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LINEAR EVOLUTION EQUATIONS OF NON-PARABOLIC TYPE WITH VARIABLE DOMAINS

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CONTENTS.

1. Introduction. We consider linear evolution equations of "hyperbolic" type, that is, non-parabolic type, in a Banach space X

$$\begin{cases} du(t)/dt = A_0(t)u(t) & 0 < t \leqq T \\ u(0) = u_0 \in D(A_0(0)) , \end{cases} \tag{1.1}$$

where $A_0(t)$ is the generator of a semigroup on X and its domain $D(A_0(t))$ depends on t.

It is our main intention to give an abstract formulation of the mixed problem (including Neumann conditions) for hyperbolic partial differential equations. For this purpose we modify Kato's formulation [5] which is the following: the space X contains a dense subspace Y $(\subset D(A_0(t)))$ which is a Banach space with respect to the stronger norm, and each $A_0(t)$ generates a semigroup on Y. Instead of Y we define a family of closed subspaces $Y(t)$ $(\subset D(A_0(t)))$ of the space Y so that $A_0(t)$ generates a semigroup on $Y(t)$. Roughly speaking, our formulation reduces to his when $Y(t) = Y$.

The basic idea is similar to [8], but the assumptions, and hence the proofs, are essentially different: the result of [8] was incomplete in the sense that it does not seem applicable to partial differential equations.

In the present paper we give only a simple application to the mixed problem for wave equations with Neumann conditions. Further applications to hyperbolic partial differential equations will be discussed in subsequent articles.

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2. Notations and statement of theorems. Let X and Y be vector spaces. Let $(X, \|\cdot\|)$ be a separable Banach space. $(Y, |||\cdot|||)$ is a Banach space such that Y is densely and continuously embedded in $(X, \|\cdot\|)$ and the unit ball $\{y \in Y; |||y||| \leqq 1\}$ of $(Y, |||\cdot|||)$ is closed in $(X, \|\cdot\|)$. Here we denote by $\|\cdot\|$ and $|||\cdot|||$ the norms of X and Y, respectively. For simplicity, $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ will be abbreviated to X and Y respectively if no confusion arises. Let $\{A(t)\}$ be a family of closed operators from X to X such that $Y \subset D(A(t))$ for each $t \in [0, T]$. Let $\{Y(t)\}$ be a family of closed subspaces of Y such that the unit ball $\{y \in Y(t); |||y||| \leqq 1\}$ of $Y(t)$ is closed in X with respect to the norm $\|\cdot\|$. Let $A_0(t)$ be the minimal closed extension of $A(t)$ to $Y(t) \to X$.

$A(t)$ and $A_0(t)$ correspond to a differential operator and the differential operator with a boundary condition, respectively.

We assume the following conditions:

(A.1) For every $t \in [0, T]$, $A_0(t)$ generates a bounded C_0-semigroup on X and $Y(t)$.

(A.2) There exist a positive constant ω and a family of monotone decreasing norms $\{\|\cdot\|_t\}_{t \in [0,T]}$ on X equivalent to $\|\cdot\|$ such that each $\exp(s(A_0(t) - \omega))$ is a contraction semigroup on X with respect to the norm $\|\cdot\|_t$.

(A.3) There exist a positive constant ω and a family of monotone decreasing norms $\{|||\cdot|||_t\}_{t \in [0,T]}$ on Y equivalent to $|||\cdot|||$ such that each $\exp(s(A_0(t) - \omega))$ is a contraction semigroup on $Y(t)$ with respect to the norm $|||\cdot|||_t$.

(A.4) $A(\cdot)$ is strongly continuous from $[0, T]$ to $B(Y, X)$, where $B(Y, X)$ is the space of bounded linear operators from Y to X. There exists a T_2-topology τ of X such that $A(t)$ is continuous from $(X, \|\cdot\|)$ to (X, τ) for any $t \in [0, T]$.

Let $C(\varepsilon, t)$ and $h(\varepsilon, t)$ be positive functions on $(0, 1] \times [0, T]$ satisfying the following:

(2.1)
- (i) $h(\varepsilon, t)$ and $C(\varepsilon, t)h(\varepsilon, t)$ decreasingly tend to zero as $\varepsilon \to 0$.
- (ii) $C(\varepsilon, s) \leqq L(t)C(\varepsilon, t)$ for any $s \in [t, t + h(\varepsilon, t)]$ where $L(t)$ is a constant which depends only on t.
- (iii) There exist positive constants M, $\tilde{M}$ and δ such that $h(M, t) \geqq \delta$ and $C(M, t) \leqq \tilde{M} < +\infty$ for any $t \in [0, T]$.

