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SQUARE-INTEGRABLE HOLOMORPHIC FUNCTIONS ON A CIRCULAR DOMAIN IN C^n

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0. Introduction. In the preceding paper [2], square-integrable holomorphic n-forms on an n-dimensional complex manifold are studied, and invariants $\mu_{0,m}$ are introduced. The purpose of this paper is to examine how $\mu_{0,m}$ are expressed when the manifold is a circular domain in the n-dimensional complex Euclidean space C^n, and to provide several examples concerning these invariants.

Let D be a circular domain in C^n which is not necessarily bounded. Let $H(D)$ be the Hilbert space of all square-integrable holomorphic functions on D, and for every integer m, let $H_m(D)$ be the subspace of $H(D)$ whose elements are m-homogeneous on D (see Definition 1.1). Then $H_m(D)$ are mutually orthogonal. If D is proper, then $H_m(D) = \{0\}$ for $m < 0$, and all elements of $H_m(D)$ for $m \geqq 0$ are actually homogeneous polynomials of degree m. Now, suppose that D is proper and has a finite volume $V(D)$. Let $K(z, \bar{w}) = \sum_{m=0}^{\infty} K_m(z, \bar{w})$ be the Bergman kernel of D, where K_m are homogeneous polynomials of degree m with respect to each of the variables z and $\bar{w}$. Then it is shown that

$$\mu_{0,m}((\partial_v)_O) = V(D)(m!)^2 K_m(v, \bar{v})$$

for $v \in C^n$, where $\partial_{(v^1,\cdots,v^n)} = \sum_j v^j \partial/\partial z^j$ (Theorem 2.2). Furthermore, if D is bounded, then every polynomial K_m is written as follows (Corollary 2.4):

$$K_m(z, \bar{w}) = (z^{I_1}, \cdots, z^{I_N})\bar{G}^{-1}(w^{I_1}, \cdots, w^{I_N})^* ,$$

where $(I_1, \cdots, I_N)$ $\left(N = \binom{n+m-1}{m}\right)$ is a numbering of the indices of the set $\{(i_1, \cdots, i_n) \in Z_+^n; i_1 + \cdots + i_n = m\}$ and $G = ((z^{I_i}, z^{I_j}))_{i,j}$ is the Gram matrix of the system $(z^{I_1}, \cdots, z^{I_N})$ of monomials with respect to the inner product on $H(D)$.

It is well-known ([7], [10]) that when a domain carries a Bergman metric g, the holomorphic sectional curvature of g does not exceed 2. In § 3, we see the following from examples:

(i) There exists a domain D in C^2 with positive, finite dimensional $H(D)$. Moreover, there exists a domain in C^2 for which the holomorphic sectional curvature of the Bergman metric is identically 2 (Proposition 3.2).

(ii) For Reinhardt domains in C^n, there is no relationship between the existence of Bergman metrics and the hyperbolicity in the sense of Kobayashi [11] (Propositions 3.1 and 3.3).

(iii) For every interval $[\alpha, \beta] \subset (-\infty, 2)$, there exists a bounded pseudoconvex Reinhardt domain in C^2 for which the image of the holomorphic sectional curvature of the Bergman metric contains $[\alpha, \beta]$ (Proposition 3.5).

1. The Hilbert space $H(D)$ for a circular domain. Let D be a domain in C^n. The set of all functions f holomorphic on D such that $\|f\|^2 = \int_D |f|^2 d\nu_n < +\infty$ is denoted by $H(D)$, where $d\nu_n$ is the Lebesgue measure on C^n. The space $H(D)$ is a separable Hilbert space with inner product $(f, g) = \int_D f\bar{g} d\nu_n$. Let $\{h_m\}$ be a complete orthonormal system of $H(D)$. Then the function $K(z, \bar{w}) = \sum_m h_m(z)\overline{h_m(w)}$ $((z, \bar{w}) \in D \times \bar{D})$ is called the *Bergman kernel* of D and the function $k(z) = K(z, \bar{z})$ is called the *Bergman function* of D.

