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EXPOSED POINTS AND EXTREMAL PROBLEMS IN H^1, II

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ABSTRACT. If $\phi \in L^\infty$, we denote by T_ϕ the functional defined on the Hardy space H^1 by

$$T_\phi(f)=\int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})d\theta/2\pi .$$

Let S_ϕ be the set of functions in H^1 which satisfy $T_\phi(f)=\|T_\phi\|$ and $\|f\|_1 \leqq 1$. If S_ϕ is not empty and weak*-compact, a description of S_ϕ was given in the first part of this paper. In this paper, the structure of S_ϕ is studied generally. Moreover, we give a characterization of exposed points, that is, g in H^1 such that $S_\phi=\{g\}$ for some ϕ.

1. Introduction. Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|d\theta$ is bounded for $0 \leqq r < 1$, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1}\int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|d\theta = \int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|d\theta ,$$

then f is said to be in the class N_+. The set of all boundary functions in N_+ is denoted by N_+ again. For $0 < p \leqq \infty$, the Hardy space H^p is defined as $N_+ \cap L^p$. If $1 \leqq p \leqq \infty$, it coincides with the space of functions in L^p whose Fourier coefficients with negative indices vanish. If h in N_+ has the form

$$h(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\log|h(e^{it})|dt/2\pi + i\alpha\right\} \qquad (z \in U)$$

for some real α, then h is called an outer function. We call q in N_+ an inner function if $|q(e^{i\theta})| = 1$ a.e. on ∂U. Each nonzero f in H^1 has a unique factorization of the form $f = qh$, where q is an inner function and h is an outer function.

If $\phi \in L^\infty$, we denote by T_ϕ the functional defined on H^1 by

$$T_\phi(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\phi(e^{i\theta})d\theta/2\pi .$$

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The norm of T_ϕ is $\|T_\phi\| = \sup\{|T_\phi(f)|: f \in S\}$. Let S_ϕ denotes the set of all $f \in S$ for which $T_\phi(f) = \|T_\phi\|$, where $S = \{f \in H^1: \|f\|_1 \leqq 1\}$.

DEFINITION. $g \in H^1$ with $\|g\|_1 = 1$ is called an exposed point of S if $S_\phi = \{g\}$ for some ϕ.

de Leeuw and Rudin [2, Theorem 8] pointed out that if g is an exposed point of S, then it is an outer function and that for every a with $|a| \leqq 1$ the function $g/(z-a)(1-\bar{a}z)$ fails to be in H^1. We can ask whether the converse is valid. Hayashi [1] gave an example which shows the converse is not true. In this paper we give a characterization for exposed points of S which is related to the sufficient condition of [2] above. From this it follows that the converse is true in some special cases. If g and g^{-1} belong to H^1, then $g/\|g\|_1$ is an exposed point of S. A more elaborate example of exposed points of S is $g/\|g\|_1$ in H^1 with a nonnegative real part [5, Theorem 3].

Let C denote the space of continuous functions on ∂U and set $A = H^\infty \cap C$. Then $H^1 = (C/zA)^*$. The author [4, Theorem 2] obtained a complete description of S_ϕ if S_ϕ is weak*-compact. In this paper, we give a general structure theorem for S_ϕ from which the description of S_ϕ in [4, Theorem 2] follows.

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2. Exposed points. Suppose S_ϕ is nonempty and set $S_\phi^0 = \{f \in S_\phi: f/(z-a)(1-\bar{a}z) \in H^1 \text{ for some } a \in \bar{U}\}$. Set $S^1 = \{f \in H^1: \|f\|_1 = 1\}$. The following lemmas are known:

LEMMA 1. (cf. [2, Theorem 1]) *If $k, h \in S^1$ and $k \neq h$, then $(k+h)/2$ is not an outer function*

LEMMA 2. (cf. [2, p. 478]) *Assuming f and g in S^1, f and g belong to the same S_ϕ if and only if* $\arg f(e^{i\theta}) = \arg g(e^{i\theta})$ *a.e. on* ∂U.

Although the following is essentially in [2, Theorems 8 and 9], we give a simpler proof.

PROPOSITION 1. *Suppose S_ϕ is nonempty. Then S_ϕ^0 is empty if and only if $S_\phi = \{f\}$ for some f in H^1.*

PROOF. If $S_\phi^0 \neq \varnothing$, then $(z-a)(1-\bar{a}z)g \in S_\phi$ for some $a \in \bar{U}$ and some $g \in H^1$. Since $(z-a)(1-\bar{a}z)/(z-c)(1-\bar{c}z) \geqq 0$ for any $c \in \bar{U}$ on

∂U, $\gamma(z-c)(1-\bar{c}z)g$ belongs to S_ϕ for some $\gamma > 0$ and $c \neq a$ by Lemma 2. Thus $S_\phi \neq \{f\}$ for any f in H^1. If there exist f and k in S_ϕ such that $f \neq k$, then by Lemma 1 $(f+k)/2 \in S_\phi$ is not an outer function. So we can write $(f+k)/2 = qh$ for a nontrivial inner function q and an outer function h. Since $\bar{q}(q-q(0))(1-\overline{q(0)}q) \geqq 0$ a.e. on ∂U, $\gamma(q-q(0)) \times (1-\overline{q(0)}q)h$ belongs to S_ϕ for some $\gamma > 0$ by Lemma 2. This implies $S_\phi^0 \neq \emptyset$. q.e.d.

