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CURVATURE OPERATOR OF THE BERGMAN METRIC ON A HOMOGENEOUS BOUNDED DOMAIN

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0. Introduction. It is well known that a symmetric bounded domain in a complex Euclidean space possesses the following two curvature properties of the Bergman metric:

(i) The sectional curvature is non-positive.

(ii) When the domain is irreducible, the curvature operator has at most two distinct eigenvalues.

The latter is shown in Calabi and Vesentini [6], and Borel [3]. Recently, it was shown in D'Atri and Miatello [10] that symmetric bounded domains are characterized by the property (i) in the category of homogeneous bounded domains. The main purpose of this paper is to show that symmetric bounded domains are characterized by the property (ii) in the category of irreducible, homogeneous bounded domains (Theorem 7.2). A theorem of this type was obtained by Itoh [15]: A compact, Kähler, simply connected homogeneous space with the second Betti number $b_2 = 1$ is Hermitian symmetric if and only if the curvature operator has at most two distinct eigenvalues. Several characterizations of symmetric bounded domains in the category of homogeneous bounded domains are discussed also in [11], [12].

Our proof of Theorem 7.2 is based on the theory of normal j-algebras. After studying curvature properties of a normal j-algebra in §§3-6, we shall prove Theorem 7.2 in §7. The proof is divided into two steps as follows: Let $(\mathfrak{g}, j)$ be a normal j-algebra corresponding to an irreducible, homogeneous bounded domain D with at most two distinct eigenvalues of the curvature operator, and let $\mathfrak{g} = \sum_{a \leq b} \mathfrak{n}_{ab} + \sum_{a \leq b} j\mathfrak{n}_{ab} + \sum_a \mathfrak{n}_{a*}$ be its root space decomposition. We first show that $\dim \mathfrak{n}_{ab} = n_{12}$ for every pair (a, b) with $a < b$, and that $\dim \mathfrak{n}_{a*} = n_{1*}$ for every a (Lemma 7.5). This means that D is quasi-symmetric in the sense of Satake [23] (cf. [10]). We next conclude that D is symmetric, by means of a criterion of Dorfmeister [12] for a quasi-symmetric bounded domain to be symmetric (Proposition 7.8).

Several by-products of our argument are given in §§8-9. Denote by HSC the holomorphic sectional curvature of the Berman metric g on a homogeneous bounded domain D, and denote by λ_D the minimum of eigenvalues of the curvature operator of g. Then min HSC, max HSC, and λ_D are biholomorphic invariants, and $\lambda_D < 0$. Set $\gamma_D = -2/\lambda_D$. When D is symmetric the following hold (Theorem 8.4, Corollary 8.5):

$$-1 \leqq \min \mathrm{HSC} = -2/\gamma_D \leqq -2/(\dim D + 1)\ ,$$
$$\max \mathrm{HSC} = -2/\gamma_D R_D \leqq -1/\dim D\ ,$$

where R_D is the rank of D.

Let B_D and C_D be the Finsler metrics on a bounded domain D of Bergman and Carathéodory, respectively. It is well known ([4], [5], [13], [14]) that

$$C_D < B_D \quad \text{on} \quad T(D) - \{\text{the zero section}\}$$

for every bounded domain D, where $T(D)$ is the holomorphic tangent bundle over D. If we assume the domain to be homogeneous or symmetric, we get more precise inequalities as follows: For every homogeneous bounded domain D,

$$2C_D^2 \leqq B_D^2 \quad \text{on} \quad T(D)$$

(Theorem 9.1); and for a symmetric bounded domain D,

$$\gamma_D C_D^2 \leqq B_D^2 \leqq \gamma_D R_D C_D^2 \quad \text{on} \quad T(D)$$

(Theorem 9.2). Furthermore, these three inequalities are sharp.

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1. Curvature operator of the Bergman metric. Let D be a bounded domain in the complex Euclidean space C^n of dimension n with the coordinate system $(z^1, \cdots, z^n)$, and set $\partial_a = \partial/\partial z^a$ $(a = 1, \cdots, n)$. Let g be *the Bergman metric* on D, i.e., for every point $p \in D$, g is a C-bilinear form on the complexification $T_p^C(D) = T_p^R(D) \otimes C$ of the real tangent space $T_p^R(D)$ at p, given by $g = 2\sum g_{a\bar{b}}\, dz^a \cdot d\bar{z}^b$, $g_{a\bar{b}} = \partial_a \overline{\partial_b} \cdot \log k$, where k is the Bergman function of D. Denote by $T_p(D)$ the holomorphic tangent space at p. Thus $T_p^C(D) = \overline{T_p(D)} + T_p(D)$ (direct sum). The restriction of g to $T_p^R(D) \times T_p^R(D)$ is a Hermitian Kähler metric (cf. [19]). Let R be the C-bilinear extension of the Riemannian curvature tensor of this Hermitian Kähler metric, and set $R_{a\bar{b}c\bar{d}} = g(R(\partial_c, \overline{\partial_d})\overline{\partial_b}, \partial_a)$. Since g is Kählerian, it follows that

$$R_{a\bar{b}c\bar{d}} = R_{c\bar{b}a\bar{d}}\ , \qquad R_{a\bar{b}c\bar{d}} = R_{a\bar{d}c\bar{b}}\ ; \tag{1.1}$$

explicitly,

$$R_{a\bar{b}c\bar{d}} = \partial_c\overline{\partial_d}\cdot g_{a\bar{b}} - \sum_{s,t} g^{\bar{s}t}(\partial_c\cdot g_{a\bar{s}})(\overline{\partial_d}\cdot g_{t\bar{b}}) \,, \tag{1.2}$$

where $(g^{\bar{s}t})$ is the inverse matrix of $(g_{a\bar{b}})$. We also note

$$R_{a\bar{b}c\bar{d}} = \overline{R_{b\bar{a}d\bar{c}}} \,. \tag{1.3}$$

For $p \in D$, we denote by $S_p^2(D)$ the 2-symmetric tensor product of the space $T_p(D)$. For brevity we set $e_{ab} = (\partial_a)_p \otimes (\partial_b)_p$ and $f_{ab} = (\partial_a)_p\cdot(\partial_b)_p = (e_{ab} + e_{ba})/2 \in S_p^2(D)$. Then every element of $S_p^2(D)$ has the form $\sum \xi^{ab}e_{ab}$, where $\xi^{ab} \in \boldsymbol{C}$ with $\xi^{ab} = \xi^{ba}$, and the set $\{f_{ab};\, a \leqq b\}$ is a basis of $S_p^2(D)$. Put $R_{ac}^{bd} = -\sum_{s,t} g^{\bar{s}b}g^{\bar{t}d}R_{a\bar{s}c\bar{t}}$, and given $X = \sum \xi^{ac}e_{ac} \in S_p^2(D)$ set $Q_p(X) = \sum_{b,d}(\sum_{a,c} R_{ac}^{bd}(p)\xi^{ac})e_{bd}$. Then, $Q_p(X) \in S_p^2(D)$ by (1.1). The endomorphism Q_p of the space $S_p^2(D)$ is called the *curvature operator* of the Bergman metric g. The space $S_p^2(D)$ is endowed with a Hermitian inner product $(\ ,\ )_p$ inherited from g, given by

$$(\sum \xi^{ab}e_{ab}, \sum \eta^{cd}e_{cd})_p = \sum \xi^{ab}\bar{\eta}^{cd}g_{a\bar{c}}(p)g_{b\bar{d}}(p) \,. \tag{1.4}$$

Since Q_p is self-adjoint with respect to $(\ ,\ )_p$ by (1.3), every eigenvalue of Q_p is real (cf [6], [15]).

LEMMA 1.1. *Let ψ be a biholomorphic mapping from a bounded domain D onto another one D', and let $p \in D$. If ξ is an eigenvalue of Q_p, then it is also an eigenvalue of $Q_{\psi(p)}$.*

PROOF. The assertion follows from the fact that ψ is an isometry with respect to the Bergman metrics of D and D'.

LEMMA 1.2. *The matrix representing Q_p with respect to the basis $(f_{11}/\sqrt{2}, f_{22}/\sqrt{2}, \cdots, f_{nn}/\sqrt{2}, f_{12}, f_{23}, \cdots, f_{1n})$ is given by*

$$\begin{bmatrix} (R_{aa}^{cc}(p))_a^c & (\sqrt{2}R_{ab}^{cc}(p))_{a<b}^c \\ (\sqrt{2}R_{aa}^{cd}(p))_a^{c<d} & (2R_{ab}^{cd}(p))_{a<b}^{c<d} \end{bmatrix}.$$

PROOF. Let $a \leqq b$. *It follows from the definitions that*

$$\begin{aligned} Q_p(f_{ab}) &= (1/2)Q_p(e_{ab} + e_{ba}) = (1/2)\sum_{c,d}(R_{ab}^{cd}(p) + R_{ba}^{cd}(p))e_{cd} \\ &= \sum_{c,d} R_{ab}^{cd}(p)e_{cd} = \sum_c R_{ab}^{cc}(p)f_{cc} + 2\sum_{c<d} R_{ab}^{cd}(p)f_{cd} \,, \end{aligned}$$

which yields the desired assertion.

We denote by SC the *scalar curvature* of the Bergman metric, i.e., $\mathrm{SC} = -2\sum g^{\bar{b}a}g^{\bar{d}c}R_{a\bar{b}c\bar{d}}$. Then by Lemma 1.2 we have the following:

LEMMA 1.3. *For $p \in D$, $SC(p) = 2$ trace Q_p.*

For $p \in D$, set

$$(1.5) \qquad \begin{cases} \lambda_D(p) = \text{the minimum of the eigenvalues of } Q_p \\ \mu_D(p) = \text{the maximum of the eigenvalues of } Q_p . \end{cases}$$

Then the functions λ_D and μ_D are biholomorphic invariants by Lemma 1.1. Furthermore, the following holds.

PROPOSITION 1.4. *Let D_i be a bounded domain in C^{n_i} $(i = 1, 2)$, and let $(p, q) \in D_1 \times D_2$.*

(i) $\lambda_{D_1 \times D_2}(p, q) = \min\{\lambda_{D_1}(p), \lambda_{D_2}(q)\},$ $\mu_{D_1 \times D_2}(p, q) = \max\{\mu_{D_1}(p), \mu_{D_2}(q)\}$.

(ii) *The curvature operator of the Bergman metric on $D_1 \times D_2$ possesses 0 as an eigenvalue with multiplicity at least $n_1 n_2$.*

PROOF. The assertions follow from the fact $g^{D_1\times D_2} = p_1^* g^{D_1} + p_2^* g^{D_2}$, where g^D means the Bergman metric on a bounded domain D and $p_i : D_1 \times D_2 \to D_i$ $(i = 1, 2)$ are the natural projections (cf [16; Theorem 3.2]).

We denote by $\mathrm{HSC}(p; X)$ the *holomorphic sectional curvature* of the Bergman metric g in the direction $X \in T_p(D) - \{0\}$, i.e.,

$$(1.6) \qquad \mathrm{HSC}(p; X) = -g(R(X, \bar{X})\bar{X}, X)/g(X, \bar{X})^2$$

(When D is one-dimensional the function HSC on D is called the *Gaussian curvature* of g).

PROPOSITION 1.5. $\mathrm{HSC}(p; \cdot) \geqq \lambda_D(p)$ *on* $T_p(D) - \{0\}$ *for every* $p \in D$, *and* $\min \mathrm{HSC}(p; \cdot) = \lambda_D(p)$ *if and only if*

(1.7) *there exists $X \in T_p(D)$ such that $X^2 \in S_p^2(D)$ is an eigenvector of Q_p subordinate to the eigenvalue $\lambda_D(p)$.*

PROOF. Let $X = \sum \xi^a(\partial_a)_p \in T_p(D) - \{0\}$. By the definitions as well as (1.4) we have

$$\begin{aligned} \mathrm{HSC}(p; X)g(X, \bar{X})^2 &= -\sum R_{\bar{a}b\bar{c}\bar{d}}(p)\xi^a\bar{\xi}^b\xi^c\bar{\xi}^d = \sum R^{st}_{ac}(p)\xi^a\bar{\xi}^b\xi^c\bar{\xi}^d g_{s\bar{b}}(p)g_{t\bar{d}}(p) \\ &= (Q_p(X^2), X^2)_p \geqq \lambda_D(p)(X^2, X^2)_p = \lambda_D(p)g(X, \bar{X})^2 . \end{aligned}$$

In the above inequality, equality holds if and only if $Q_p(X^2) = \lambda_D(p)X^2$. The proof is complete.

The Bergman metric g^U on the unit disk $U = \{z \in C; |z| < 1\}$ in C is called the *Poincaré metric* and given by $g^U = 4dz\cdot d\bar{z}/(1 - |z|^2)^2$, since the Bergman function of U is $1/\pi(1 - |z|^2)^2$. It follows from (1.2) and (1.6) that the Gaussian curvature of g^U is identically -1.