Now, suppose that D is *circular*, i.e., $e^{i\theta}D \subset D$ for every $\theta \in \boldsymbol{R}$. We denote by $\pi: C^n - \{O\} \to P^{n-1}$ the canonical projection defining the complex projective space P^{n-1}. Take a mapping ψ from P^{n-1} to the unit sphere S^{2n-1} in C^n such that $\pi \circ \psi = 1_{P^{n-1}}$, and consider a domain $V = \{(\zeta, r) \in P^{n-1} \times \boldsymbol{R}_+;\ r\psi(\zeta) \in D\}$ in $P^{n-1} \times \boldsymbol{R}_+$, where $\boldsymbol{R}_+ = \{r \in \boldsymbol{R};\ r \geqq 0\}$ endowed with the relative topology. The set V is independent of the choice of ψ, and D is reproduced in terms of V as follows:

$$D = \{re^{i\theta}\psi(\zeta);\ (\zeta, r) \in V,\ \theta \in \boldsymbol{R}\}\ . \tag{1.1}$$

Conversely, for every domain V in $P^{n-1} \times \boldsymbol{R}_+$, the set D defined by (1.1) is a circular domain in C^n. We call V the *representative domain* for the circular domain D.

DEFINITION 1.1. Let m be an integer. A holomorphic function f on D is called *m-homogeneous* if $f(\lambda z) = \lambda^m f(z)$ for all $\lambda \in C$ and $z \in D$ with $|\lambda| \in I(z)$, where $I(z)$ denotes the connected component of the set $\{r \in \boldsymbol{R}_+ - \{0\};\ rz \in D\}$ containing 1 for $z \in D$. Denote by $H_m(D)$ the space of all functions of $H(D)$ which are m-homogeneous.

Let v be the volume element on P^{n-1} induced from the Fubini-Study metric, and set $U = \{\pi(z);\ z = (z^1, \cdots, z^n) \in C^n,\ z^n \neq 0\}$, $u^j(\zeta) = z^j/z^n$ for $\zeta = \pi(z) \in U$, and $u = (u^1, \cdots, u^{n-1}): U \to C^{n-1}$. Then, letting $|u|^2 = \sum |u^j|^2$, we have

$$v|_U = (1 + |u|^2)^{-n} u^* d\nu_{n-1} .$$

Let α be the mapping from U into S^{2n-1} given by

$$\alpha = (1 + |u|^2)^{-1/2}(u, 1) .$$

We get the following by elementary calculation.

LEMMA 1.2. *Let D be a circular domain in $\boldsymbol{C}^n$ with representative domain $V \subset \boldsymbol{P}^{n-1} \times \boldsymbol{R}_+$, $f \in H_l(D)$ and $g \in H_m(D)$. If $l \neq m$, then $(f, g) = 0$, while if $l = m$, then*

$$(f, g) = 2\pi \int_{(\zeta, r) \in V, \zeta \in U} f(r\alpha(\zeta)) \overline{g(r\alpha(\zeta))} r^{2n-1} v(\zeta) \wedge dr .$$

We also note the following.

LEMMA 1.3. *Let f be a holomorphic function on a circular domain D. For every $z \in D$, let $\sum_{m \in \boldsymbol{Z}} f_m(z)\lambda^m$ be the Laurent expansion around 0 of the function $\{\lambda \in \boldsymbol{C}; |\lambda| \in I(z)\} \ni \lambda \mapsto f(\lambda z) \in \boldsymbol{C}$ (see Definition 1.1). Then the function f_m is holomorphic on D and m-homogeneous for every m, and the series $\sum_m f_m$ converges to f uniformly on every compact subset of D.*

By virtue of Lemmas 1.2 and 1.3 we can show the following by the same argument as in Skwarczyński [13; Theorem 0.8].

PROPOSITION 1.4. *Let D be a circular domain in $\boldsymbol{C}^n$ and B_m complete orthogonal systems of the space $H_m(D)$ for $m \in \boldsymbol{Z}$. Then the union $\bigcup_m B_m$ is a complete orthogonal system of $H(D)$.*

A circular domain D is called *proper* if D contains the origin O. By definition, we immediately get the following:

LEMMA 1.5. *For $m \geqq 0$ (resp. $m < 0$), every m-homogeneous function on a proper circular domain D is the restriction to D of a homogeneous polynomial of degree m (resp. is 0). In particular,*

$$\dim H_m(D) \begin{cases} = 0 , & m < 0 \\ \leqq \dbinom{n+m-1}{m} , & m \geqq 0 . \end{cases}$$

When a circular domain D is *starlike*, i.e., $\lambda D \subset D$ for all $\lambda \in [0, 1]$, there exists a unique $(0, +\infty]$-valued function R defined on $\boldsymbol{P}^{n-1}$ such that the representative domain V of D is given by

$$V = \{(\zeta, r) \in \boldsymbol{P}^{n-1} \times \boldsymbol{R}_+ ; r < R(\zeta)\} .$$

The function R is lower semi-continuous, and D is represented in terms of R as follows, where we let $|\cdot|$ be the Euclidean norm on C^n:

$$D = \{z \in C^n - \{O\};\ |z| < R \circ \pi(z)\} \cup \{O\} .$$

Moreover, it is convenient to consider the upper semi-continuous function $\varphi = -\log R \circ \pi(\cdot, 1) + \log(1 + |\cdot|^2)^{1/2}$ on C^{n-1}, which is plurisubharmonic for pseudoconvex D (cf. [1]).