(A.5) For any $\varepsilon > 0$, $t \in [0, T)$, $x \in Y(t)$ and $s \in (t, t + h(\varepsilon, t)]$ there exists $y \in Y(s)$ such that

$$\|x - y\|_s \leqq \varepsilon C_1 |s - t| \cdot |||x|||_t \tag{2.2}$$

$$|||x - y|||_s \leqq C_2 C(\varepsilon, t)|s - t| \cdot |||x|||_t \,. \tag{2.3}$$

Moreover, when $s = t + h(\varepsilon, t)$ and $x \in D(A_0(t))_{Y(t)\to Y(t)}$, y satisfies

$$|||y|||_s \leqq (1 + C_3 h(\varepsilon, t))|||[I - h(\varepsilon, t)A_0(t)]x|||_t \tag{2.4}$$

in addition to (2.2) and (2.3).

These constants C_i $(i = 1, 2, 3)$ do not depend on $\varepsilon, t, s, x, y, Y(t), Y(s)$.

REMARK. Our condition (A.2) is equivalent to the stability condition of Kato [5].

THEOREM 1. *If the conditions* (A.1)-(A.5) *hold, then there exists a unique solution* $u(t) \in Y(t)$ *to the equation*

$$\begin{cases} u(t) = u_0 + \int_0^t A_0(s)u(s)ds \quad \textit{for} \quad t \in [0, T] \,, \\ u_0 \in Y(0) \,. \end{cases} \tag{2.5}$$

In stead of the condition (A.4), we may assume the following:

(A.4)′ $A(\cdot)$ is norm-continuous from $[0, T]$ to $B(Y, X)$ and $D(A(t)) = Y$ for $t \in [0, T]$.

THEOREM 2. *If the conditions* (A.1)-(A.3), (A.4)′, (A.5) *hold, then we have the same conclusion as in Theorem* 1.

3. Construction of an approximating sequence. We shall construct an approximating sequence $\{u^\varepsilon(t) \mid \varepsilon \downarrow 0\}$. For an arbitrary fixed constant $0 < \varepsilon \leqq 1$ we define u^ε, v^ε inductively as follows.

(a) Let

$$t_0 = 0 \tag{3.1}$$

$$u^\varepsilon(t_0) = v^\varepsilon(t_0) = u_0 \,. \tag{3.2}$$

(b) Using (2.1) and (A.4) we take $\varepsilon_1 > 0$ small enough to satisfy (3.ε_1):

$$\text{Max}\,\{C(\varepsilon_1, t_0)h(\varepsilon_1, t_0), \varepsilon_1, \|[A(t_0 + h(\varepsilon_1, t_0)) - A(t_0)]u_0\|_{t_0}\} \leqq \varepsilon \tag{3.ε_1}$$

and we set

$$h_1 < \min(h(\varepsilon_1, t_0), (2\omega)^{-1}) \tag{3.3}$$

$$u^\varepsilon(t_1) = (I - h_1 A_0(t_0))^{-1} u_0 \in Y(t_0) \,, \tag{3.4}$$

where $t_1 = t_0 + h_1$.

We set

$$t_{l+1} = t_l + h_{l+1} \quad l \in N . \tag{3.5}$$

(c) Assume that $\{\varepsilon_j, h_j, u^\varepsilon(t_j), t_j\}$ are determined for $j = 1, 2, \cdots, k$. Then using (A.5) we take $v^\varepsilon(t_k) \in Y(t_k)$ satisfying (3.$\sharp_k$):

$$\begin{cases} |||u^\varepsilon(t_k) - v^\varepsilon(t_k)|||_{t_k} \leqq C_1\varepsilon_k h_k |||u^\varepsilon(t_k)|||_{t_{k-1}} \\ |||u^\varepsilon(t_k) - v^\varepsilon(t_k)|||_{t_k} \leqq C_2 C(\varepsilon_k, t_{k-1}) h_k |||u^\varepsilon(t_k)|||_{t_{k-1}} \\ |||v^\varepsilon(t_k)|||_{t_k} \leqq (1 + C_3 h_k)|||[I - h_k A_0(t_{k-1})]u^\varepsilon(t_k)|||_{t_{k-1}} . \end{cases} \tag{3.$\sharp_k$}$$