2. Normal j-algebras. Suppose that a bounded domain D is homogeneous, i.e., the group $\mathrm{Aut}(D)$ of all biholomorphic transformations of

D acts on D transitively. Aut(D) is a Lie group. A Lie group is called *triangular* if its Lie algebra $\mathfrak{h}$ is triangular in the sense that it is solvable and every eigenvalue of $\operatorname{ad} x$ is real for every $x \in \mathfrak{h}$. Let G be a maximal triangular analytic Lie subgroup of Aut(D), and let $\mathfrak{g}$ be its Lie algebra. It is well known (Vinberg [25]) that G acts on D simply transitively, i.e., for every $p \in D$, the mapping $\Phi: G \ni f \mapsto f(p) \in D$ becomes a diffeomorphism, and that

(2.1) every maximal triangular analytic Lie subgroup of Aut(D) is conjugate to G.

Fix a point $p \in D$. Then we get two $\boldsymbol{R}$-linear isomorphisms $\rho: \mathfrak{g} \ni x \mapsto x_e \in T_e(G)$ and $\Phi_*: T_e(G) \to T_p^{\boldsymbol{R}}(D)$, where $T_e(G)$ is the tangent space at the identity mapping $e \in G$, and get the unique endomorphism $j \in \operatorname{End}(\mathfrak{g})$ of $\mathfrak{g}$ such that $(\Phi_* \circ \rho) \circ j = J \circ (\Phi_* \circ \rho)$, where $J \in \operatorname{End}(T_p^{\boldsymbol{R}}(D))$ is the complex structure tensor of D at p. Set $\langle x, y \rangle = g(\Phi_* \circ \rho(x), \Phi_* \circ \rho(p))$ for $x, y \in \mathfrak{g}$, where g is the Bergman metric on D (see §1). Then $\langle\ ,\ \rangle$ is a j-invariant inner product on $\mathfrak{g}$. Let $\kappa \in \mathfrak{g}^*$ be the form (called the *Koszul form*) on $\mathfrak{g}$ given by

$$\kappa(x) = (1/2) \operatorname{trace} (\operatorname{ad} jx - j \circ \operatorname{ad} x) , \qquad x \in \mathfrak{g} . \tag{2.2}$$

Then it is known (Koszul [21]) that $\langle x, y \rangle = \kappa([jx, y])$ for $x, y \in \mathfrak{g}$.

The Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ with complex structure j obtained in the above manner possesses the following three properties:

(j1) $\mathfrak{g}$ is a finite dimensional triangular Lie algebra.

(j2) $[jx, jy] = j[jx, y] + j[x, jy] + [x, y]$ for $x, y \in \mathfrak{g}$.

(j3) There exists a form $\omega \in \mathfrak{g}^*$ such that $\omega([jx, jy]) = \omega([x, y])$ ($x, y \in \mathfrak{g}$) and $\omega([jx, x]) > 0$ ($x \in \mathfrak{g} - \{0\}$), i.e., such that $\langle x, y \rangle_\omega = \omega([jx, y])$ is a j-invariant inner product on $\mathfrak{g}$.

DEFINITION 2.1 (Pyatetskii-Shapiro [22; p. 51]). A Lie algebra $\mathfrak{g}$ over $\boldsymbol{R}$ with complex structure j is called a *normal j-algebra* if $(\mathfrak{g}, j)$ satisfies the above three conditions (j1)-(j3), and a form ω in (j3) is called *admissible*. We say that a normal j-algebra $(\mathfrak{g}, j)$ is *isomorphic* to another one $(\mathfrak{g}', j')$ if there exists a Lie algebra isomorphism ψ from $\mathfrak{g}$ onto $\mathfrak{g}'$ such that $\psi \circ j = j' \circ \psi$.

By (2.1) there corresponds, up to isomorphism, a unique normal j-algebra to every homogeneous bounded domain. In fact, this correspondence gives the bijection from biholomorphic equivalence classes of homogeneneous bounded domains in $\boldsymbol{C}^n$ onto isomorphic equivalence classes of normal j-algebras of dimension $2n$ (cf. [22; pp. 66-73]).

Now, let $(\mathfrak{g}, j)$ be a normal j-algebra.

DEFINITION 2.2 (Takeuchi [24; pp. I-37, I-38]). An element $x \in \mathfrak{g} - \{0\}$ is called an *idempotent* if $[jx, x] = x$. An idempotent x is called *primitive* if the 1-eigenspace $\{y \in \mathfrak{g}; [jx, y] = y\}$ of $\operatorname{ad} jx$ coincides with $\boldsymbol{R}x$.

DEFINITION 2.3. Let PI be the set of all primitive idempotents. It is known that PI is a finite set (cf. (n1) below). Put

$$\mathfrak{a} = \sum_{x \in \mathrm{PI}} \boldsymbol{R}jx , \quad \mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] ,$$

$$\mathfrak{n}(\alpha) = \{x \in \mathfrak{n}; [h, x] = \alpha(h)x \ (h \in \mathfrak{a})\} \quad \text{for} \quad \alpha \in \mathfrak{a}^* .$$

Then, $[\mathfrak{n}(\alpha), \mathfrak{n}(\beta)] \subset \mathfrak{n}(\alpha + \beta)$ for $\alpha, \beta \in \mathfrak{a}^*$. The cardinality of PI is called the *rank* of $(\mathfrak{g}, j)$, or the *rank* of the corresponding homogeneous bounded domain. Every element of $\Delta = \{\alpha \in \mathfrak{a}^*; \mathfrak{n}(\alpha) \neq \{0\}\}$ is called a *root*.

The following properties (n1)-(n3) are fundamental (cf. [22; Theorem 2, p. 61] together with [24; p. I-38]):

(n1) The set PI is non-empty and linearly independent. Furthermore, $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ is a direct sum of $\mathfrak{a}$ and $\mathfrak{n}$ (as vector space). In particular, $\mathfrak{n} = \sum_{\alpha \in \Delta} \mathfrak{n}(\alpha)$ (direct sum).

(n2) Let $\Delta_0 \subset \Delta$ be the dual basis of $\mathfrak{a}^*$ of the basis $j\mathrm{PI} = \{jx; x \in \mathrm{PI}\}$ of $\mathfrak{a}$. Then the following hold:

$$\Delta \subset \{(\alpha + \beta)/2; \alpha, \beta \in \Delta_0\} \cup \{(\alpha - \beta)/2; \alpha, \beta \in \Delta_0, \alpha \neq \beta\} \cup \{\alpha/2; \alpha \in \Delta_0\} ,$$

$$j\mathfrak{n}((\alpha + \beta)/2) = \mathfrak{n}((\alpha - \beta)/2) + \mathfrak{n}((\beta - \alpha)/2) \text{ for } \alpha, \beta \in \Delta_0 \text{ with } \alpha \neq \beta, \text{ and}$$

$$j\mathfrak{n}(\alpha) \subset \mathfrak{a} , \quad j\mathfrak{n}(\alpha/2) = \mathfrak{n}(\alpha/2) \quad \text{for} \quad \alpha \in \Delta_0 .$$

In particular, $\dim \mathfrak{n}(\alpha) = 1$ for every $\alpha \in \Delta_0$.

(n3) For $\alpha, \beta, \gamma \in \Delta_0$, the following hold:

$$\alpha \neq \beta \quad \text{implies} \quad (\alpha - \beta)/2 \notin \Delta \quad \text{or} \quad (\beta - \alpha)/2 \notin \Delta , \quad \text{and}$$

$$(\alpha - \beta)/2 , \quad (\beta - \gamma)/2 \in \Delta \quad \text{imply} \quad (\alpha - \gamma)/2 \in \Delta .$$

Let R be the rank of $(\mathfrak{g}, j)$, which is positive by (n1), and let $\mathrm{PI} = \{r_1, \cdots, r_R\}$ with $\Delta_0 = \{\varepsilon_1, \cdots, \varepsilon_R\}$, i.e., $\varepsilon_a(jr_b) = \delta_{ab}$. By (n3) the relation $(\alpha - \beta)/2 \in \Delta \cup \{0\}$ for $\alpha, \beta \in \Delta_0$ defines an order on Δ_0. So, renumbering $r_1, \cdots, r_R$, if necessary, we may assume that

$$(\varepsilon_a - \varepsilon_b)/2 \in \Delta \quad \text{implies} \quad a < b . \tag{2.3}$$

DEFINITION 2.4 Put $\mathscr{L} = \sum_{\alpha, \beta \in \Delta_0} \mathfrak{n}((\alpha + \beta)/2)$, $\mathscr{U} = \sum_{\alpha \in \Delta_0} \mathfrak{n}(\alpha/2)$. Furthermore, fix a numbering $r_1, \cdots, r_R$ of PI so that (2.3) holds, and set $\mathfrak{n}_{ab} = \mathfrak{n}((\varepsilon_a + \varepsilon_b)/2)$, $\mathfrak{n}_{a*} = \mathfrak{n}(\varepsilon_a/2)$ $(a, b \in \{1, \cdots, R\})$. Thus, $\mathscr{L} = \sum_{1 \leq a \leq b \leq R} \mathfrak{n}_{ab}$, $\mathscr{U} = \sum_{a=1}^{R} \mathfrak{n}_{a*}$, $j\mathscr{L} = \mathfrak{a} + \sum_{1 \leq a < b \leq R} j\mathfrak{n}_{ab}$, $\mathfrak{g} = \mathscr{L} + j\mathscr{L} + \mathscr{U}$ (direct sums), and $j\mathscr{U} = \mathscr{U}$.

Set $r = \sum_{x \in \mathrm{PI}} x = \sum_{a=1}^{R} r_a \in \mathscr{L}$ and denote by $\mathfrak{g}(\xi)$ the ξ-eigenspace of $\operatorname{ad} jr$ in $\mathfrak{g}$. Then, $[\mathfrak{g}(\xi), \mathfrak{g}(\xi')] \subset \mathfrak{g}(\xi + \xi')$. The following is easily shown by (n2).

LEMMA 2.5. *$\mathscr{L} = \mathfrak{g}(1)$, $\mathscr{U} = \mathfrak{g}(1/2)$, and $j\mathscr{L} = \mathfrak{g}(0)$. In particular, $[\mathscr{L}, \mathscr{L} + \mathscr{U}] = \{0\}$, $[\mathscr{L}, j\mathscr{L}] \subset \mathscr{L}$, $[\mathscr{U}, \mathscr{U}] \subset \mathscr{L}$, $[\mathscr{U}, j\mathscr{L}] \subset \mathscr{U}$, and $[j\mathscr{L}, j\mathscr{L}] \subset j\mathscr{L}$.*

From Lemma 2.5 and (j2) it follows that

$$(2.4) \qquad [x, ju] = j[x, u] , \quad [ju, jv] = [u, v] \quad \text{for} \quad x \in j\mathscr{L}, u, v \in \mathscr{U} .$$

LEMMA 2.6 ([22; Lemma 2, p. 60], [24; item (vi), P. I-33]). *For $b \in \{1, \cdots, R\}$ and $x \in \mathfrak{n}(\alpha)$ with $\alpha \in \Delta$,*

$$[r_b, jx] = \begin{cases} -x , & \alpha = (\varepsilon_a + \varepsilon_b)/2 \ \text{ for some } a \text{ with } \ a \leqq b \\ 0 , & \text{otherwise.} \end{cases}$$

PROOF. Let $\alpha = (\varepsilon_a + \varepsilon_b)/2$ for some a with $a \leqq b$. It follows from (j2) that $[jr_b, jx] = j[jr_b, x] + j[r_b, jx] + [r_b, x]$. Since $[r_b, x] = 0$, $[jr_b, x] = (\delta_{ab} + 1)x/2$, and $[jr_b, jx] = (\delta_{ab} - 1)jx/2$, we have $[r_b, jx] = -x$. When $\alpha \in \Delta - \{(\varepsilon_a + \varepsilon_b)/2; a = 1, \cdots, b\}$ we have $jx \in \mathfrak{n}(\beta)$ for some $\beta \in \Delta - \{(\varepsilon_a - \varepsilon_b)/2; a = 1, \cdots, b-1\}$. Since $\varepsilon_b + \beta \notin \Delta$ by (n2), we have $[r_b, jx] = 0$; therefore, the proof is complete.

We shall show the following.

PROPOSITION 2.7. *A form $\psi \in \mathfrak{g}^*$ is admissible if and only if*

$$(2.5) \qquad \psi|_{j\mathscr{L} \cap \mathfrak{n} + j(j\mathscr{L} \cap \mathfrak{n}) + \mathscr{U}} = 0, \quad \text{and} \quad \psi(x) > 0 \quad \text{for} \quad x \in \mathrm{PI} .$$

PROOF. We first note $j\mathscr{L} \cap \mathfrak{n} = \sum_{a<b} j\mathfrak{n}_{ab}$ and $j(j\mathscr{L} \cap \mathfrak{n}) = \sum_{a<b} \mathfrak{n}_{ab}$. Let ψ be an admissible form, and let $a \in \{1, \cdots, R\}$. For every $x \in \sum_{b=a+1}^{R} (\mathfrak{n}_{ab} + j\mathfrak{n}_{ab}) + \mathfrak{n}_{a*}$, we have $[jr_a, x] = x/2$. Then by (j3) and Lemma 2.6, $\psi(x) = 2\psi([jr_a, x]) = -2\psi([r_a, jx]) = -2\psi(0) = 0$. Furthermore, $\psi(r_a) = \psi([jr_a, r_a]) > 0$ by (j3). Thus, (2.5) holds. The "if" part of the assertion is contained in the following lemma.