PROPOSITION 1.6. *Let D be a starlike circular domain in C^n, and φ the function defined above. Then for $f, g \in H_m(D)$ with $m \geqq 0$, we have*

$$(f, g) = \frac{\pi}{m+n} \int_{C^{n-1}} f(\cdot, 1)\overline{g(\cdot, 1)} e^{-2(m+n)\varphi} d\nu_{n-1} ,$$

where f and g are regarded as polynomials (see Lemma 1.5).

PROOF. By Lemma 1.2 we have

$$(f, g) = \frac{\pi}{m+n} \int_U f \circ \alpha \overline{g \circ \alpha} R^{2(m+n)} v .$$

Since $\alpha \circ \pi(\cdot, 1) = (1 + |\cdot|^2)^{-1/2}(\cdot, 1)$ and $\pi(\cdot, 1)^* v|_U = (1 + |\cdot|^2)^{-n} d\nu_{n-1}$, the change of variables yields the desired formula.

Finally, let D be a *Reinhardt domain* in C^n, i.e., D is a domain in C^n such that $(e^{i\theta^1}z^1, \cdots, e^{i\theta^n}z^n) \in D$ for all $(z^1, \cdots, z^n) \in D$ and $\theta^j \in \boldsymbol{R}$. Of course, D may be unbounded. Let Ω be the real representative domain of D: $\Omega = \{(|z^1|, \cdots, |z^n|);\ (z^1, \cdots, z^n) \in D\} \subset \boldsymbol{R}^n_+$. We recall the following two properties of D:

(R_1) For a pair of functions $z^I, z^J \in H(D)$ ($I, J \in \boldsymbol{Z}^n$), one has $(z^I, z^J) = 0$, if $I \neq J$, while if $I = J = (i_1, \cdots, i_n)$, then

$$(z^I, z^I) = (2\pi)^n \int_\Omega (r^1)^{2i_1+1} \cdots (r^n)^{2i_n+1} dr^1 \wedge \cdots \wedge dr^n .$$

(R_2) Every holomorphic function on D can be expanded in a Laurent series around O, which converges uniformly on every compact subset of D.

By making use of the facts (R_1) and (R_2) we obtain the following improvement of [13; Theorem 0.8]:

(R_3) The set $\{z^I;\ I \in \boldsymbol{Z}^n\} \cap H(D)$ is a complete orthogonal system of the space $H(D)$.

2. Invariants $\mu_{0,m}$ of a proper circular domain. Let D be a domain in C^n with the natural coordinate system $(z^1, \cdots, z^n)$. Set $\partial_j = \partial/\partial z^j$ $(j = 1, \cdots, n)$, and $\partial^I = \partial_1^{i_1} \cdots \partial_n^{i_n}$, $|I| = i_1 + \cdots + i_n$ for $I = (i_1, \cdots, i_n) \in \boldsymbol{Z}^n_+$, where ∂_j^0 means the identity operator acting on functions on D.

Every holomorphic tangent vector $X \in T_z(D)$ at $z \in D$ is written as $X = (\partial_v)_z$, where $\partial_v = \sum_j v^j \partial_j$ with $v = (v^1, \cdots, v^n) \in \boldsymbol{C}^n$. For every $m \in \boldsymbol{Z}_+$, $z \in D$, and $X = (\partial_v)_z \in T_z(D)$, set

$$A_m(z) = \{f \in H(D);\ \partial^I f(z) = 0 \text{ for all } I \in \boldsymbol{Z}_+^n \text{ with } |I| < m\}\ ,$$
$$\mu_m(X) = \max\{|(\partial_v)^m f(z)|^2;\ f \in A_m(z),\ \|f\| = 1\} \quad \text{(cf. [2])}\ .$$

For $j = 0, 1$, we consider the following conditions ([10]):

(B.j) For every $z \in D$ and every non-zero $\binom{n+j-1}{j}$-dimensional vector $(\xi_I)_{|I|=j}$, there exists a function $f \in H(D)$ such that $\sum_I \xi_I \partial^I f(z) \neq 0$.