(d) Take $\varepsilon_{k+1} > 0$ small enough to satisfy

$$\begin{aligned} \text{Max}\, \{&C(\varepsilon_{k+1}, t_k) h(\varepsilon_{k+1}, t_k),\ \varepsilon_{k+1},\ \|[A(t_k + h(\varepsilon_{k+1}, t_k)) - A(t_k)]u^\varepsilon(t_k)\|_{t_k}, \\ &|||(I - h(\varepsilon_{k+1}, t_k) A_0(t_k))^{-1} v^\varepsilon(t_k) - v^\varepsilon(t_k)|||_{t_k}\} \leqq \varepsilon \end{aligned} \tag{3.ε_{k+1}}$$

and set

$$h_{k+1} = h(\varepsilon_{k+1}, t_k) \ ; \quad t_{k+1} = t_k + h_{k+1} \tag{3.6}$$

$$u^\varepsilon(t_{k+1}) = (I - h_{k+1} A_0(t_k))^{-1} v^\varepsilon(t_k) \in Y(t_k) . \tag{3.7}$$

The sequence $\{t_j\}$ may be limited by $t_\omega = \lim t_k < T$, since h_k may decrease rapidly. If $u^\varepsilon(t_\omega) \in Y_{t_\omega}$, we can start from t_ω again; $t_{\omega+1} = t_\omega + h_\omega, \cdots$. We shall show $u^\varepsilon(t_\omega) \in Y(t_\omega)$. We need the following Lemma.

LEMMA 3.1. *If* $x_k \in Y(t_k)$ *with* $|||x_k|||_{t_k} \leqq L$ *and if* $t_k \to t_\omega$ *and* $x_k \to x_\omega$ *as* $k \to \infty$ *then* $x_\omega \in Y(t_\omega)$.

PROOF. There exists $k_0 > 0$ such that $t_\omega - t_k < \delta$ for any $k \geqq k_0$. From (2.1) and (A.5) for any $k \geqq k_0$ there exists $y_k \in Y(t_\omega)$ such that

$$\begin{aligned} &\|x_k - y_k\|_{t_\omega} \leqq MC_1 |t_\omega - t_k| \cdot |||x_k|||_{t_k} \leqq MC_1 L(t_\omega - t_k) \\ &|||x_k - y_k|||_{t_\omega} \leqq C_2 C(M, t_k)(t_\omega - t_k)|||x_k|||_{t_k} \leqq C_2 C(M, t_k)(t_\omega - t_k)L \leqq C' . \end{aligned}$$

Therefore

$$\|y_k - x_\omega\|_{t_\omega} \leqq \|y_k - x_k\|_{t_\omega} + \|x_k - x_\omega\|_{t_\omega} \to 0 \quad \text{as} \quad t_k \to t_\omega .$$

Since the right hand side tends to zero as $t_k \to t_\omega$, it follows that $y_k \to x_\omega$ in X. By $|||y_k|||_{t_\omega} \leqq C' + |||x_k|||_{t_\omega} \leqq C' + L$ and the closedness of $\{y \in Y(t);\ |||y|||_t \leqq 1\}$, we get $x_\omega \in Y(_\omega)$.

The boundedness of $\{u^\varepsilon\}$ is shown in Proposition 3.1.

In the following, the constants C_i $(i \in N)$ do not depend on ε, δ, s, t, x, y, $Y(t)$, $Y(s)$.

From the definition of $\{u^\varepsilon(t_k)\}$ and $\{v^\varepsilon(t_k)\}$ we get the following:

PROPOSITION 3.1.

$$|||u^\varepsilon(t_k)|||_{t_{k-1}} \leqq C_4 |||u_0|||_{t_0} \quad \text{for} \quad \forall k \in N \tag{3.8}$$

$$|||v^\varepsilon(t_k)|||_{t_k} \leqq C_4 |||u_0|||_{t_0} \quad \text{for} \quad \forall k \in N . \tag{3.9}$$

PROOF. Since $|||u^\varepsilon(t_k)|||_{t_{k-1}} \leqq (1-\omega h_k)^{-1}|||v^\varepsilon(t_{k-1})|||_{t_{k-1}}$ by (A.3), it suffices to show (3.9).

From (3.7) and (3.$\#_k$) it follows that

$$\begin{aligned}|||v^\varepsilon(t_{k-1})|||_{t_{k-1}} &\leqq (1+C_3h_{k-1})|||[I-h_{k-1}A_0(t_{k-2})]u^\varepsilon(t_{k-1})|||_{t_{k-2}}\\ &= (1+C_3h_{k-1})|||v^\varepsilon(t_{k-2})|||_{t_{k-2}}\,.\end{aligned}$$

Hence

$$\begin{aligned}|||v^\varepsilon(t_{k-1})|||_{t_{k-1}} &\leqq \prod_{j=1}^{k-1}(1+C_3h_j)|||v^\varepsilon(t_0)|||_{t_0}\\ &\leqq \exp(C_3t_k)|||u_0|||_{t_0}\\ &\leqq \exp(C_3T)|||u_0|||_{t_0}\,.\end{aligned}$$

q.e.d.

