\|^2 - \mathrm{Ric}(Y, Y) , \tag{1.1}$$

where Δ is the Laplacian, $\{V_i\}$ an orthonormal frame. Suppose that $\|X\|$ is not identically zero. Then $\|X\|$ has a relative maximum at a point z in $\bar{M}-D$, since $\|X\|$ goes to zero at infinity. At z, the left hand side of (1.1) is nonpositive and the right hand side is strictly positive, since the Ricci tensor of the canonical metric is negative, a contradiction. Hence $X \equiv 0$ and $f_t \equiv f_0$. The group of automorphisms of $\bar{M}-D$ is discrete in a compact set and is a finite group. q.e.d.

REMARK 1. Yau [22] obtained the following result: There exists a simply-connected compact complex manifold which admits an Einstein-Kaehler metric with negative Ricci curvature. For example, consider any smooth hypersurface of degree $d \geqq n+2$ in $\boldsymbol{P}^n$ $(n \geqq 3)$. Then this is a compact complex manifold satisfying the condition of Fact A. In particular, the above space does not admit any Riemannian metrics of nonpositive sectional curvature. Therefore Theorem 3 is regarded as a noncompact version of the above result of Yau.

REMARK 2. The differential geometric aspects of the results in [15] will be treated in the forthcoming article, which contains a sufficient condition for the simultaneous resolution of quotient singularities in terms of "Chern classes". This corresponds to the existence of a complete Einstein-Kaehler V-metric with negative constant holomorphic curvature.

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Added in proof. The formula for $\delta = -k(M)/3$ in [10, Proposition 4] (see [10, p. 398 and the formula (4) on p. 393]) was incorrect and should read as follows:

$$\begin{aligned} k(M) = 3\Big\{&\sum_n \{n(n+2)/(n+1)\}\times\#(\text{type } A_n) + \sum_m \{(4m^2-4m-9)/4(m-2)\} \\ &\times\#(\text{type } D_m,\ m \geqq 4) + (167/24)\times\#(\text{tpye } E_6) + (383/48) \\ &\times\#(\text{type } E_7) + (1079/120)\times\#(\text{type } E_8)\Big\}\ . \end{aligned}$$

As a result, the proof of the proposition was wrong from p. 398 line 14 on. However, it follows as a special case from Theorem 2 in my paper "Einstein-Kähler metrics on open Satake V-surfaces with isolated quotient singularities", to appear in Math. Ann.. Thanks are due to David Morrison, who pointed out the error.