Now, the Bergman kernel K of D is characterized by the following reproducing property: $K(\cdot, \bar{z}) \in H(D)$ and $f(z) = (f, K(\cdot, \bar{z}))$ for all $z \in D$ and $f \in H(D)$. The reproducing property of K implies the following (cf. [2], [4], [5]): If $z \in D$, $I \in \boldsymbol{Z}_+^n$, and $f \in H(D)$, then $\bar{\partial}^I K(\cdot, \bar{z}) \in H(D)$,

$$\partial^I f(z) = (f, \bar{\partial}^I K(\cdot, \bar{z}))\ , \tag{2.1}$$

$$(\partial_v)^m f(z) = (f, (\overline{\partial_v})^m K(\cdot, \bar{z})) \quad \text{and} \tag{2.2}$$

$$\|(\overline{\partial_v})^m K(\cdot, \bar{z})\|^2 = (\partial_v)^m (\overline{\partial_v})^m k(z) \tag{2.3}$$

for $v \in \boldsymbol{C}^n$ and $m \in \boldsymbol{Z}_+$, where k is the Bergman function of D. It follows form (2.1) and (2.2) that

$$A_m(z) = \{\bar{\partial}^I K(\cdot, \bar{z});\ I \in \boldsymbol{Z}_+^n,\ |I| < m\}^\perp\ , \tag{2.4}$$

$$\mu_m(X) = \max\{|(f, (\overline{\partial_v})^m K(\cdot, \bar{z}))|^2;\ f \in A_m(z),\ \|f\| = 1\} \tag{2.5}$$

for $m \in \boldsymbol{Z}_+$ and $X = (\partial_v)_z \in T_z(D)$.

If D satisfies the condition (B.0), then for every positive integer $m \in \boldsymbol{N}$, the function $\mu_{0,m} = \mu_m/\mu_0$ on the holomorphic tangent bundle $T(D)$ is a biholomorphically invariant Finsler pseudometric on D of order $2m$ ([2; § 4]).

From now on, we suppose that D is a proper circular domain. We first note the following.

LEMMA 2.1. *Let D be a proper circular domain with Bergman kernel K. Then*

$$H_m(D) = \operatorname{span}_{\boldsymbol{C}}\{\bar{\partial}^I K(\cdot, O);\ I \in \boldsymbol{Z}_+^n,\ |I| = m\}$$

for $m \in \boldsymbol{Z}_+$.

PROOF. Let B_m be a complete orthonormal system of $H_m(D)$ for every $m \in \boldsymbol{Z}_+$. By Proposition 1.4 we have

$$K(z, \bar{w}) = \sum_{j=0}^{\infty} \sum_{h \in B_j} h(z)\overline{h(w)}\ . \tag{2.6}$$

Let $I \in \boldsymbol{Z}_+^n$ with $|I| = m$. It follows from (2.6) that

$$\bar{\partial}^I K(\cdot, O) = \sum_{j=0}^{\infty} \sum_{h \in B_j} h \overline{\partial^I h(O)} = \sum_{h \in B_m} I! \, \overline{h_I} h \,, \tag{2.7}$$

where $I! = i_1! \cdots i_n!$ for $I = (i_1, \cdots, i_n)$ and $h(w) = \sum_I h_I w^I$; therefore $\bar{\partial}^I K(\cdot, O) \in H_m(D)$ so that $\mathrm{span}_C \{\bar{\partial}^I K(\cdot, O); |I| = m\}$ is contained in $H_m(D)$. To prove the opposite inclusion, we fix a numbering $(h_1, \cdots, h_L)$ of the elements of the set B_m and a numbering $(I_1, \cdots, I_N)$ of the indices of $\{I \in \boldsymbol{Z}_+^n; |I| = m\}$ (note that $L \leqq N$). Write $h_j(z) = \sum_{l=1}^N a_{jl} z^{I_l}$ $(j = 1, \cdots, L)$, and set $f_l = \bar{\partial}^{I_l} K(\cdot, O)$. Since $\{h_j\}$ is linearly independent, by a change of the numbering (I_l), we may assume that the matrix $(a_{jl})_{1 \leqq j, l \leqq L}$ is non-singular. From (2.7) it follows that $f_l = \sum_{j=1}^L I_l! \, \overline{a_{jl}} h_j$ $(l = 1, \cdots, L)$. Since $(a_{jl})_{1 \leqq j, l \leqq L}$ is non-singular, every h_j is a linear combination of $\{f_1, \cdots, f_L\}$. Hence the proof is complete.

The following is the main theorem of this section.