Let

$$u^\varepsilon(t) = \begin{cases} u_0 & t=t_0\\ u^\varepsilon(t_k) & t_{k-1}<t\leqq t_k\,.\end{cases} \tag{3.10}$$

PROPOSITION 3.2.

$$\|u^\varepsilon(t_k)-u^\varepsilon(t_l)\|_{t_k} \leqq C_4[|t_k-t_l| + \varepsilon|t_{k-1}-t_{l-1}|]|||u_0|||_{t_0} \tag{3.11}$$

$$\|v^\varepsilon(t_k)-v^\varepsilon(t_l)\|_{t_k} \leqq 2C_4|t_k-t_l|\cdot|||u_0|||_{t_0} \quad for \quad t_l\leqq t_k\,. \tag{3.12}$$

PROOF. From (3.$\#_k$), (3.7) and (3.8) we obtain

$$\begin{aligned}\|u^\varepsilon(t_k)-u^\varepsilon(t_l)\|_{t_k} &\leqq \sum_{j=l+1}^{k}\|u^\varepsilon(t_j)-u^\varepsilon(t_{j-1})\|_{t_j}\\ &\leqq \sum(\|u^\varepsilon(t_j)-v^\varepsilon(t_{j-1})\|_{t_j}+\|v^\varepsilon(t_{j-1})-u^\varepsilon(t_{j-1})\|_{t_j})\\ &\leqq \sum(C_1\varepsilon_{j-1}h_{j-1}|||u^\varepsilon(t_{j-1})|||_{t_{j-2}}+Ch_j|||u^\varepsilon(t_j)|||_{t_j})\\ &\leqq C''C_4|||u_0|||_{t_0}[|t_k-t_l|+\varepsilon|t_{k-1}-t_{l-1}|]\,.\end{aligned}$$

Similarly, we can prove (3.12) using the inequality

$$\|v^\varepsilon(t_j)-v^\varepsilon(t_{j-1})\|_{t_j} \leqq \|v^\varepsilon(t_j)-u^\varepsilon(t_j)\|_{t_j}+\|u^\varepsilon(t_j)-v^\varepsilon(t_{j-1})\|_{t_j}\,.$$

q.e.d.

By transfinite induction we shall construct $u^\varepsilon(t)$ on the whole interval $[0, T]$. Ω denotes the first uncountable ordinal number and α any fixed ordinal number smaller than Ω. Assume that $u^\varepsilon(t_\beta)$ is defined for all $\beta<\alpha$ and that $\sup_{\beta<\alpha} t_\beta < T$. If α is an isolated ordinal number, then t_α and $u^\varepsilon(t_\alpha)$ are defined, since $t_{\alpha-1}$ and $u^\varepsilon(t_{\alpha-1})$ are already defined. If α is a limit ordinal number, we put $t_\alpha=\sup_{\beta<\alpha}t_\beta$. Let $\beta_1<\beta_2<\cdots$ be such a sequence of ordinal numbers that $\beta_k<\alpha$ and $\tilde t_{\beta_k}\to t_\alpha$. We put $t_k=\tilde t_{\beta_k}$, and $x_k=u^\varepsilon(t_k)$. By Proposition 3.2, x_k converges to some $x_\omega\in X$. Applying Lemma 3.1, we get $x_\omega\in Y(t_\alpha)$ and we define $u^\varepsilon(t_\alpha)=x_\omega$.

We see easily that there exists an ordinal number $\alpha_0<\Omega$ such that

$t_{\sigma_0} = T$, since $t_{\beta-1} - t_\beta > 0$ an Ω is an uncountable ordinal number.
Thus desired approximate sequences $\{u^\varepsilon(t)\}$ are obtained.