LEMMA 3. *If S_ϕ^0 is nonempty, then* $\|T_{z\phi}\| = \|T_\phi\|$, $zS_{z\phi} \subset S_\phi$ *and* $S_\phi^0 = \{\gamma(z-a)(1-\bar{a}z)k \in S^1;\ \gamma > 0,\ |a| \leqq 1,\ k \in S_{z\phi}\}$.

PROOF. Since S_ϕ^0 is nonempty, from the first part of the proof of Proposition 1 it follows that there exists $k \in S^1$ with $zk \in S_\phi^0$. Then $k \in S_{z\phi}$, because $T_\phi(zk) = \|T_\phi\|$ and $\|T_{z\phi}\| = \|T_\phi\|$. Hence $S_\phi^0 = \{\gamma(z-a)(1-\bar{a}z)k \in S^1 : \gamma > 0, |a| \leqq 1$ and $k \in S_{z\phi}\}$ and $zS_{z\phi} \subset S_\phi$. q.e.d.

PROPOSITION 2. *If S_ϕ^0 is not empty, then S_ϕ is the L^1-closure of S_ϕ^0.*

PROOF. If $f \in S_\phi$ and $f = qh$, where q is a nontrivial inner function and h is an outer function, then there is a positive constant γ_α such that $\gamma_\alpha(q-\alpha)(1-\bar{\alpha}q)h \in S_\phi$ for any complex number α. By a theorem of Frostman in [3, p. 119], there is a sequence $\{\alpha_n\}$ such that $\alpha_n \to 0$ and $q - \alpha_n$ has zeros in U. So if we set $F_n = \gamma_n(q-\alpha_n) \times (1-\bar{\alpha}_n q)h$ and $\gamma_n = \gamma_{\alpha_n}$, then $F_n/(z-a_n)(1-\bar{a}_n z) \in H^1$ for some $a_n \in U$, hence $F_n \in S_\phi^0$. Since $\alpha_n \to 0$, we have $(q-\alpha_n)(1-\bar{\alpha}_n q) \to q$ a.e. on ∂U and $\gamma_n \to 1$. Thus f can be approximated by functions in S_ϕ^0. When $f \in S_\phi$ is an outer function, there is $g \in S_\phi$ with $g \neq f$ by Proposition 1 because $S_\phi^0 \neq \emptyset$. Then $\lambda f + (1-\lambda)g$ belongs to S_ϕ for $0 < \lambda < 1$, it is not an outer function by Lemma 1 and it can be approximated by functions in S_ϕ^0 by what was just proved. On the other hand, f can be approximated by $\lambda f + (1-\lambda)g$ as $\lambda \to 1$ and hence the proposition follows. q.e.d.

COROLLARY 1. *If S_ϕ is weak*-compact and S_ϕ^0 is nonempty, then* $S_\phi^0 = S_\phi$.

PROOF. By Lemma 3, $S_\phi^0 = \{\gamma(z-a)(1-\bar{a}z)k \in S^1 : \gamma > 0,\ |a| \leqq 1$ and $k \in S_{z\phi}\}$. Since $zS_{z\phi} \subset S_\phi$ and S_ϕ is weak*-compact, $S_{z\phi}$ is weak*-compact, too. We shall show that S_ϕ^0 is closed in the L^1-topology. Then $S_\phi^0 = S_\phi$ by Proposition 2. If $\|\gamma_j(z-a_j)(1-\bar{a}_j z)k_j - f\|_1 \to 0$ as $j \to \infty$, where $\gamma_j > 0$, $|a_j| \leqq 1$, $k_j \in S_{z\phi}$ and $f \in S_\phi$, then $\gamma_j(s-a_j) \times (1-\bar{a}_j s)k_j(s) \to f(s)$ for any s in U. There are subsequences a_{j_n} and k_{j_n} of a_j and k_j such that $a_{j_n} \to a$ and $k_{j_n} \to k$ in the weak*-topology as $n \to \infty$. Hence $\gamma_{j_n} \to f(s)/(s-a)(1-\bar{a}s)k(s)$ for any s in U such that $(s-a)(1-\bar{a}s)k(s) \neq 0$. Since $k \not\equiv 0$, we have $\gamma_{j_n} \to \gamma$ as $n \to \infty$. Thus $f = \gamma(z-a)(1-\bar{a}z)k$ and $f \in S_\phi^0$. q.e.d.

If q is a singular inner function and $\phi = \bar{q}$, then $q \in S_\phi$ but $q \notin S_\phi^0$. Hence S_ϕ^0 may be a proper subset of S_ϕ. Now we shall give a characterization of exposed points of S.