LEMMA 2.8. *If $\psi \in \mathfrak{g}^*$ satisfies (2.5) and $\langle\ ,\ \rangle_\psi$ is defined by $\langle x, y\rangle_\psi = \psi([jx, y])$, then*

(i) *$\psi([jx, jy] - [x, y]) = 0$ for $x, y \in \mathfrak{g}$; therefore $\langle\ ,\ \rangle_\psi$ is a symmetric j-invariant bilinear form on $\mathfrak{g}$;*

(ii) *the decomposition $\mathfrak{g} = \sum_{a\leqq b} \mathfrak{n}_{ab} + \sum_{a\leqq b} j\mathfrak{n}_{ab} + \sum_a \mathfrak{n}_{a*}$ is orthogonal with respect to $\langle\ ,\ \rangle_\psi$; and*

(iii) *$\psi([jx, x]) > 0$ for $x \in \mathfrak{g} - \{0\}$.*

PROOF. (i) Let $x, y \in \mathscr{L}$ and $u, v \in \mathscr{U}$. Since $[x, y] = [x, u] = 0$, from (j2) and (2.4) we get

$$[jx, jy] - [x, y] = [jx, jy] \in j\mathscr{L} \cap \mathfrak{n} ,$$
$$[jx, jjy] - [x, jy] = j([jx, jy] + [x, jjy]) \in j(j\mathscr{L} \cap \mathfrak{n}) ,$$

$$[jx, ju] - [x, u] = [jx, ju] \in \mathscr{U}\,, \quad \text{and}$$
$$[ju, jv] - [u, v] = 0\,;$$

therefore, the desired assertion follows from (2.5).

(ii) Since $[\mathscr{L}, \mathscr{L} + \mathscr{U}] = \{0\}$, for $x, y \in \mathscr{L}$ and $u \in \mathscr{U}$ we see $\langle jy, x\rangle_\psi = \psi([-y, x]) = \psi(0) = 0$, $\langle u, x\rangle_\psi = \psi([ju, x]) = \psi(0) = 0$, and $\langle jx, u\rangle_\psi = \psi([-x, u]) = \psi(0) = 0$; therefore, the subspaces $\mathscr{L}$, $j\mathscr{L}$, and $\mathscr{U}$ are mutually orthogonal. Furthermore, for $x \in \mathfrak{n}_{ab}$ and $y \in \mathfrak{n}_{cd}$ with $a \leqq b$, $c \leqq d$, and $(a, b) \neq (c, d)$, we have $\langle jx, jy\rangle_\psi = \langle x, y\rangle_\psi = \psi([jx, y]) \in \psi(\delta_{ab}(\delta_{ac} + \delta_{ad})\mathfrak{n}_{cd} + (1 - \delta_{ab})\delta_{bc}\mathfrak{n}_{ad} + (1 - \delta_{ab})\delta_{bd}\mathfrak{n}_{ac}) = \{0\}$; therefore, $j\mathfrak{n}_{ab}$ $(a \leqq b)$ (resp. $\mathfrak{n}_{ab}$ $(a \leqq b)$) are mutually orthogonal. On the other hand, for $u \in \mathfrak{n}_{a*}$ and $v \in \mathfrak{n}_{b*}$ with $a \neq b$ we see $\langle u, v\rangle_\psi = \psi([ju, v]) \in \psi(\mathfrak{n}_{ab}) = \{0\}$; therefore, $\mathfrak{n}_{a*}$ are mutually orthogonal. Thus, (ii) follows.

(iii) Let $x \in \mathfrak{g} - \{0\}$. To prove the assertion, by (i) and (ii) we may assume $x \in \mathfrak{n}_{ab}$ $(a \leqq b)$ or $x \in \mathfrak{n}_{a*}$. We can then find $\xi \in \boldsymbol{R}$ such that $[jx, x] = \xi r_a$. Taking an admissible form ω, we get $\xi\omega(r_a) = \omega([jx, x]) > 0$ by (j3). Since $\omega(r_a) > 0$, we see $\xi > 0$. So, $\psi([jx, x]) = \xi\psi(r_a) > 0$, as desired. The proof is now complete.

COROLLARY 2.9 ([22; Lemma 2, p. 60], [24; Corollary 1, p. I-41]). *For every admissible form ω, the decomposition $\mathfrak{g} = \sum_{a\leqq b}\mathfrak{n}_{ab} + \sum_{a\leqq b} j\mathfrak{n}_{ab} + \sum_a \mathfrak{n}_{a*}$ is orthogonal with respect to the inner product $\langle\ ,\ \rangle_\omega$. In particular, the subspaces $\mathscr{L}$, $j\mathscr{L}$, and $\mathscr{U}$ are mutually orthogonal with respect to $\langle\ ,\ \rangle_\omega$.*

From this corollary we also get the following.

COROLLARY 2.10 (D'Atri [7; p. 407]). *Let ω and ω' be admissible forms. Then $\langle\ ,\ \rangle_\omega = \langle\ ,\ \rangle_{\omega'}$ if and only if $\omega|_{j\mathfrak{a}} = \omega'|_{j\mathfrak{a}}$.*

PROOF. For $x, y \in \mathfrak{n}(\alpha)$ with $\alpha \in \Delta$, we have $\langle x, y\rangle_\omega = \omega([jx, y])$ with $[jx, y] \in j\mathfrak{a}$, while $\langle jr_a, jr_a\rangle_\omega = \omega(r_a)$ with $r_a \in j\mathfrak{a}$. So, the desired assertion follows from Corollary 2.9.

Now, given an admissible form ω, set

$$\omega_a = \omega(r_a) \quad (a = 1, \cdots, R)\,. \tag{2.6}$$

Corollary 2.10 says that R positive parameters $\omega_1, \cdots, \omega_R$ determine $\langle\ ,\ \rangle_\omega$ uniquely. With respect to the Koszul form κ of (2.2), by Lemma 2.6 it is shown ([24; p. II-37]) that

$$\kappa_a = 1 + (1/2)\sum_{b\neq a} n_{ab} + (1/4)n_{a*} \tag{2.7}$$

for any $a = 1, \cdots, R$, where

$$n_{ab} = \dim \mathfrak{n}_{ab}\,, \quad n_{a*} = \dim \mathfrak{n}_{a*}\,. \tag{2.8}$$

3. Levi-Civita connection of an invariant Kähler metric. Let D be a homogeneous bounded domain, $(\mathfrak{g}, j)$ be the corresponding normal j-algebra, and G be the analytic Lie subgroup of $\mathrm{Aut}(D)$ whose Lie algebra is $\mathfrak{g}$ (see § 2). Then for every admissible form ω, there exists a unique G-invariant Hermitian metric ${}^{\omega}g$ on D such that

$$\langle x, y\rangle_{\omega} = {}^{\omega}g(\Phi_{*}\circ\rho(x), \Phi_{*}\circ\rho(y)) \quad \text{for} \quad x, y \in \mathfrak{g}\ , \tag{3.1}$$

where Φ and ρ are mappings given in § 2. The metric ${}^{\omega}g$ is Kählerian, because the closedness of the Kähler form ${}^{\omega}\psi(X, Y) = {}^{\omega}g(X, JY)$ $(X, Y \in X(D))$ is directly shown by (3.1). Of course, the Kähler metric ${}^{\kappa}g$ with respect of the Koszul form κ coincides with the Bergman metric on D.

We denote by ${}^{\omega}\nabla_{x}y$ $(x, y \in g)$ the bilinear mapping on $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$ induced from the Levi-Civita connection ∇ of the Kähler metric ${}^{\omega}g$ via $\Phi_{*}\circ\rho$, i.e., $(\Phi_{*}\circ\rho)({}^{\omega}\nabla_{x}y) = \nabla_{(\Phi_{*}\circ\rho)(x)}\Phi_{*}(y)$ for $x, y \in \mathfrak{g}$. It is given by

$$ {}^{\omega}\nabla_{x}y = ([x, y] - (\mathrm{ad}\, x)^{\omega}y - (\mathrm{ad}\, y)^{\omega}x)/2\ , \tag{3.2}$$

where $(\mathrm{ad}\, x)^{\omega}$ is the adjoint operator of $\mathrm{ad}\, x$ with respect to $\langle\ ,\ \rangle_{\omega}$. The formula (3.2) is equivalent to the following two properties:

$$\langle {}^{\omega}\nabla_{x}y, z\rangle_{\omega} = -\langle y, {}^{\omega}\nabla_{x}z\rangle_{\omega} \tag{3.3}$$

$$ {}^{\omega}\nabla_{x}y - {}^{\omega}\nabla_{y}x = [x, y]\ . \tag{3.4}$$

The Kählerian property of ${}^{\omega}g$ is equivalent to

$$ {}^{\omega}\nabla_{x}jy = j\,{}^{\omega}\nabla_{x}y\ . \tag{3.5}$$

LEMMA 3.1. *For $x \in \mathscr{L}$, the following hold*:
(i) ${}^{\omega}\nabla_{jx} = (\mathrm{ad}\, jx - (\mathrm{ad}\, jx)^{\omega})/2$ *on* $\mathscr{L} + \mathscr{U}$;
(ii) $(\mathrm{ad}\, jx)^{\omega}\mathscr{L} \subset \mathscr{L}$, $(\mathrm{ad}\, jx)^{\omega}\mathscr{U} \subset \mathscr{U}$;
(iii) ${}^{\omega}\nabla_{jx}\mathscr{L} \subset \mathscr{L}$, ${}^{\omega}\nabla_{jx}\mathscr{U} \subset \mathscr{U}$.

PROOF. Let $y \in \mathscr{L} + \mathscr{U}$, and $z \in \mathfrak{g}$. Then we have $\langle(\mathrm{ad}\, y)^{\omega}jx, z\rangle_{\omega} = \langle jx, [y, z]\rangle_{\omega} = 0$ (Lemma 2.5, Corollary 2.9); therefore, $(\mathrm{ad}\, y)^{\omega}jx = 0$. So, (i) follows from (3.2). Furthermore, $\langle(\mathrm{ad}\, jx)^{\omega}y, z\rangle_{\omega} = \langle y, [jx, z]\rangle_{\omega} = 0$ when $y \in \mathscr{L}$ and $z \in j\mathscr{L} + \mathscr{U}$, or when $y \in \mathscr{U}$ and $z \in j\mathscr{L} + \mathscr{L}$. Thus, (ii) holds. Lastly, (iii) follows from (ii) together with Lemma 2.5. The proof is complete.

COROLLARY 3.2 (D'Atri [7], [8]). ${}^{\omega}\nabla_{h} = 0$ *for* $h \in \mathfrak{a}$.

PROOF. To prove the assertion, it is sufficient by Lemma 3.1 (i) and (3.5) to show $\mathrm{ad}\, h = (\mathrm{ad}\, h)^{\omega}$ on $\mathscr{L} + \mathscr{U}$ for every $h \in \mathfrak{a}$. But, for $y \in \mathfrak{n}(\alpha)$ and $z \in \mathfrak{g}$ we have

$$\langle(\operatorname{ad} h)^{\omega} y, z\rangle_{\omega} = \begin{cases} \delta_{\alpha\beta}\beta(h)\langle y, z\rangle_{\omega}, & z \in \mathfrak{n}(\beta) \\ 0, & z \in \mathfrak{a} \end{cases}$$

and

$$\langle(\operatorname{ad} h) y, z\rangle_{\omega} = \begin{cases} \delta_{\alpha\beta}\alpha(h)\langle y, z\rangle_{\omega}, & z \in \mathfrak{n}(\beta) \\ 0, & z \in \mathfrak{a}, \end{cases}$$

as desired.

LEMMA 3.3. *For $x, y \in \mathscr{L}$ and $u \in \mathscr{U}$, the following hold:*
(i) ${}^{\omega}\nabla_x = j\circ(\operatorname{ad} jx + (\operatorname{ad} jx)^{\omega})/2$ *on* $\mathscr{L} + \mathscr{U}$;
(ii) ${}^{\omega}\nabla_x \mathscr{L} \subset j\mathscr{L}$, ${}^{\omega}\nabla_x \mathscr{U} \subset \mathscr{U}$, ${}^{\omega}\nabla_u \mathscr{L} \subset \mathscr{U}$;
(iii) ${}^{\omega}\nabla_x y = {}^{\omega}\nabla_y x$, ${}^{\omega}\nabla_u x = {}^{\omega}\nabla_x u$;
(iv) ${}^{\omega}\nabla_{ju} x = j\,{}^{\omega}\nabla_u x$.

PROOF. Let $z \in \mathscr{L} + \mathscr{U}$. Since $[x, z] = 0$ (Lemma 2.5), we get ${}^{\omega}\nabla_x z = {}^{\omega}\nabla_z x$ by (3.4). So, (iii) holds. From (3.5), (3.4), and Lemma 3.1 (i), we have ${}^{\omega}\nabla_x z = {}^{\omega}\nabla_z x = -j\,{}^{\omega}\nabla_z jx = -j({}^{\omega}\nabla_{jx} z + [z, jx]) = j(\operatorname{ad} jx + (\operatorname{ad} jx)^{\omega})z/2$; therefore, (i) holds. Lastly, (ii) follows from Lemma 3.1 (ii), and (iv) follows from (iii) and (3.5). The proof is complete.