THEOREM 2.2. *Let D be a proper circular domain in $\boldsymbol{C}^n$ with finite volume $V(D)$ with respect to the Lebesgue measure on $\boldsymbol{C}^n$, and B_m complete orthonormal systems of $H_m(D)$ for $m \in \boldsymbol{Z}_+$. Then D satisfies* (B.0), *and the invariants $\mu_{0,m}$ on the space $T_O(D)$ are given by*

$$\mu_{0,m}((\partial_v)_O) = V(D)(m!)^2 \sum_{h \in B_m} |h(v)|^2 \,, \quad v \in \boldsymbol{C}^n \,.$$

To prove this theorem, we use the following well-known fact (cf. [2; Lemma 3.8]).

LEMMA 2.3. *Let $\{x_1, \cdots, x_m\}$ $(m \geqq 0)$ be a linearly independent system of a pre-Hilbert space H over $\boldsymbol{C}$, and $x_{m+1} \in H$. Then the maximum of the set $\{|(y, x_{m+1})|^2; y \in H, (y, x_j) = 0 \ (j = 1, \cdots, m), \|y\| = 1\}$ coincides with $G(x_1, \cdots, x_{m+1})/G(x_1, \cdots, x_m)$, where $G(x_1, \cdots, x_k)$ denotes the Gramian of the system $(x_1, \cdots, x_k)$, that is, $G(x_1, \cdots, x_k) = \det((x_i, x_j))_{i,j}$ with the convention $G(\varnothing) = 1$.*

PROOF OF THEOREM 2.2. By Lemma 2.1 and (2.4) we have $A_m(O) = (\bigcup_{j=0}^{m-1} B_j)^\perp$. Since $\bigcup_{j=0}^{m-1} B_j$ is an orthogonal system, Lemma 2.3, together with (2.3) and (2.5), yields the following:

$$\mu_m((\partial_v)_O) = \|(\overline{\partial_v})^m K(\cdot, O)\|^2 = (\partial_v)^m (\overline{\partial_v})^m k(O) \,.$$

On the other hand, (2.6) implies

$$k(z) = \sum_{j=0}^{\infty} \sum_{h \in B_j} |h(z)|^2 \,.$$

For $I, J \in \boldsymbol{Z}_+^n$ with $|I| = |J| = m$, we have

$$\partial^I \bar{\partial}^J k(O) = \sum_{h \in B_m} I!\, J!\, h_I \overline{h_J} \ ,$$

where $h(z) = \sum_I h_I z^I$, so that we get

$$(\partial_v)^m (\overline{\partial_v})^m k(O) = (m!)^2 \sum_{|I|=|J|=m} v^I \bar{v}^J \partial^I \bar{\partial}^J k(O)/I!\, J! = (m!)^2 \sum_{h \in B_m} |h(v)|^2 \ .$$

Thus, $\mu_m((\partial_v)_O) = (m!)^2 \sum_{h \in B_m} |h(v)|^2$. Furthermore, B_0 consists only of a constant function $V(D)^{-1/2}$, so that (B.0) holds and $\mu_0((\partial_v)_O) = k(O) = V(D)^{-1}$. The proof is now complete.

When a proper circular domain D is bounded, the set of all monomials of degree m forms a basis of $H_m(D)$ for every $m \in \boldsymbol{Z}_+$. In that case, we have the following.

COROLLARY 2.4. *Let D be a bounded, proper circular domain in $\boldsymbol{C}^n$. For every $m \in \boldsymbol{Z}_+$, set*

$$K_m(z, \bar{w}) = (z^{I_1}, \cdots, z^{I_N}) \bar{G}^{-1} (w^{I_1}, \cdots, w^{I_N})^* \ ,$$

where $(I_1, \cdots, I_N)$ is a numbering of the set $\{I \in \boldsymbol{Z}_+^n;\ |I| = m\}$ and G is the Gram matrix of the system $(z^{I_1}, \cdots, z^{I_N})$. Then the invariants $\mu_{0,m}$ on $T_O(D)$ are given by

$$\mu_{0,m}((\partial_v)_O) = V(D)(m!)^2 K_m(v, \bar{v}) \ , \quad v \in \boldsymbol{C}^n \ .$$

PROOF. By Theorem 2.2 the proof is reduced to the following lemma.

LEMMA 2.5. *If $(f_1, \cdots, f_N)$ is a linearly independent system of $H(D)$, and $\{g_1, \cdots, g_N\}$ is an orthonormal basis of the subspace spanned by $\{f_1, \cdots, f_N\}$, then*

$$\sum_{j=1}^N g_j(z) \overline{g_j(w)} = (f_1(z), \cdots, f_N(z)) \bar{G}^{-1} (f_1(w), \cdots, f_N(w))^* \ ,$$

where G is the Gram matrix of the system $(f_1, \cdots, f_N)$.