THEOREM 3. *Let g be a nonzero function in H^1 with $\|g\|_1 = 1$. g is an exposed point of S if and only if g cannot be approximated by any k in H^1 which satisfies the following conditions:*
(1) $\arg k(e^{i\theta}) = \arg g(e^{i\theta})$ a.e. on ∂U and (2) $k/(z-a)\times(1-\bar{a}z)$ belongs to H^1 for some $a \in \bar{U}$.

PROOF. If g is an exposed point of S, then $S_\phi = \{g\}$ for some ϕ by definition. Thus $S_\phi^0 = \emptyset$ by Proposition 1. Hence the proof of the "only if" part follows. If g is not an exposed point of S, then S_ϕ^0 is dense in S_ϕ in the L^1-topology with $\phi = |g|/g$ by Proposition 2. Hence the proof of the "if" part follows. q.e.d.

We can give a simpler characterization of exposed points of S under some condition. Suppose g is a nonzero function in S such that S_ϕ is weak*-compact for $\phi = |g|/g$. Then g is an exposed point of S if and only if $g/(z-a)(1-\bar{a}z)$ fails to be in H^1 for any $a \in \bar{U}$. The "only if" part is known in [2]. For the "if" part, use Corollary 1.

3. The description of S_ϕ. Let us denote by Z_+ the set of all nonegative integers. The structure of S_ϕ for $\phi = \bar{z}^n$ $(n \in Z_+)$ is known completely as follows. We omit the proof, since it is straightforward.

$$S_{\bar{z}^n} = \{\gamma \prod (z - a_j)(1 - \bar{a}_j z) \in S^1 \colon \gamma > 0,\, a_j \in \bar{U}\}\,.$$

In this section we consider S_ϕ in general. For any $\phi \in L^\infty$ $\|T_\phi\| \geqq \|T_{z\phi}\| \geqq \|T_{z^2\phi}\| \geqq \cdots$.

THEOREM 4. *Let n be Z_+. Suppose $S_{z^l\phi} \neq \emptyset$ for any $l \in Z_+$ with $0 \leqq l \leqq n$. Then the following are equivalent.*

(1) $\|T_\phi\| = \|T_{z^n\phi}\|$.

(2) *S_ϕ is the L^1-closure of the set of all $f \in S^1$ of the form $f = \gamma p h$, with $\gamma > 0$, $p \in S_{\bar{z}^n}$ and $h \in S_{z^n\phi}$.*

PROOF. (2) $\Rightarrow$ (1). If $h \in S_{z^n\phi}$ then $z^n h \in S_\phi$ because $z^n \in S_{\bar{z}^n}$, hence $\|T_\phi\| = \|T_{z^n\phi}\|$. (1) $\Rightarrow$ (2). The proof is by induction on n. If $n = 0$ then (1) $\Rightarrow$ (2) is true trivially. Assume (2) follows from (1) for n. We shall prove (1) $\Rightarrow$ (2) for $n+1$. If $\|T_{z^n\phi}\| = \|T_{z^{n+1}\phi}\|$ then $zS_{z^{n+1}\phi} \subset S_{z^n\phi}$, hence $S_{z^n\phi}^0 \neq \emptyset$. By Lemma 3 $S_{z^n\phi}^0 = \{\gamma p_1 k \colon \gamma > 0,\ p_1 \in S_{\bar{z}}$ and $k \in S_{z^{n+1}\phi}\}$ and by Proposition 2 $S_{z^n\phi}$ is the L^1-closure of $S_{z^n\phi}^0$. This and the hypothesis of induction show that S_ϕ is the L^1-closure of the set of all $f \in S^1$ of the form $f = \gamma p_1 p_n h$, with $\gamma > 0$, $p_1 \in S_{\bar{z}}$, $p_n \in S_{\bar{z}^n}$ and $h \in S_{z^{n+1}\phi}$. Hence (2) follows because $\gamma' p_1 p_n \in S_{\bar{z}^{n+1}}$ for some $\gamma' > 0$. q.e.d.

COROLLARY 2. *Let n be in Z_+. Suppose $S_{z^l\phi} \neq \emptyset$ for any $l \in Z_+$ with $0 \leqq l \leqq n$. Then the following are equivalent.*

(1) $\|T_\phi\| = \|T_{z^n\phi}\|$ *and* $S_{z^n\phi} = \{g\}$.

(2) *S_ϕ consists of all $f \in S^1$ of the form $f = \gamma pg$ with $\gamma > 0$, $p \in S_{\bar{z}^n}$ and an exposed $g \in S_{z^n\phi}$.*

COROLLARY 3. (cf. [4, Theorem 1] and [1]) *If $\phi = \bar{z}^n |g|/g$ $(n \in Z_+)$ and $g/\|g\|_1$ is an exposed point of S, then S_ϕ consists of all $f \in S^1$ of the form $f = \gamma pg$ with $\gamma > 0$ and $p \in S_{\bar{z}^n}$.*

From Corollary 3 the description of S_ϕ follows in case S_ϕ is weak*-compact [4, Theorem 2]. If $\|T_\phi\| > \|T_{z^n\phi}\|$ for some $n \in Z_+ \setminus \{0\}$ then S_ϕ is nonempty and S_ϕ is weak*-compact (see [4, p. 228]).

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