COROLLARY 3.4. *For $x, y \in \mathfrak{n}_{ab}$ with $a \leqq b$, the following hold:*

$$(\operatorname{ad} jx)^{\omega} y = \langle x, y\rangle_{\omega} r_b/\omega_b, \quad (\operatorname{ad} jx) y = \langle x, y\rangle_{\omega} r_a/\omega_a;$$
$${}^{\omega}\nabla_{jx} y = (r_a/2\omega_a - r_b/2\omega_b)\langle x, y\rangle_{\omega}, \quad {}^{\omega}\nabla_x y = (jr_a/2\omega_a + jr_b/\omega_b)\langle x, y\rangle_{\omega}$$

(*see* (2.6)).

PROOF. The first two formulas are easily shown by definition. The last two formulas follow from Lemma 3.1 (i) and Lemma 3.3 (i).

LEMMA 3.5. *For $u, v \in \mathscr{U}$, the following hold:*
(i) ${}^{\omega}\nabla_u v = x_1 + jx_2$, where $x_1 = [u, v]/2$, $x_2 = [ju, v]/2 \in \mathscr{L}$;
(ii) ${}^{\omega}\nabla_{ju} v = -j\,{}^{\omega}\nabla_u v$.

PROOF. We first note $(\operatorname{ad} u)^{\omega} v \in j\mathscr{L}$. Indeed, for $y \in \mathscr{L} + \mathscr{U}$ we have $\langle(\operatorname{ad} u)^{\omega} v, y\rangle_{\omega} = \langle v, [u, y]\rangle_{\omega} = 0$ (Lemma 2.5, Corollary 2.9), as desired. It follows from (3.2) and (3.5) that

$$\begin{cases} {}^{\omega}\nabla_u v = [u, v]/2 - ((\operatorname{ad} u)^{\omega} v + (\operatorname{ad} v)^{\omega} u)/2 \\ {}^{\omega}\nabla_u v = -j\,{}^{\omega}\nabla_u jv = -j[u, jv]/2 + j((\operatorname{ad} u)^{\omega} jv + (\operatorname{ad} jv)^{\omega} u)/2 \end{cases}$$

Comparing the $\mathscr{L}$- and $j\mathscr{L}$-components of ${}^{\omega}\nabla_u v$, we have $(\operatorname{ad} u)^{\omega} v + (\operatorname{ad} v)^{\omega} u = j[u, jv] = -j[ju, v]$ (by (2.4)). So, (i) follows. The assertion (ii) follows from (i) and (2.4). The proof is complete.

By Lemmas 3.1, 3.3, and 3.5 we obtain the following (see also (3.5)).

PROPOSITION 3.6. *For every pair of x and y in $\mathscr{L} \cup j\mathscr{L} \cup \mathscr{U}$, the subspace, where the element ${}^{\omega}\nabla_x y$ belongs, is given by the following table*:

$x \backslash y$	$\mathscr{L}$	$j\mathscr{L}$	$\mathscr{U}$
$\mathscr{L}$	$j\mathscr{L}$	$\mathscr{L}$	$\mathscr{U}$
$j\mathscr{L}$	$\mathscr{L}$	$j\mathscr{L}$	$\mathscr{U}$
$\mathscr{U}$	$\mathscr{U}$	$\mathscr{U}$	$\mathscr{L} + j\mathscr{L}$

For example, if $x \in \mathscr{L}$ and if $y \in \mathscr{L}$, then ${}^{\omega}\nabla_x y \in j\mathscr{L}$.

4. Riemannian curvature tensor of an invariant Kähler metric (1). Given an admissible form ω, denote by ${}^{\omega}R$ the bilinear mapping on $\mathfrak{g} \times \mathfrak{g}$ into End($\mathfrak{g}$) induced from the Riemannian curvature tensor of the Kähler metric ${}^{\omega}g$ via $\Phi_* \circ \rho$ (see §§ 2-3), i.e., for $x, y \in \mathfrak{g}$,

$$(4.1) \qquad {}^{\omega}R(x, y) = [{}^{\omega}\nabla_x, {}^{\omega}\nabla_y] - {}^{\omega}\nabla_{[x,y]} .$$

It follows from (3.5) that

$$(4.2) \qquad {}^{\omega}R(jx, jy) = {}^{\omega}R(x, y) , \quad j \circ {}^{\omega}R(x, y) = {}^{\omega}R(x, y) \circ j .$$

We now extend $\langle \, , \rangle_\omega$ to a unique complex symmetric bilinear form on the complexification $\mathfrak{g}^C = \mathfrak{g} \otimes C$ of $\mathfrak{g}$, and extend ${}^{\omega}\nabla$ to the complex bilinear mapping on $\mathfrak{g}^C \times \mathfrak{g}^C$ into $\mathfrak{g}^C$. Thus, (4.1) is valid also for $x, y \in \mathfrak{g}^C$ as an endomorphism of $\mathfrak{g}^C$ (cf. [19]). Observe that the space $\mathfrak{g}^{1,0} = \{z \in \mathfrak{g}^C; jz = iz\}$, where j is extended to a complex linear endomorphism of $\mathfrak{g}^C$, corresponds to the holomorphic tangent space $T_p(D)$ via $\Phi_* \circ \rho$ (see § 2), and denote by χ the mapping from $\mathfrak{g}$ onto $\mathfrak{g}^{1,0}$, given by

$$(4.3) \qquad \chi(x) = (x - ijx)/2 , \qquad x \in \mathfrak{g} .$$

Given $x_a \in \mathfrak{g}$ ($a = 1, 2, 3, 4$), set

$${}^{\omega}R_{x_1\bar{x}_2x_3\bar{x}_4} = \langle {}^{\omega}R(\chi(x_3), \overline{\chi(x_4)})\overline{\chi(x_2)}, \chi(x_1)\rangle_\omega .$$

Since $\chi(jx) = i\chi(x)$ for $x \in \mathfrak{g}$, we have

$${}^{\omega}R_{jx_1\bar{x}_2x_3\bar{x}_4} = i\,{}^{\omega}R_{x_1\bar{x}_2x_3\bar{x}_4} , \quad {}^{\omega}R_{x_1\overline{jx}_2x_3\bar{x}_4} = -i\,{}^{\omega}R_{x_1\bar{x}_2x_3\bar{x}_4} , \quad \text{etc.}$$

Using (4.2) we get

$$(4.4) \qquad \begin{aligned} 4\,{}^{\omega}R_{x_1\bar{x}_2x_3\bar{x}_4} &= {}^{\omega}S(x_1, x_2, x_3, x_4) + i\,{}^{\omega}S(x_1, x_2, x_3, jx_4) \\ &= {}^{\omega}S(x_1, x_2, x_3, x_4) + i\,{}^{\omega}S(x_1, jx_2, x_3, x_4) , \end{aligned}$$

where

$$ {}^{\omega}S(x_1, x_2, x_3, x_4) = \langle {}^{\omega}R(x_3, x_4)x_2, x_1\rangle_{\omega} - \langle {}^{\omega}R(x_3, jx_4)jx_2, x_1\rangle_{\omega} . $$

By (3.3) and (4.1) we have

$$ (4.5) \quad \langle {}^{\omega}R(x_3, x_4)x_2, x_1\rangle_{\omega} = \langle {}^{\omega}\nabla_{x_3}x_2, {}^{\omega}\nabla_{x_4}x_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{x_4}x_2, {}^{\omega}\nabla_{x_3}x_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{[x_3,x_4]}x_2, x_1\rangle_{\omega} . $$

LEMMA 4.1. *For* $y_a \in \mathscr{L}$ and $u_b \in \mathscr{U}$, *the following hold*:

(i) ${}^{\omega}R_{y_1\bar{u}_2y_3\bar{u}_4} = 0$;

(ii) ${}^{\omega}R_{y_1\bar{u}_2y_3\bar{y}_4} = 0$;

(iii) ${}^{\omega}R_{u_1\bar{u}_2y_3\bar{u}_4} = 0$.

PROOF. Since $[\mathscr{L}, \mathscr{U}] = \{0\}$ (Lemma 2.5), it follows from (4.5) that

$$ \langle {}^{\omega}R(y_3, u_4)u_2, y_1\rangle_{\omega} = \langle {}^{\omega}\nabla_{y_3}u_2, {}^{\omega}\nabla_{u_4}y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{u_4}u_2, {}^{\omega}\nabla_{y_3}y_1\rangle_{\omega} , $$

$$ \langle {}^{\omega}R(y_3, ju_4)ju_2, y_1\rangle_{\omega} = \langle {}^{\omega}\nabla_{y_3}ju_2, {}^{\omega}\nabla_{ju_4}y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{ju_4}ju_2, {}^{\omega}\nabla_{y_3}y_1\rangle_{\omega} . $$

By (3.5), Lemma 3.3 (iv), and Lemma 3.5 (ii), the right hand sides of both formulas above coincide; therefore, ${}^{\omega}S(y_1, u_2, y_3, u_4) = 0$, which also yields ${}^{\omega}S(y_1, u_2, y_3, ju_4) = 0$. Thus, (i) holds. Next, by (4.5) we have

$$ \langle {}^{\omega}R(y_3, y_4)u_2, y_1\rangle_{\omega} = \langle {}^{\omega}\nabla_{y_3}u_2, {}^{\omega}\nabla_{y_4}y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{y_4}u_2, {}^{\omega}\nabla_{y_3}y_1\rangle_{\omega} $$

and

$$ \langle {}^{\omega}R(y_3, jy_4)ju_2, y_1\rangle_{\omega} = \langle {}^{\omega}\nabla_{y_3}ju_2, {}^{\omega}\nabla_{jy_4}y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{jy_4}ju_2, {}^{\omega}\nabla_{y_3}y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{[y_3,jy_4]}ju_2, y_1\rangle_{\omega} . $$

Since $[y_3, jy_4] \in \mathscr{L}$ (Lemma 2.5), every term on the right hand sides of the above two formulas vanishes (Corollary 2.9, Proposition 3.6). So, ${}^{\omega}S(y_1, u_2, y_3, y_4) = {}^{\omega}S(y_1, ju_2, y_3, y_4) = 0$, which imply (ii). Similarly, we have $\langle {}^{\omega}R(y_3, u_4)u_2, u_1\rangle_{\omega} = 0$ and ${}^{\omega}S(u_1, u_2, y_3, u_4) = {}^{\omega}S(u_1, ju_2, y_3, u_4) = 0$, which yield (iii). The proof is complete.

LEMMA 4.2. *For* $y_a \in \mathscr{L}$, *the following two formulas hold*:

(i) $4{}^{\omega}R_{y_1\bar{y}_2y_3\bar{y}_4} = \langle j{}^{\omega}\nabla_{y_3}y_2, [y_1, jy_4]\rangle_{\omega} + \langle j{}^{\omega}\nabla_{y_1}y_2, [y_3, jy_4]\rangle_{\omega} + \langle j{}^{\omega}\nabla_{y_3}y_1, (\operatorname{ad} jy_4){}^{\omega}y_2\rangle_{\omega}$;

(ii) $4{}^{\omega}R_{y_1\bar{y}_2y_3\bar{y}_4} = \langle {}^{\omega}\nabla_{jy_3}y_2, [jy_4, y_1]\rangle_{\omega} + \langle {}^{\omega}\nabla_{jy_3}y_1, (\operatorname{ad} jy_4){}^{\omega}y_2\rangle_{\omega} - \langle y_1, ({}^{\omega}\nabla_{[jy_3,jy_4]} + j{}^{\omega}\nabla_{[jy_3,y_4]})y_2\rangle_{\omega}$.

PROOF. Proposition 3.6 together with Corollary 2.9 implies

$$ {}^{\omega}S(y_1, y_2, y_3, y_4) = \langle {}^{\omega}\nabla_{y_3}y_2, ({}^{\omega}\nabla_{y_4} + j{}^{\omega}\nabla_{jy_4})y_1\rangle_{\omega} - \langle {}^{\omega}\nabla_{y_3}y_1, ({}^{\omega}\nabla_{y_4} - j{}^{\omega}\nabla_{jy_4})y_2\rangle_{\omega} + \langle y_1, j{}^{\omega}\nabla_{[y_3,jy_4]}y_2\rangle_{\omega} $$

and ${}^{\omega}S(y_1, jy_2, y_3, y_4) = 0$. But, from Lemmas 3.1 and 3.3, we get ${}^{\omega}\nabla_{jy_4} -$

$j^{\omega}\nabla_{y_4} = \mathrm{ad}\, jy_4$, ${}^{\omega}\nabla_{jy_4} + j^{\omega}\nabla_{y_4} = -(\mathrm{ad}\, jy_4)^{\omega}$ on $\mathscr{L}$. We thus obtain (i). Similarly, we have

$$\begin{aligned}{}^{\omega}S(jy_1, jy_2, jy_3, jy_4) = \langle {}^{\omega}\nabla_{jy_3}y_2, ({}^{\omega}\nabla_{jy_4} - j^{\omega}\nabla_{y_4})y_1\rangle_{\omega} \\ - \langle {}^{\omega}\nabla_{jy_3}y_1({}^{\omega}\nabla_{jy_4} + j^{\omega}\nabla_{y_4})y_2\rangle_{\omega} - \langle y_1, ({}^{\omega}\nabla_{[jy_3, jy_4]} \\ + j^{\omega}\nabla_{[jy_3, y_4]})y_2\rangle_{\omega}\end{aligned}$$

and ${}^{\omega}S(jy_1, jjy_2, jy_3, jy_4) = 0$. Since ${}^{\omega}R_{y_1\bar{y}_2y_3\bar{y}_4} = {}^{\omega}R_{jy_1\overline{jy_2}jy_3\overline{jy_4}}$, the same argument as above shows (ii). The proof is now complete.