PROOF. Let $g_j = \sum_{i=1}^N a_{ij} f_i$ $(j = 1, \cdots, N)$, and set $A = (a_{ij})$. Since $(g_i, g_j) = \delta_{ij}$, we have $I = {}^t A G \bar{A}$; therefore $I = \bar{A}\, {}^t A G$, or $I = A A^* \bar{G}$. Hence we have

$$\begin{aligned} \sum_{j=1}^N g_j(z) \overline{g_j(w)} &= (g_1(z), \cdots, g_N(z))(g_1(w), \cdots, g_N(w))^* \\ &= (f_1(z), \cdots, f_N(z)) A A^* (f_1(w), \cdots, f_N(w))^* \\ &= (f_1(z), \cdots, f_N(z)) \bar{G}^{-1} (f_1(w), \cdots, f_N(w))^* \ . \end{aligned}$$

3. Examples. When a domain D satisfies the conditions (B.0) and (B.1) in § 2, it is called *B-hyperbolic*. In that case, there exists a unique Hermitian metric g (called the *Bergman metric*) on D such that $\mu_{0,1}(X) =$

$g(X, \bar{X})$ for $X \in T_p(D)$, and the holomorphic sectional curvature $HSC(X)$ of the Bergman metric in the direction $X \in T_p(D) - \{0\}$ satisfies the following ([2; Theorem 4.4], [7; p. 525)]:

$$HSC(X) = 2 - \mu_{0,2}(X)/g(X, \bar{X})^2 \; . \tag{3.1}$$

We say that a manifold M is *K-hyperbolic* if M is hyperbolic in the sense of Kobayashi [11]. Every bounded domain is both B- and K-hyperbolic, and satisfies $HSC < 2$.

We first consider the following one-parameter family of unbounded proper Reinhardt domains in $\boldsymbol{C}^2$.

EXAMPLE 1. $D_s = \{(z^1, z^2) \in \boldsymbol{C}^2; |z^1| < 1, |z^2|^2 < (1 - |z^1|^2)^s\}$ $(s < 0)$.

By Lemma 1.5 we have $(z^1)^m(z^2)^n \notin H(D_s)$ for $m, n \in \boldsymbol{Z}$ with $m < 0$ or $n < 0$. By (R_1) in § 1 we have

$$\|(z^1)^m(z^2)^n\|^2 = \frac{\pi^2}{n+1}\int_0^1 t^m(1-t)^{s(n+1)}dt \; , \quad m, n \in \boldsymbol{Z}_+ \; ,$$

so that if $m, n \in \boldsymbol{Z}_+$ then

$$(z^1)^m(z^2)^n \in H(D_s) \Leftrightarrow n < -1/s - 1 \; . \tag{3.2}$$

In particular, $H(D_s) = \{0\}$ if $s \leqq -1$. Suppose that $-1 < s < 0$. Put $N(s) = -[1/s + 2]\,(\in \boldsymbol{Z}_+)$. Then $n < -1/s - 1$ if and only if $n \leqq N(s)$; in this case, one has

$$\|(z^1)^m(z^2)^n\|^2 = \frac{\pi^2}{n+1}\frac{m!}{(s(n+1)+m+1)\cdots(s(n+1)+1)} \; .$$

By the formula

$$(1-x)^{-\alpha} = \sum_{m=0}^{\infty}\frac{(\alpha+m-1)(\alpha+m-2)\cdots\alpha}{m!}x^m \; , \qquad |x| < 1 \; , \quad \alpha \in \boldsymbol{R} \; , \tag{3.3}$$

the Bergman kernel $K(z, \bar{w})$ of D_s is written as

$$K(z, \bar{w}) = \pi^{-2}(1 - z^1\bar{w}^1)^{-s-2}\sum_{n=0}^{N(s)} a_{n+1}U_s(z, \bar{w})^n \; ,$$

where $a_n = n^2 s + n$ and $U_s(z, \bar{w}) = (1 - z^1\bar{w}^1)^{-s}z^2\bar{w}^2$. It is easily shown that the image of the function U_s on $D_s \times \overline{D_s}$ is the whole $\boldsymbol{C}$; therefore the Bergman kernel K vanishes at some point in $D_s \times \overline{D}_s$. On the other hand, the image of the function $u_s(z) = U_s(z, \bar{z})$ on D_s is the interval $[0, 1)$. Therefore, making use of (3.3) again, we obtain the following expression for the Bergman function $k(z) = K(z, \bar{z})$ of D_s:

$$k(z) = \frac{F_s(u_s(z))}{\pi^2(1-|z^1|^2)^{s+2}(1-u_s(z))^3},$$

where F_s is a polynomial given by

$$F_s(u) = (s+1) + (s-1)u - a_{N+2}u^{N+1} - (2s - a_{N+1} - a_{N+2})u^{N+2} - a_{N+1}u^{N+3}$$

with $N = N(s)$.