COROLLARY 4.3. *For $y_a \in \mathscr{L}$ with $jy_3 \in \mathfrak{a}$, it holds that* $4^{\omega}R_{y_1\bar{y}_2y_3\bar{y}_4} = \langle [j[y_3, jy_4], y_1], y_2\rangle_{\omega}/2 - \langle [j[y_3, jy_4], y_2], y_1\rangle_{\omega}/2 + \langle [j[jy_3, y_4], y_1], y_2\rangle_{\omega}$.

PROOF. The assertion follows from Lemma 4.2 (ii) and Corollary 3.2, since $[jy_3, jy_4] = j[jy_3, y_4] + j[y_3, jy_4]$, and since ${}^{\omega}\nabla_{j[jy_3, y_4]} + j^{\omega}\nabla_{[jy_3, y_4]} = -(\mathrm{ad}\, j[jy_3, y_4])^{\omega}$ on $\mathscr{L}$.

LEMMA 4.4. *For $u_a \in \mathscr{U}$, it holds that*

$$\begin{aligned}4^{\omega}R_{u_1\bar{u}_2u_3\bar{u}_4} = 2(\langle {}^{\omega}\nabla_{u_3}u_2, {}^{\omega}\nabla_{u_4}u_1\rangle_{\omega} + \langle {}^{\omega}\nabla_{u_3}u_4, {}^{\omega}\nabla_{u_2}u_1\rangle_{\omega}) \\ + 2i(\langle j^{\omega}\nabla_{u_3}u_2, {}^{\omega}\nabla_{u_4}u_1\rangle_{\omega} + \langle j^{\omega}\nabla_{u_3}u_4, {}^{\omega}\nabla_{u_2}u_1\rangle_{\omega})\ .\end{aligned}$$

PROOF. By (3.5) and Lemma 3.5 (ii), we have

$${}^{\omega}S(u_1, u_2, u_3, u_4) = 2\langle {}^{\omega}\nabla_{u_3}u_2, \omega\nabla_{u_4}u_1\rangle_{\omega} - \langle ({}^{\omega}\nabla_{[u_3, u_4]}u_2) - j^{\omega}\nabla_{[u_3, ju_4]}u_2), u_1\rangle_{\omega}\ .$$

But, since $[\mathscr{U}, \mathscr{U}] \subset \mathscr{L}$, by Lemma 3.5 (i) we have

$${}^{\omega}\nabla_{[u_3, u_4]}u_2 - j^{\omega}\nabla_{[u_3, ju_4]}u_2 = {}^{\omega}\nabla_{u_2}([u_3, u_4] - j[u_3, ju_4]) = 2^{\omega}\nabla_{u_2}{}^{\omega}\nabla_{u_3}u_4\ .$$

So, (3.3) implies

$${}^{\omega}S(u_1, u_2, u_3, u_4) = 2(\langle {}^{w}\nabla_{u_3}u_2, {}^{\omega}\nabla_{u_4}u_1\rangle_{\omega} + \langle {}^{\omega}\nabla_{u_3}u_4, {}^{\omega}\nabla_{u_2}u_1\rangle_{\omega})\ ,$$

which yields the desired assertion.

COROLLARY 4.5. *If $u_b \in \mathfrak{n}_{a*}$ for every b, then* $4^{\omega}R_{u_1\bar{u}_2u_3\bar{u}_4} = (\langle u_3, u_2\rangle_{\omega}\langle u_1, u_4\rangle_{\omega} + \langle u_3, u_4\rangle_{\omega}\langle u_1, u_2\rangle_{\omega})/2\omega_a$ (see (2.6)).

PROOF. Since $[u_1, u_2] = \langle u_1, ju_2\rangle_{\omega}r_a/\omega_a$, the desired assertion follows from Lemmas 3.5 and 4.4.

5. Riemannian Curvature tensor of an invariant Kähler metric (2). In this section we examine when ${}^{\omega}R_{x_1\bar{x}_2x_3\bar{x}_4}$ vanish.

LEMMA 5.1. *Let $a, b \in \{1, \cdots, R\}$ with $a \leqq b$. For $y \in \mathfrak{n}_{cd}$ with $s \leqq d$, and for $y' \in \mathfrak{n}_{st}$ with $s \leqq t$, if ${}^{\omega}R_{r_a\bar{y}r_b\bar{y}'} \neq 0$, then $(c, d) = (s, t) = (a, b)$. Furthermore, if $y, y' \in \mathfrak{n}_{ab}$, then*

$$4^{\omega}R_{r_a\bar{y}r_b\bar{y}'} = (1 + \delta_{ab})\langle y, y'\rangle_{\omega}/2\ .$$

PROOF. Since $[r_b, jy'] = -\delta_{bt}y'$ (Lemma 2.6), and since $[jr_b, y'] = (\delta_{bs} + \delta_{bt})y'/^2$, Corollary 4.3 implies that

$$4^{\omega}R_{r_a\bar{y}r_b\bar{y}'} = \delta_{bs}\delta_{at}\langle y, y'\rangle_{\omega}/2 + \delta_{bt}\langle [jy', y], r_a\rangle_{\omega}/2 .$$

From Jacobi's identity we have

$$\begin{aligned}\langle r_a, [jy', y]\rangle_{\omega} &= \langle y', [jr_a, y]\rangle_{\omega} + \langle jy, [jr_a, jy']\rangle_{\omega} \\ &= (\delta_{ac} + \delta_{ad} + \delta_{as} - \delta_{at})\langle y, y'\rangle_{\omega}/2 .\end{aligned}$$

Thus,

$$4^{\omega}R_{r_a\bar{y}r_b\bar{y}'} = (2\delta_{bs}\delta_{at} + \delta_{bt}(\delta_{ac} + \delta_{ad} + \delta_{as} - \delta_{at}))\langle y, y'\rangle_{\omega}/4 .$$

So, ${}^{\omega}R_{r_a\bar{y}r_b\bar{y}'} \neq 0$ implies $\langle y, y'\rangle \neq 0$; therefore, $(c, d) = (s, t)$ and $4^{\omega}R_{r_a\bar{y}r_b\bar{y}'} = (\delta_{bc}\delta_{ad} + \delta_{bd}\delta_{ac})\langle y, y'\rangle_{\omega}/2$. Since $\delta_{bc}\delta_{ad} + \delta_{bd}\delta_{ac} \neq 0$ implies $(c, d) = (a, b)$, all the desired assertions follow.

LEMMA 5.2. *Let $a, b \in \{1, \cdots, R\}$ with $a < b$, and let $m, m' \in \mathfrak{n}_{ab}$. For $y \in \mathfrak{n}_{cd}$ with $c \leqq d$, and for $y' \in \mathfrak{n}_{st}$ with $s \leqq t$,* if ${}^{\omega}R_{m\bar{y}m'\bar{y}'} \neq 0$, *then one of the following three cases occurs:*

(i)$_1$ $(c, d) = (a, a)$ *and* $(s, t) = (b, b)$,

(i)$_2$ $(c, d) = (b, b)$ *and* $(s, t) = (a, a)$,

(ii) $(c, d) = (s, t)$ *with* $d = b, c < b$.

Furthermore, if $y, y' \in \mathfrak{n}_{ab}$, then

$$\begin{aligned}4^{\omega}R_{m\bar{y}m'\bar{y}'} &= (\langle m', y\rangle_{\omega}\langle m, y'\rangle_{\omega} + \langle m', y'\rangle_{\omega}\langle m, y\rangle_{\omega})/2\omega_a \\ &\quad - \langle m, m'\rangle_{\omega}\langle y, y'\rangle_{\omega}/2\omega_b\end{aligned}$$

(*see* (2.6)).

To prove Lemma 5.2, we need the following well-known fact (cf. [22; p. 63]).

LEMMA 5.3. *If $x, x' \in j\mathfrak{n}_{ab}$ with $a < b$, and if $y, y' \in \sum_{c=b+1}^{R}(\mathfrak{n}_{bc} + j\mathfrak{n}_{bc}) + \mathfrak{n}_{b*}$, then*

$$\langle [x, y], [x', y']\rangle_{\omega} + \langle [x, y'], [x', y]\rangle_{\omega} = \langle x, x'\rangle_{\omega}\langle y, y'\rangle_{\omega}/\omega_b .$$

PROOF OF LEMMA 5.2. Let $m, m' \in \mathfrak{n}_{ab}$, $y \in \mathfrak{n}_{cd}$, and $y' \in \mathfrak{n}_{st}$. By Lemma 4.2 (i) we have

$$\begin{aligned}(5.1) \qquad 4^{\omega}R_{m\bar{y}m'\bar{y}'} &= \langle j^{\omega}\nabla_{m'}y, [m, jy']\rangle_{\omega} + \langle j^{\omega}\nabla_m y, [m', jy']\rangle_{\omega} \\ &\quad + \langle j^{\omega}\nabla_{m'}m, (\mathrm{ad}\, jy')^{\omega}y\rangle_{\omega} .\end{aligned}$$

Since ${}^{\omega}\nabla_{m'}m = (jr_a/2\omega_a + jr_b/2\omega_b)\langle m, m'\rangle_{\omega}$ (Corollary 3.4), by Lemma 2.6 we have

$$(5.2) \qquad \langle j^{\omega}\nabla_{m'}m, (\mathrm{ad}\, jy')^{\omega}y\rangle_{\omega} = -(\delta_{at}/2\omega_a + \delta_{bt}/2\omega_b)\langle m, m'\rangle_{\omega}\langle y, y'\rangle_{\omega} .$$

On the other hand, it follows from Lemma 3.3 (i) that

$$(5.3)\quad \begin{aligned}\langle j^{\omega}\nabla_{m'}y, [m, jy']\rangle_{\omega} &= \langle j^{\omega}\nabla_{y}m', [m, jy']\rangle_{\omega}\\ &= \langle [m', jy], [m, jy']\rangle_{\omega}/2 - \langle m', [jy, [m, jy']]\rangle_{\omega}/2 .\end{aligned}$$

Substituting (5.2) and (5.3) into (5.1), we have

$$(5.4)\quad \begin{aligned}4{}^{\omega}R_{m\bar{y}m'\bar{y}'} &= \langle [m', jy], [m, jy']\rangle_{\omega}/2 - \langle m', [jy, [m, jy']]\rangle_{\omega}/2\\ &\quad + \langle [m, jy], [m', jy']\rangle_{\omega}/2 - \langle m, [jy, [m', jy']]\rangle_{\omega}/2\\ &\quad - (\delta_{at}/2\omega_a + \delta_{bt}/2\omega_b)\langle m, m'\rangle_{\omega}\langle y, y'\rangle_{\omega} .\end{aligned}$$

We divide the proof into four cases.

(a) Suppose $s = t$, and let $y' = r_s$. Then (5.4) becomes

$$(5.5)\quad \begin{aligned}4{}^{\omega}R_{m\bar{y}m'\bar{r}_s} &= -(\delta_{as} + \delta_{bs})(\langle [m', jy], m\rangle_{\omega} + \langle [m, jy], m'\rangle_{\omega})/2\\ &\quad - (\delta_{as}/2\omega_a + \delta_{bs}/2\omega_b)\langle m, m'\rangle_{\omega}\langle y, r_s\rangle_{\omega} .\end{aligned}$$

If $s \neq a$ and $s \neq b$, then ${}^{\omega}R_{m\bar{y}m'\bar{r}_s} = 0$. So, suppose $s = a$ or b. It then follows from (5.5) that

$$(5.6)\quad \begin{aligned}4{}^{\omega}R_{m\bar{y}m'\bar{r}_s} &= -\langle [m', jy], m\rangle_{\omega}/2 - \langle [m, jy], m'\rangle_{\omega}/2\\ &\quad - \langle m, m'\rangle_{\omega}\langle y, r_s\rangle_{\omega}/2\omega_s .\end{aligned}$$

Since $[m', jy]$ and $[m, jy]$ belong to $\mathfrak{n}((\varepsilon_a + \varepsilon_b + \varepsilon_c - \varepsilon_d)/2)$, if $c \neq d$ then ${}^{\omega}R_{m\bar{y}m'\bar{r}_s}$ vanishes. So, in addition, suppose $c = d$, and let $y = r_c$. Then, (5.6) gives

$$4{}^{\omega}R_{m\bar{r}_c m'\bar{r}_s} = (\delta_{ac} + \delta_{bc} - \delta_{cs})\langle m, m'\rangle_{\omega}/2 ;$$

therefore, ${}^{\omega}R_{m\bar{r}_c m'\bar{r}_s} \neq 0$ implies that $(s, c) = (a, b)$ or (b, a).

(b) Suppose $s < t, t \neq a$, and $t \neq b$. Then, $[m, jy'] = [m', jy'] = 0$ because $\mathfrak{n}((\varepsilon_a + \varepsilon_b + \varepsilon_s - \varepsilon_t)/2) = \{0\}$. So, (5.4) implies ${}^{\omega}R_{m\bar{y}m'\bar{y}'} = 0$.

(c) Suppose $s < t = a$. Since $[m, jy'], [m', jy'] \in \mathfrak{n}_{sb}$ $(s < b)$, the elements $[jy, [m, jy']]$ and $[jy, [m', jy']]$ belong to the subspace $\delta_{cd}\mathfrak{n}_{sb} + (1 - \delta_{cd})\delta_{ds}\mathfrak{n}_{cb}$ and are orthogonal to m' and m, respectively. So, by (5.4) we have

$$\begin{aligned}4{}^{\omega}R_{m\bar{y}m'\bar{y}'} &= \langle [m', jy], [m, jy']\rangle_{\omega}/2 + \langle [m, jy], [m', jy']\rangle_{\omega}/2\\ &\quad - \langle m, m'\rangle_{\omega}\langle y, y'\rangle_{\omega}/2\omega_a .\end{aligned}$$

When $(c, d) \neq (s, a)$, we have $\langle [m', jy], [m, jy']\rangle_{\omega} = \langle [m, jy], [m', jy']\rangle_{\omega} = 0$ and $\langle y, y'\rangle_{\omega} = 0$, which imply ${}^{\omega}R_{m\bar{y}m'\bar{y}'} = 0$; while when $(c, d) = (s, a)$ it follows from Lemma 5.4 that ${}^{\omega}R_{m\bar{y}m'\bar{y}'} = 0$.

(d) Suppose $s < t = b$. Since $[m, jy'], [m', jy'] \in \mathfrak{n}_{as}$, the elements $[jy, [m, jy']]$ and $[jy, [m', jy']]$ belong to $\delta_{cd}\mathfrak{n}_{as} + (1 - \delta_{cd})(\delta_{da}\mathfrak{n}_{cs} + \delta_{ds}\mathfrak{n}_{ca})$ and are orthogonal to m' and m, respectively. So, by (5.4) we have

(5.7) $$4\,{}^{\omega}R_{m\bar{y}m'\bar{y}'} = \langle [m', jy], [m, jy'] \rangle_{\omega}/2 + \langle [m, jy], [m', jy'] \rangle_{\omega}/2 - \langle m, m' \rangle_{\omega} \langle y, y' \rangle_{\omega}/2\omega_b \,.$$

Thus, if $(c, d) \neq (s, b)$ then ${}^{\omega}R_{m\bar{y}m'\bar{y}'} = 0$ as in case (c). This completes the proof of the first assertion. The second assertion follows from (5.7) together with Corollary 3.4. The proof is now complete.

Now, we recall the mapping $F: \mathscr{U} \times \mathscr{U} \to \mathscr{L}^C$ ([22; p. 68]) given by

(5.8) $$F(u, v) = [ju, v]/4 + i[u, v]/4 \quad \text{for} \quad u, v \in \mathscr{U}.$$

We get the following formula by Lemmas 3.5 and 4.4:

(5.9) $$4\,{}^{\omega}R_{u\bar{v}u'\bar{v}'} = 8(\langle F(u', v), F(u, v') \rangle_{\omega} + \langle F(u', v'), F(u, v) \rangle_{\omega})$$

for $u, u', v, v' \in \mathscr{U}$, where $\langle\ ,\ \rangle_{\omega}$ is extended to a complex bilinear form on $\mathfrak{g}^C$ (see §4).

LEMMA 5.4. *Suppose that* $u_a \in \mathfrak{n}(\alpha_a/2)$ $(\alpha_a \in \Delta_0)$ *satisfy* ${}^{\omega}R_{u_1\bar{u}_2u_3\bar{u}_4} \neq 0$. *Then one of the following cases occurs*:

(i) $\alpha_1 = \alpha_3 \neq \alpha_2 = \alpha_4$,

(ii)$_2$ $\alpha_1 = \alpha_2 \neq \alpha_3 = \alpha_4$,

(ii)$_2$ $\alpha_1 = \alpha_4 \neq \alpha_2 = \alpha_3$,

(iii) $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4$.

PROOF. Since $F(u, v) \in \mathfrak{n}^C_{ab}$ for every $(u, v) \in \mathfrak{n}_{a*} \times \mathfrak{n}_{b*}$, and since $\langle z, w \rangle_{\omega} = 0$ for every $(z, w) \in \mathfrak{n}^C_{ab} \times \mathfrak{n}^C_{cd}$ with $\{a, b\} \neq \{c, d\}$, it follows from (5.9) that if ${}^{\omega}R_{u_1\bar{u}_2u_3\bar{u}_4} \neq 0$ then $\alpha_1 + \alpha_4 = \alpha_2 + \alpha_3$ or $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4$. This yields that at least one of the following holds: $(\alpha_1, \alpha_4) = (\alpha_3, \alpha_2)$, $(\alpha_1, \alpha_3) = (\alpha_2, \alpha_4)$, $(\alpha_1, \alpha_2) = (\alpha_4, \alpha_3)$. So, the assertion follows.

6. Splitting of the curvature operator. In this section we shall study the endomorphism Q of the 2-symmetric tensor product $S^2(\mathfrak{g}^{1,0})$ of the space $\mathfrak{g}^{1,0}$ (see §4), induced from the curvature operator Q_p of the Bergman metric on D via $\Phi_* \circ \rho$ (see §2). We call Q the *curvature operator* of $(\mathfrak{g}, j)$. We thus deal only with the Koszul form κ as the admissible form of $(\mathfrak{g}, j)$, and use the simple notation $\langle\ ,\ \rangle$, ∇, etc., instead of $\langle\ ,\ \rangle_{\kappa}$, ${}^{\kappa}\nabla$, etc. Let $\chi: \mathfrak{g} \to \mathfrak{g}^{1,0}$ be the mapping given by (4.3). Then we have direct sum decompositions

(6.1) $$\mathfrak{g}^{1,0} = \chi(\mathscr{L}) + \chi(\mathscr{U}) \,;$$
$$\chi(\mathscr{L}) = \sum_{1 \leq a \leq b \leq R} \chi(\mathfrak{n}_{ab}) \,, \quad \chi(\mathscr{U}) = \sum_{a=1}^{R} \chi(\mathfrak{n}_{a*}) \,,$$

where R is the rank of $(\mathfrak{g}, j)$. Since $\langle \chi(x), \overline{\chi(y)} \rangle = \langle x, y \rangle/2$ for every $x, y \in \mathfrak{g}$, it follows from Corollary 2.9 that

(6.2) the decompositions (6.1) are orthogonal with respect to the Hermitian inner product $\langle \cdot \, , \bar{\cdot} \rangle$ on $\mathfrak{g}^{1,0}$.

The following is an immediate consequence of Lemma 1.2.

LEMMA 6.1. *Let $x_1, \cdots, x_n, jx_1, \cdots, jx_n$ be an orthogonal basis of $\mathfrak{g}$ with respect to $\langle \, , \rangle$. Then the matrix representing Q, with respect to the basis $(h_{11}/\sqrt{2}, h_{22}/\sqrt{2}, \cdots, h_{nn}/\sqrt{2}, h_{12}, h_{23}, \cdots, h_{1n})$, where $h_{ab} = \chi(x_a)\cdot\chi(x_b) \in S^2(\mathfrak{g}^{1,0})$, is given by*

$$\begin{bmatrix} (R_{aa}^{cc})_a^c & (\sqrt{2}\, R_{ab}^{cc})_{a<b}^c \\ (\sqrt{2}\, R_{aa}^{cd})_a^{c<d} & (2\, R_{ab}^{cd})_{a<b}^{c<d} \end{bmatrix}$$

where

$$\begin{aligned} R_{ac}^{bd} &= -R_{x_a \bar{x}_b x_c \bar{x}_d} / \langle \chi(x_b), \overline{\chi(x_b)} \rangle \langle \chi(x_d), \overline{\chi(x_d)} \rangle \\ &= -4R_{x_a \bar{x}_b x_c \bar{x}_d} / \langle x_b, x_b \rangle \langle x_d, x_d \rangle \, . \end{aligned}$$

Let E be a Q-invariant C-subspace of $S^2(\mathfrak{g}^{1,0})$. Then the orthogonal complement of E with respect to $\langle \, ,^- \rangle$ is also Q-invariant. If $E = E_1 + \cdots + E_N$ is a direct sum decomposition of E into Q-invariant subspaces, then we say that the curvature operator $Q|_E$ on E *splits into* $E_1, \cdots, E_N$.

By observing Lemma 6.1 and (6.2), we obtain the following from Lemma 4.1.

PROPOSITION 6.2. *The curvature operator of* $(\mathfrak{g}, j)$ *splits into the three subspaces* $\chi(\mathscr{L})^2$, $\chi(\mathscr{L})\cdot\chi(\mathscr{U})$, *and* $\chi(\mathscr{U})^2$.

Similarly, by Lemmas 5.1 and 5.2 we get the following Propositions 6.3 and 6.4.

PROPOSITION 6.3. *The curvature operator on the invariant subspace* $\chi(\mathscr{L})^2$ *splits into* $2R$ *subspaces*

$$(\mathscr{L}_a) = \chi(\mathfrak{n}_{aa})^2 \qquad (a = 1, \cdots, R) \, ,$$

$$(\mathscr{L}^b) = \sum_{a=1}^{b-1} (\chi(\mathfrak{n}_{aa})\cdot\chi(\mathfrak{n}_{bb}) + \chi(\mathfrak{n}_{ab})^2) \qquad (b = 2, \cdots, R) \, ,$$

$$(\mathscr{L}^*) = \sum \{\chi(\mathfrak{n}_{ab})\cdot\chi(\mathfrak{n}_{cd});\, (a, b) \neq (c, d), (a, c) \neq (b, d)\} \, .$$

PROPOSITION 6.4. *The curvature operator of* $(\mathfrak{g}, j)$ *has* $-1/\kappa_1, \cdots, -1/\kappa_R$ *as eigenvalues, with eigenvectors* $\chi(r_1)^2, \cdots, \chi(r_R)^2$, *respectively* (*see* (2.7)).

For every $a, b \in \{1, \cdots, R\}$ with $a < b$ and with $N = n_{ab} > 0$, let $m_1, \cdots, m_N$ be an orthogonal basis of $\mathfrak{n}_{ab}$ normalized by $[jm_c, m_c] = r_a$ ($c = 1, \cdots, N$). Thus, $\langle m_c, m_d \rangle = \kappa_a \delta_{cd}$. Put $f_{ab} = (\chi(r_a)\cdot\chi(r_b), \chi(m_1)^2/\sqrt{2}, \cdots, \chi(m_N)^2/\sqrt{2}, \chi(m_1)\cdot\chi(m_2), \chi(m_2)\cdot\chi(m_3), \cdots, \chi(m_1)\cdot\chi(m_N))$, a basis of $\chi(\mathfrak{n}_{aa})\cdot$

$\chi(\mathfrak{n}_{bb})+\chi(\mathfrak{n}_{ab})^2$, and consider $(1+N+N(N-1)/2)$-square matrix

$$M_{ab}=-\kappa_a^{-1}\begin{bmatrix}0 & (1/\sqrt{2})e_N & 0\\ (1/\sqrt{2})^t e_N & I_N-(\kappa_b^a/2)E_N & 0\\ 0 & 0 & I_{N(N-1)/2}\end{bmatrix},$$

where $e_N=(1,\cdots,1)$ (N-times), $E_N=(\xi_{st})$ with $\xi_{st}=1(s,t\in\{1,\cdots,N\})$, I_N is the identity matrix of order N, etc., and $\kappa_b^a=\kappa_a/\kappa_b$. When $n_{ab}=0$, let f_{ab} be a single element $(\chi(r_a)\cdot\chi(r_b))$, and M_{ab} be a $(1,1)$-matrix (0). Then Lemmas 5.1 and 5.2 together with Lemma 6.1 and (6.2) imply the following.

PROPOSITION 6.5. *For every $k\in\{2,\cdots,R\}$, the matrix L_k representing the curvature operator on the invariant subspace $(\mathscr{L}^k)$ in Proposition 6.3, with respect to the basis $(f_{1k},f_{2k},\cdots,f_{k-1,k})$, has the form*

$$L_k=\begin{bmatrix}M_{1k} & L_{2k}^{1k} & \cdots & L_{k-1,k}^{1k}\\ L_{1k}^{2k} & M_{2k} & \cdots & L_{k-1,k}^{2k}\\ \vdots & \vdots & \ddots & \vdots\\ L_{1k}^{k-1,k} & L_{2k}^{k-1,k} & \cdots & M_{k-1,k}\end{bmatrix},$$

where L_{tk}^{sk} is a $(1+n_{sk}(n_{sk}+1)/2,\ 1+n_{tk}(n_{tk}+1)/2)$-matrix whose components of the first row and the first column are all zero.