Now, all the domains D_s are K-hyperbolic by virtue of the following theorem formulated by Sibony [12; p. 366] and essentially due to Kiernan [9]:

(K-S) Let E, M be two complex manifolds, and f a surjective holomorphic mapping from E onto M. Suppose that M is K-hyperbolic and admits an open covering $\{U_\nu\}$ such that $f^{-1}(U_\nu)$ is K-hyperbolic for all ν. Then E is K-hyperbolic.

It is well-known that the domain $\boldsymbol{C} - \{0, 1\}$ is K-hyperbolic ([11]) and not B-hyperbolic (in fact $H(\boldsymbol{C} - \{0, 1\}) = \{0\}$). We have found such an example among Reinhardt domains.

PROPOSITION 3.1. *The domain D_s with $s \leqq -1/2$ is K-hyperbolic, but not B-hyperbolic.*

Example 1 suggests the existence of a Reinhardt domain D in $\boldsymbol{C}^2$ with positive finite dimensional $H(D)$. The following is such.

EXAMPLE 2. $D_{s,t} = D_s \cup \{(z^1, z^2); (z^2, z^1) \in D_t\}$ $(s, t < 0)$. From (3.2) it follows that

$$(3.4) \qquad (z^1)^m(z^2)^n \in H(D_{s,t}) \Leftrightarrow m < -1/t - 1, \quad n < -1/s - 1$$

for $m, n \in \boldsymbol{Z}_+$.

PROPOSITION 3.2. *If $-1/2 < s \leqq -1/3$ and $-1/2 < t \leqq -1/3$, then the domain $D_{s,t}$ is B-hyperbolic, and the holomorphic sectional curvature of the Bergman metric is identically* 2.

PROOF. In view of (3.4), the assumptions for s and t imply that the space $H(D_{s,t})$ contains all polynomials of degree $\leqq 1$, and contains no polynomial of degree $\geqq 2$; therefore the properties (B.0) and (B.1) hold and $\mu_2 = 0$, so that $\mu_{0,2} = 0$. By (3.1) we get $HSC = 2$.

EXAMPLE 3. $D^s = D_{s,s} \cup \{(z^1, z^2) \in \boldsymbol{C}^2;\ |z^1| \geqq 1,\ |z^2| \geqq 1,\ (|z^1|^{-2/s} - 1) \times (|z^2|^{-2/s} - 1) < 1\}$ $(s < 0)$. Similarly to (3.4), we have

$$(3.5) \qquad (z^1)^m(z^2)^n \in H(D^s) \Leftrightarrow m, n < -1/s - 1$$

for $m, n \in \boldsymbol{Z}_+$.

PROPOSITION 3.3. *The domain $D = D^s \cup \{(z^1, z^2) \in \boldsymbol{C}^2;\ |z^1| < 1,\ |z^2| < 2\}$ with $-1/2 < s < 0$ is B-hyperbolic but not K-hyperbolic.*

PROOF. The assumption for s and (3.5) imply that all the polynomials of degree $\leqq 1$ belong to both $H(D^s)$ and $H(D)$; therefore D satisfies (B.0) and (B.1). Furthermore, since D contains a complex line $C\times\{1\}$, it is not K-hyperbolic.

By Propositions 3.1 and 3.3 we see that for Reinhardt domains there is, in general, no relationship between K-hyperbolicity and B-hyperbolicity. It is noted that if a domain is B-hyperbolic, and if the holomorphic sectional curvature of the Bergman metric is bounded from above by a negative constant, then the domain is K-hyperbolic (cf. [11; p. 61]).

REMARK 3.4. The following domain ([14; p. 415]) also satisfies the same property as D in Proposition 3.3:

$$D = \{(z^1, z^2) \in \boldsymbol{C}^2;\ |z^2|^2 < \exp(-|z^1|^{2/s})\} \quad (s > 0)\ .$$

Indeed, all polynomials belong to $H(D)$, and the Bergman kernel is given by

$$K(z, \bar{w}) = \frac{1}{\pi^2}\sum_{m,n=0}^{\infty}\frac{(n+1)^{s(m+1)+1}}{s\Gamma(s(m+1))}(z^1\bar{w}^1)^m(z^2\bar{w}^2)^n\ ,$$

while D contains a complex line $C\times\{0\}$.

Finally, we give an example of a bounded pseudoconvex Reinhardt domain for which the holomorphic sectional curvature of the Bergman metric possesses a positive value. Let D be a bounded proper Reinhardt domain in C^2 with a real representative domain Ω. For $m, n \in \boldsymbol{Z}_+$, set

$$a_{mn} = \left(\int_{\Omega}(r^1)^{2m+1}(r^2)^{2n+1}dr^1\wedge dr^2\right)^{-1}\ . \tag{3.6}$$

Then the formula (3.1), together with Theorem 2.2, implies the following:

(R_4) The holomorphic sectional curvature HSC of the Bergman metric on D at the origin O is given by

$$HSC((\partial_v)_O) = 2 - 4a_{00}(a_{20}x^2 + a_{11}xy + a_{02}y^2)(a_{10}x + a_{01}y)^{-2}$$

for $v = (v^1, v^2) \in C^2 - \{0\}$ with $x = |v^1|^2$, $y = |v^2|^2$.

(R_5) If $a_{01} = a_{10}$, $a_{02} = a_{20}$, and $2a_{20} \leqq a_{11}$, then

$$\begin{cases}\min\limits_{v\neq 0} HSC((\partial_v)_O) = 2 - a_{00}(2a_{20} + a_{11})/a_{10}^2 \\ \max\limits_{v\neq 0} HSC((\partial_v)_O) = 2 - 4a_{00}a_{20}/a_{10}^2\ .\end{cases}$$

EXAMPLE 4 ([3]). The domain $D(N) = \{(z^1, z^2) \in \boldsymbol{C}^2;\ |z^1|^{2/N} + |z^2|^{2/N} < 1\}$ ($N \in \boldsymbol{N}$) is pseudoconvex, and the values a_{mn} of (3.6) for this domain are

$$a_{mn} = \frac{4(m+n+2)(N(m+n+2)-1)!}{N(N(m+1)-1)!\,(N(n+1)-1)!}\ .$$

Since $2a_{20} \leqq a_{11}$, by the formula in (R_5) we have

$$\min_{v \neq 0} HSC((\partial_v)_o) < 2 - \frac{8}{9}\prod_{j=1}^{N}\left(1 + \frac{N}{3N-j}\right) < 2 - \frac{8}{9}\left(\frac{4}{3}\right)^N ,$$

$$\max_{v \neq 0} HSC((\partial_v)_o) = 2 - \frac{32}{9}\prod_{j=1}^{N}\left(1 - \left(\frac{N}{3N-j}\right)^2\right) > 2 - 4\left(\frac{8}{9}\right)^{N+1} .$$

From this we get the following.

PROPOSITION 3.5. *For any interval* $[\alpha, \beta] \subset (-\infty, 2)$, *there exists a bounded pseudoconvex Reinhardt domain in* C^2 *for which* $\inf HSC < \alpha$ *and* $\sup HSC > \beta$.

REMARK 3.6. It is well-known that there exist homogeneous, bounded domains for which $\max HSC \geqq 0$. For example, the Siegel domain $D[q]$ in C^{3+q}, $q = 3, 4, \cdots$, considered in D'Atri [6; § 4] satisfies $\min HSC = -2/3$ and $\max HSC = 1/3 - 2/(q+3)$.

Now, let C be the Carathéodory metric on a bounded domain D. Then the following is well-known (Hahn [8], Burbea [4], [5]):

$$C^2 < \mu_{0,1} \quad \text{on} \quad T(D) - \{\text{the zero section}\} . \tag{3.7}$$

Moreover, the following is also known ([4; Theorem 2]):

$$4C^4 < (2 - HSC)\mu_{0,1}^2 \quad \text{on} \quad T(D) - \{\text{the zero section}\}. \tag{3.8}$$

The assertion (3.8) is equivalent to $4C^4 < \mu_{0,2}$ by (3.1). As a corollary to Proposition 3.5 we get the following assertion concerning the opposite inequality of (3.7):

COROLLARY 3.7. *For any* $\alpha > 0$, *there exists a bounded pseudoconvex Reinhardt domain in* C^2 *for which* $C^2 \ngeqq \alpha\mu_{0,1}$.

PROOF. It follows from (3.8) that

$$\inf_{X \in T(D), X \neq 0} C(X)^2/\mu_{0,1}(X) \leqq 2^{-1}(2 - \sup HSC)^{1/2} .$$

Hence, the desired assertion follows from Proposition 3.5.

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