From Lemma 5.4 we conclude the following.

PROPOSITION 6.6. *The curvature operator on the invariant subspace $\chi(\mathscr{U})^2$ splits into $1+R(R-1)/2$ subspaces $(\mathscr{U}_*)=\sum_{a=1}^R\chi(\mathfrak{n}_{a*})^2$ and $(\mathscr{U}_{ab})=\chi(\mathfrak{n}_{a*})\cdot\chi(\mathfrak{n}_{b*})$ $(a<b)$ (some of which may be $\{0\}$).*

For every $k\in\{1,\cdots,R\}$, let $u_1^k,\cdots,u_h^k,ju_1^k,\cdots,ju_h^k(h=n_{k*})$ be an orthonormal system of $(\mathfrak{n}_{k*},\langle\ ,\ \rangle)$, and set $f_k=(\chi(u_1^k)^2/\sqrt{2},\cdots,\chi(u_h^k)^2/\sqrt{2},\chi(u_1^k)\cdot\chi(u_2^k),\cdots,\chi(u_1^k)\cdot\chi(u_h^k))$, a basis of $\chi(\mathfrak{n}_{k*})^2$. Furthermore, for every pair (k,l) with $k\neq l$, we consider an $(n_{k*}(n_{k*}+1)/2,\ n_{l*}(n_{l*}+1)/2)$-matrix

$$L_l^k=-4\begin{bmatrix}(R_{u_a\bar{v}_c u_a\bar{v}_c})_a^c & (\sqrt{2}R_{u_a\bar{v}_c u_b\bar{v}_c})_{a<b}^c\\ (\sqrt{2}R_{u_a\bar{v}_c u_a\bar{v}_d})_a^{c<d} & (2R_{u_a\bar{v}_c u_b\bar{v}_d})_{a<b}^{c<d}\end{bmatrix}, \tag{6.3}$$

where u_a and v_c mean u_a^l and u_c^k, respectively. Then Lemma 5.4 together with Lemma 6.1 and (6.2) implies the following (see also Corollary 4.5).

PROPOSITION 6.7. *The matrix L representing the curvature operator on the invariant subspace $(\mathscr{U}_*)$ in Proposition 6.6, with respect to the basis $(f_1,\cdots,f_R)$, is given by*

$$L = \begin{bmatrix} -\kappa_1^{-1} I_{H_1} & L_2^1 & \cdots & L_R^1 \\ L_1^2 & -\kappa_2^{-1} I_{H_2} & \cdots & L_R^2 \\ \vdots & \vdots & \ddots & \vdots \\ L_1^R & L_2^R & \cdots & -\kappa_R^{-1} I_{H_R} \end{bmatrix},$$

where $H_k = n_{k*}(n_{k*} + 1)/2$.

7. Characterization of symmetric bounded domains by the curvature operator. A homogeneous bounded domain in C^n is called *irreducible* if it is not biholomorphically equivalent to a product of any two homogeneous bounded domains of lower dimension. In the case of symmetric bounded domains the irreducibility in the above sense coincides with the irreducibility as a Riemannian manifold, with respect to the Bergman metric.

In this section we shall show the following main theorem of this paper.

THEOREM 7.1. *Suppose that the curvature operator of the Bergman metric on a homogeneous bounded domain D in C^n has at most two distinct eigenvalues. Then D is irreducible symmetric, or D is biholomorphic to a product of several balls of the same dimension.*

By combining Theorem 7.1 with the result of Calabi and Vesentini [6; Chap. 3, § 2] and Borel [3; Proposition 3.4], we get the following.

THEOREM 7.2. *An irreducible homogeneous bounded domain D in C^n is symmetric if and only if the curvature operator of the Bergman metric on D has at most two distinct eigenvalues.*

Now, let D be a homogeneous bounded domain of rank R, and let $(\mathfrak{g}, j)$ be the corresponding normal j-algebra with the Koszul admissible form κ. To prove Theorem 7.1 we employ the following Lemmas 7.3 and 7.4

LEMMA 7.3 (D'Atri and Miatello [10; Proposition 3]). *The domain D is irreducible, quasi-symmetric in the sense of Satake* [23] *if and only if $R = 1$, or $n_{ab} = n_{12} > 0$ for every a, b with $a < b$ and $n_{a*} = n_{1*}$ for every a (see* (2.8)).

Put $\mathrm{End}(\mathscr{U}, j) = \{f \in \mathrm{End}(\mathscr{U});\ j \circ f = f \circ j\}$, and define a linear mapping $\varphi\colon \mathscr{L} \to \mathrm{End}(\mathscr{U}, j)$ by

$$\langle \chi(\varphi(y)u), \overline{\chi(v)} \rangle /2 = \langle E(u, v), y \rangle \quad \text{for} \quad u, v \in \mathscr{U}, y \in \mathscr{L}$$

(see (4.3), (5.8)), or

$$\langle \varphi(y)u, v \rangle = \langle [ju, v], y \rangle \quad \text{for} \quad u, v \in \mathscr{U}, y \in \mathscr{L}. \tag{7.1}$$

Denote by $\varphi^c: \mathscr{L}^C \to \mathrm{End}(\mathscr{U}, j)$ the complex linear extension of φ, i.e., $\varphi^c(x+iy)u = \varphi(x)u + j\varphi(y)u$ for $x, y \in \mathscr{L}$ and $u \in \mathscr{U}$ (cf. [9; p. 41]).

LEMMA 7.4 (Dorfmeister [12; Satz 3.4, p. 95]). *When D is quasi-symmetric in the sense of Satake* [23], *D is symmetric if and only if $\varphi^c(F(v, u))v = 0$ for every pair $(u, v) \in \mathfrak{n}_{a*} \times \mathfrak{n}_{b*}$ with $a \neq b$.*

Now, suppose that

(7.2) the curvature operator of $(\mathfrak{g}, j)$ has at most two distinct eigenvalues (see §6).

If $R = 1$, then D is a ball ([22; p. 52]); therefore the conclusion of Theorem 7.1 trivially holds. So, suppose

$$R \geqq 2\,. \tag{7.3}$$

LEMMA 7.5. *Assumptions* (7.2) *and* (7.3) *imply $\kappa_1 = \cdots = \kappa_R$ and $n_{ab} = n_{12}$ for every pair (a, b) with $a < b$. Moreover, the curvature operator has precisely two distinct eigenvalues $-1/\kappa_1$ and $n_{12}/2\kappa_1$.*

PROOF. Let us denote by V the set all eigenvalues of Q. By Proposition 6.4, $\{-1/\kappa_1, \cdots, -1/k_R\} \subset V$; while by assumption (7.2) the cardinarity $\sharp V$ of V is 1 or 2. Put

$$\xi = \begin{cases} -1/\kappa_1\,, & \sharp V = 1 \\ \text{the value in } \ V - \{-1/\kappa_1\}\,, & \sharp V = 2\,. \end{cases}$$

Then the matrix M representing Q satisfies

$$M^2 - (-\kappa_1^{-1} + \xi)M - \xi\kappa_1^{-1} I = 0\,. \tag{7.4}$$

So every matrix L_b $(b = 2, \cdots, R)$ in Proposition 6.5 also satisfies the equation (7.4) with L_b instead of M; in view of the (1, 1)-component of the a-th diagonal block of this equation for every $a \in \{1, \cdots, b-1\}$, we have

$$n_{ab}/2\kappa_a^2 - \xi/\kappa_1 = 0\,; \tag{7.5}$$

therefore, $\xi \geqq 0$ by (7.3). This implies $\sharp V = 2$ and $\kappa_1 = \cdots = \kappa_R$. Once more by (7.5) we have $n_{ab} = 2\kappa_1\xi$ for every a, b with $a < b$. The proof is now complete.

PROPOSITION 7.6. *Assumptions* (7.2) *and* (7.3) *imply that D is irreducible and quasi-symmetric in the sense of Satake* [23], *or D is a product of R copies of an $(n_{1*}+1)$-dimensional ball.*

PROOF. From Lemma 7.5 together with (2.7) it follows that $n_{ab} = n_{12}$ for every (a, b) wite $a < b$, and that $n_{a*} = n_{1*}$ for every a. First, suppose $n_{12} = 0$. Then $\mathfrak{g}$ is a direct sum of R mutually isomorphic j-ideals $\mathfrak{n}_{aa} + j\mathfrak{n}_{aa} + \mathfrak{n}_{a*}$ $(a = 1, \cdots, R)$. Since the normal j-algebra $\mathfrak{n}_{11} + j\mathfrak{n}_{11} + \mathfrak{n}_{1*}$ corresponds to an $(n_{1*} + 1)$-dimensional ball B, we have $D = B \times \cdots \times B$ (R-times) (cf. [22; pp. 52, 64]). Next, suppose $n_{12} > 0$. Then Lemma 7.3 implies that D is irreducible and quasi-symmetric in the sense of Satake [23]. The proof is complete.

LEMMA 7.7. *When D is quasi-symmetric in the sense of Satake* [23], *the following four conditions are mutually equivalent* (*see* (5.8), (7.1)):

(s1) *D is symmetric.*

(s2) $\varphi^c(F(v, u))v = 0$ *for* $u \in \mathfrak{n}_{a*}$, $v \in \mathfrak{n}_{b*}$ *with* $a \neq b$.

(s3) $\langle F(v, u'), F(v, u)\rangle = 0$ *for* $u, u' \in \mathfrak{n}_{a*}$, $v \in \mathfrak{n}_{b*}$ *with* $a \neq b$.

(s4) $R_{v\bar{u}v\bar{u}'} = 0$ *for* $u, u' \in \mathfrak{n}_{a*}$, $v \in \mathfrak{n}_{b*}$ *with* $a \neq b$.

PROOF. Lemma 7.4 asserts the equivalence (s1)$\Leftrightarrow$(s2). By (7.1) we have

$$\langle \varphi^c(F(v, u))v, u'\rangle = \mathrm{Re}\langle F(v, u), F(v, u')\rangle ,$$
$$\langle \varphi^c(F(v, u))v, ju'\rangle = \mathrm{Im}\,\langle F(v, u), F(v, u')\rangle$$

for every $u, u', v \in \mathscr{U}$; therefore, the equivalence (s2) $\Leftrightarrow$ (s3) follows. Lastly, the equivalence (s3) $\Leftrightarrow$ (s4) follows from (5.9). So, the proof is complete.

We now show the following, which proves Theorem 7.1 completely by Lemma 7.7.

PROPOSITION 7.8. *Assumptions* (7.2) *and* (7.3) *imply the assertion* (s4) *in Lemm* 7.7.

PROOF. We consider the matrix L given in Proposition 6.7 with $\kappa_a = \kappa_1$, $H_a = H_1$ for every $a = 1, \cdots, R$. Since $-1/\kappa_1$ and $\xi = n_{12}/2\kappa_1$ are all eigenvalues of the curvature operator (Lemma 7.5), the matrix L satisfies the equation (7.4) with L instead of M; in view of the k-th diagonal block of this equation, we get

$$\Big(\sum_{a \neq k} L_a^k\, {}^t\overline{L_a^k} + \kappa_1^{-2} I_{H_1}\Big) + (-\kappa_1^{-1} + \xi)\kappa_1^{-1} I_{H_1} - \xi\kappa_1^{-1} I_{H_1} = 0\ ,$$

or $\sum_{a \neq k} L_a^k\, {}^t\overline{L_a^k} = 0$. This implies that $L_a^k = 0$ for every (k, a) with $a \neq k$. It follows from (6.3) that $R_{v\bar{u}v'\bar{u}'} = 0$ for $u, u' \in \mathfrak{n}_{a*}$, $v, v' \in \mathfrak{n}_{b*}$ *with* $a \neq b$, as desired.

8. Holomophic sectional curvature of the Bergman metric.

Let D be a homogeneous bounded domain. Then the scalar curvature SC of the Bergman metric on D is identically $-2n$ (cf. [16; Theorem 4.1]), and both λ_D and μ_D in (1.5) become constant functions by Lemma 1.1. Furthermore, the constant λ_D is negative (Lemma 1.3). Set

$$\gamma_D = -2/\lambda_D \quad (>0) \tag{8.1}$$

(cf. [6; p. 499], [3; p. 508]). We first note the following.

PROPOSITION 8.1. *Let D be an irreducible symmetric bounded domain of rank R in C^n with normal j-algebra $\mathfrak{g} = \sum_{a\leq b} \mathfrak{n}_{ab} + \sum_{a\leq b} j\mathfrak{n}_{ab} + \sum_a \mathfrak{n}_{a*}$. Then* $\dim \mathfrak{n}_{ab} = n_{12} > 0$ $(a < b)$ *and* $\dim \mathfrak{n}_{a*} = n_{1*} = 2m$ $(a = 1, \cdots, R)$ *provided that $R \geqq 2$, and the following hold:*

(i) *When $R = 1$ the eigenvalue of the curvature operator of the Bergman metric is $\lambda_D = -1/\kappa_1$; while when $R \geqq 2$ the eigenvalues are precisely $\lambda_D = -1/\kappa_1$ and $n_{12}/2\kappa_1$.*

(ii) *The condition* (1.7) *holds.*

(iii) *The invariant γ_D is an integer between 2 and $n+1$; $\gamma_D = 2$ if and only if D is a disk in C, and $\gamma_D = n+1$ if and only if D is a ball.*

(iv) *$R\gamma_D \leqq 2n$, and the equality holds if and only if $m = 0$, i.e., D is biholomorphic to a Siegel domain of the first kind* (cf. [22]).

PROOF. We first employ Theorem 7.2. Then Lemma 7.5 implies (i), and Proposition 6.4 implies (ii). It follows from the definition of γ_D and from (2.7) that

$$\gamma_D = \begin{cases} 2 + (R-1)n_{12} + m\ , & R \geqq 2 \\ 2 + m\ , & R = 1\ . \end{cases} \tag{8.2}$$

By observing the dimensions in the decomposition (6.1) we get

$$n = \begin{cases} R + R(R-1)n_{12}/2 + Rm\ , & R \geqq 2 \\ R + Rm\ , & R = 1\ . \end{cases} \tag{8.3}$$

From these formulas we obtain the assertions (iii) and (iv). The proof is complete.

It is well known (cf. [3], [6]) that

(8.4) for every triple (n, γ, R) of positive integers, there exists, up to biholomorphic equivalence, at most one irreducible symmetric bounded domain D such that

$$(\dim D, \gamma_D, \operatorname{rank} D) = (n, \gamma, R)\ .$$

PROPOSITION 8.2 *For every triple $(n, n_{12}, m) \in N\times N\times Z_+$, there exists, up to biholomorphic equivalence, at most one irreducible symmetric bounded domain D such that* $\operatorname{rank} D \geqq 2$ *and* $(\dim D, \dim \mathfrak{n}_{ab}, \dim \mathfrak{n}_{a*}) = (n, n_{12}, 2m)$.

PROOF. Let $R = \operatorname{rank} D \geqq 2$. It follows from (8.3) that

$$n_{12}R^2 + (2 - n_{12} + 2m)R - 2n = 0 .$$

Since there exists at most one integer $R \geqq 2$ satisfying the above quadratic equation, the desired assertion follows from (8.2) and (8.4).

PROPOSITION 8.3. *A symmetric bounded domain is irreducible if and only if* 0 *is not an eigenvalue of the curvature operator of the Bergman metric.*

PROOF. The "only if" part follows from Proposition 8.1 (i), and the "if" part from Proposition 1.4 (ii).

THEOREM 8.4. *The holomorphic sectional curvature* HSC *of the Bergman metric on a symmetric bounded domain D of rank R_D satisfies*

$$\begin{cases} \min \mathrm{HSC} = \lambda_D = -2/\gamma_D \\ \max \mathrm{HSC} = \lambda_D/R_D = -2/R_D\gamma_D \end{cases}$$

(*see* (8.1)).

PROOF. Proposition 8.1 (ii) together with Proposition 1.4 (i) implies that the condition (1.7) holds also for D not necessarily irreducible. Therefore, Proposition 1.5 yields $\min \mathrm{HSC} = \lambda_D$. But, it is well known (cf. [17; p. 41]) that HSC is negative and $\min \mathrm{HSC} = R_D \max \mathrm{HSC}$. So, the proof is complete.

COROLLARY 8.5. *Under the notation of Theorem* 8.4 *the following hold*:

(i) $\max \mathrm{HSC} \leqq -1/\dim D$, *and the equality holds if and only if* $D = D_1 \times \cdots \times D_l$, *where every D_i is an irreducible symmetric bounded domain biholomorphic to Siegel domain of the first kind, and* $\gamma_{D_i} = \gamma_{D_1}$ $(i = 1, \cdots, l)$.

(ii) $\min \mathrm{HSC} \geqq -1$, *and the equality holds if and only if D is a disk U in C, or D is a product of U and a symmetric bounded domain.*

(iii) $\min \mathrm{HSC} \geqq -2/(\dim D + 1)$, *and the equality holds if and only if D is a ball.*

PROOF. When D is irreducible, the assertions follow from Theorem 8.4 together with Proposition 8.1 (iii) and (iv). Let D_i $(i = 1, 2)$ be symmetric bounded domains. Then, $\gamma_{D_1 \times D_2} = \min\{\gamma_{D_1}, \gamma_{D_2}\}$ (Proposition 1.4 (i)) and $R_{D_1 \times D_2} = R_{D_1} + R_{D_2}$. From these we obtain the assertions also for D reducible.

Combining Theorem 8.4 with (8.4), we get the following.

COROLLARY 8.6. *Let D and D' be irreducible symmetric bounded domains of the same dimension. If* $\min \mathrm{HSC}^D = \min \mathrm{HSC}^{D'}$ *and* $\max \mathrm{HSC}^D =$

max $\mathrm{HSC}^{D'}$, *then* D *is biholomorphic to* D'. *Here,* HSC^D *means the holomorphic sectional curvature of the Bergman metric on* D, *etc.*

REMARK 8.7. Among the domains $D(r) = \{z \in \boldsymbol{C}^2; |z^1| < 1, |z^2|^2 < (1 - |z^1|^2)^r\}$ $(0 \leqq r < +\infty)$, the same characterization as in Corollary 8.6 holds ([1], [2]): If $r, r' \in [0, +\infty)$, $\inf \mathrm{HSC}^{D(r)} = \inf \mathrm{HSC}^{D(r')}$, and $\sup \mathrm{HSC}^{D(r)} = \sup \mathrm{HSC}^{D(r')}$, then $r = r'$.

9. Carathéodory and Kobayashi metrics. In this section a *Finsler metric* on a bounded domain D stands for a non-negative real valued function F on the holomorphic tangent bundle $T(D)$ of D satisfying

(f1) $F(\xi X) = |\xi| F(X)$,

(f2) $F(X) = 0$ implies $X = 0$

for every $X \in T(D)$ and $\xi \in \boldsymbol{C}$. We do not assume F to be continuous. Let B_D be the Finsler metric induced from the Bergman metric g on D, i.e., $B_D(X) = g(X, \bar{X})^{1/2}$ for $X \in T(D)$. For the unit disk $U = \{z \in \boldsymbol{C}; |z| < 1\}$, we have

$$B_U(\xi\partial/\partial z) = \sqrt{2}|\xi|/(1 - |z|^2) , \quad (z, \xi) \in U \times \boldsymbol{C}$$

(see §1). Let C_D be the Finsler metric of Carathéodory on D (or, simply, the *Carathéodory metric* on D), i.e.,

$$C_D(X) = \sup\{B_U(f_*X)/\sqrt{2};\ f \in \mathrm{Hol}(D, U)\}$$

fox $X \in T(D)$, where $\mathrm{Hol}(D_1, D_2)$ means the set of all holomorphic mappings from D_1 into D_2. Let K_D be the Finsler metric of Kobayashi on D (or, simply, the *Kobayashi metric* on D), i.e.,

$$K_D(X) = \inf\{B_U(Y)/\sqrt{2};\ Y \in T(U),\ f \in \mathrm{Hol}(U, D) \quad \text{with} \quad f_*Y = X\}$$

for $X \in T(D)$. These definitions of C_D and K_D coincide with the usual ones ([4], [5], [13], [14]); while in [18; §2], $\sqrt{2}C_D$ and $\sqrt{2}K_D$ are used as the definitions of C_D and K_D. From the Schwarz lemma to the effect that $f^*B_U \leqq B_U$ for every $f \in \mathrm{Hol}(U, U)$, it follows that $C_D \leqq K_D$. It is immediately seen from the definitions that for a Finsler metric F on D.

(9.1) $C_D \leqq F$ if and only if $f^*B_U \leqq \sqrt{2}F$ for every $f \in \mathrm{Hol(D}, U)$,

and

(9.2) $F \leqq K_D$ if and only if $\sqrt{2}f^*F \leqq B_U$ for every $f \in \mathrm{Hol}(U, D)$.

Now, the following is well known (Hahn [13], [14], Burbea [4], [5]):

$$C_D < B_D \quad \text{on} \quad T(D) - \{\text{the zero section}\} \tag{9.3}$$

for every bounded domain D. When D is homogeneous (resp. symmetric), we get a more precise result than (9.3), as in the following Theorem 9.1 (resp. Theorem 9.2).

THEOREM 9.1. *It holds that* $2C_D^2 \leqq B_D^2$ *for every homogeneous bounded domain* D. *Furthermore, this inequality is sharp, i.e., there exist a homogeneous bounded domain* D *and* $X \in T_p(D) - \{0\}$ *such that* $2C_D^2(X) = B_D^2(X)$.

THEOREM 9.2. *For a symmetric bounded domain* D, *it holds that* $K_D = C_D$ *and* $\gamma_D C_D^2 \leqq B_D^2 \leqq \gamma_D R_D C_D^2$, *where* γ_D *is the invariant in* (8.1) and R_D *is the rank of* D. *For every such domain each inequality is sharp, i.e., for every such* D, *there exist* $X_1, X_2 \in T_p(D) - \{0\}$ *such that* $\gamma_D C_D^2(X_1) = B_B^2(X_1)$ *and* $B_D^2(X_2) = \gamma_D R_D C_D^2(X_2)$.

To prove the above two theorems, we use a result in the previous section as well as the following two results.

LEMMA 9.3 (Yau [26; Theorem 2]). *Let* (M, g^M) *be a complete Kähler manifold whose Ricci curvature is bounded from below by a constant* $-\alpha$. *Let* (N, g^N) *be a Hermitian manifold whose holomorphic bisectional curvature is bounded from above by a negative constant* $-\beta$. *Suppose that there exists a non-constant holomorphic mapping from* M *into* N. *Then* $\alpha \geqq 0$ *and* $f^*g^N \leqq (\alpha/\beta)g^M$ *for every* $f \in \mathrm{Hol}(M, N)$.

LEMMA 9.4 (Kobayashi [17; Theorem 4.1, p. 42]). *Let* D *be a symmetric bounded domain with the Bergman metric* g^D *whose holomorphic sectional curvature is bounded from below by a negative constant* $-\alpha$. *Let* (N, g^N) *be a Hermitian manifold whose holomorphic sectional curvature is bounded from above by a negative constant* $-\beta$. *Then* $f^*g^N \leqq (\alpha/\beta)g^D$ *for every* $f \in \mathrm{Hol}(D, N)$.

PROOF OF THEOREM 9.1. (The first assertion.) Let D be a homogeneous bounded domain. Then the Ricci curvature of the Bergman metric on D is identically -1 ([16; Theorem 4.1]). Furthermore, the holomorphic bisectional curvature of the Bergman metric on the unit disk U coincides with its Gaussian curvature, and is identically -1, as was seen in §1. So, Lemma 9.3 implies that $f^*B_U \leqq B_D$ for every $f \in \mathrm{Hol}(D, U)$; therefore, by (9.1) we obtain the first assertion of Theorem 9.1.

PROOF OF THEOREM 9.2. Let D be a symmetric bounded domain. It is well known ([17; p. 52]) that $K_D = C_D$. The following is also known (Korányi [20]): There exists $\alpha > 0$ such that for every $X \in T_p(X)$ there corresponds surjectively a non-negative real vector $(\xi_1, \cdots, \xi_R) \in \boldsymbol{R}_+^R$ of dimension $R = R_D$ with the properties $B_D(X) = \alpha(\sum_a \xi_a^2)^{1/2}$ and $C_D(X) = \max\{\xi_1, \cdots, \xi_R\}$. From this it follows that

$$\alpha^2 C_D^2 \leqq B_D^2 \leqq R_D \alpha^2 C_D^2 \quad \text{on} \quad T(D) \tag{9.4}$$

and that each inequality is sharp. We shall show $\alpha^2 = \gamma_D$. By Theorem 8.4 the holomorphic sectional curvature HSC of the Bergman metric on D satisfies

$$\text{HSC} \geqq -2/\gamma_D\ , \tag{9.5}$$

$$\text{HSC} \leqq -2/R_D\gamma_D\ . \tag{9.6}$$

By (9.5), Lemma 9.4 implies that $\gamma_D f^*B_U^2 \leqq 2B_D^2$ for every $f \in \text{Hol}(D, U)$. So, by (9.1) we have $\gamma_D C_D^2 \leqq B_D^2$. From the sharpness of the first inequality of (9.4) it follows that $\gamma_D \leqq \alpha^2$. Similarly, Lemma 9.4 together with (9.2) and (9.6) shows $B_D^2 \leqq R_D\gamma_D K_D^2 = R_D\gamma_D C_D^2$. Combining this with the sharpness of the second inequality of (9.4), we have $\gamma_D \geqq \alpha^2$; therefore $\alpha^2 = \gamma_D$, as desired.

PROOF OF THEOREM 9.1. (The second assertion.) We shall show the sharpness of the inequality in Theorem 9.1. For this it is sufficient (Theorem 9.2) to find a symmetric bounded domain D so that $\gamma_D = 2$, or $\min \text{HSC} = -1$ (Theorem 8.4). The unit disk U, or the product of U and a symmetric bounded domain possesses the desired property (Corollary 8.5 (ii)). Thus, Theorem 9.1 is completely proved.

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