4. Strong convergence of $\{u^\varepsilon(t)\}$ in X. Our main purpose of this section is the following:

PROPOSITION 4.1. *The sequence $\{u^\varepsilon(t)\}$ is convergent in X with respect to the norm $\|\cdot\|$ uniformly in $t \in [0, T]$ as ε tends to 0.*

For two functions $u^\varepsilon(t)$ and $u^{\varepsilon'}(t)$ depending on $\{t_l\}$ and $\{t_{l'}\}$, we give another approximate function $u^\delta(s)$ depending on $\{s_j\}$ such that

$$\min\{\varepsilon, \varepsilon'\} \geqq \delta > 0 , \quad \{t_l\} \cup \{t_{l'}\} \subset \{s_j\} , \quad t_{k-1} = s_l < s_{l+1} < \cdots < s_{l+j} = t_k .$$

For the proof of Proposition 4.1 it suffices to show:

PROPOSITION 4.2.

$$\|u^\varepsilon(t_k) - u^\delta(t_k)\|_{t_k} \leqq \|u^\varepsilon(t_{k-1}) - u^\delta(t_{k-1})\|_{t_{k-1}} + C_5(\varepsilon + \delta)(h_k + h_{k-1}) . \tag{4.1}$$

For the proof we need the following lemmas.

For $x \in Y(t)$, we define the set $W(\varepsilon, t, x, s)$ consisting of all elements $y \in Y(s)$ which satisfy the following:

$$\|y - x\|_s \leqq C_1 \varepsilon |s - t| \cdot |||x|||_t \tag{4.2}$$

$$|||y - x|||_s \leqq C_2 C(\varepsilon, t) |s - t| \cdot |||x|||_t . \tag{4.3}$$

Set

$$E \equiv u^\varepsilon(t_k) - v^\varepsilon(t_{k-1}) ; \qquad \tilde{k}_l = s_l - s_{l-1} . \tag{4.4}$$

For fixed k, l, we inductively define a sequence of triplets $\{v_j, w_j, \hat{u}_j\}$ by

$$v_0 \equiv v_0(t_{k-1}; t_{k-1}) = v^\varepsilon(t_{k-1}) \tag{4.5}$$

$$w_1 \equiv w_1(s_{l+1}; s_l) = 0 \tag{4.6}$$

$$\hat{u}_1 \equiv \hat{u}_1(s_{l+1}; s_l) = v_0 + \tilde{k}_{l+1} h_k^{-1} E + w_1 \tag{4.7}$$

$$v_i \equiv v_i(s_{l+i}; s_{l+i}) \in W(\varepsilon_k, s_{l+i-1}, \hat{u}_i, s_{l+i}) \quad \text{for} \quad i = 1, 2, \cdots, j \tag{4.8}$$

$$\begin{aligned} \hat{u}_i - v_{i-1} &\equiv \hat{u}_i(s_{l+1}; s_{l+i-1}) - v_{i-1} \\ &\in W(\varepsilon_k, t_{k-1}, \tilde{k}_{l+1} h_k^{-1} E, s_{l+i-1}) \quad \text{for} \quad i = 2, \cdots, j \end{aligned} \tag{4.9}$$

$$w_i \equiv w_i(s_{l+i}) = \hat{u}_i - v_{i-1} - \tilde{k}_{l+1} h_k^{-1} E \quad \text{for} \quad i = 2, \cdots, j \tag{4.10}$$

$$w'_i \equiv w'_i(s_{l+i}) = v_i - \hat{u}_i \quad \text{for} \quad i = 1, \cdots, j . \tag{4.11}$$

REMARK. $z \equiv z(s; t)$ means z belongs to the space $Y(t)$ at "s-time".

For simplicity, set

(4.12) $$C(k) = C(\varepsilon_k, t_{k-1}) \times C_2 .$$

LEMMA 4.1.

(4.13) $$|||w_i|||_{s_{l+i-1}} \leqq C(k)\tilde{k}_{l+i}|||E|||_{t_{k-1}}$$

(4.14) $$\|w_i\|_{s_{l+i-1}} \leqq C_1\varepsilon_k\tilde{k}_{l+i}|||E|||_{t_{k-1}} .$$

PROOF. From (4.10), (4.9), (4.4) and (4.3), we get

$$|||w_i|||_{s_{l+i-1}} = |||\hat{u}_i - v_{i-1} - \tilde{k}_{l+i}h_k^{-1}E|||_{s_{l+i-1}} \leqq C(k)\sum_{m=1}^{i-1}\tilde{k}_{l+m}|||\tilde{k}_{l+1}h_k^{-1}E|||_{t_{k-1}}$$
$$\leqq C(k)\tilde{k}_{l+1}|||E|||_{t_{k-1}} ,$$

since $\sum_{m=1}^{i-1}\tilde{k}_{l+m} \leqq \sum_{m=1}^{j}\tilde{k}_{l+m} = h_k$, Thus we get (4.13).

From (4.10), (4.9), (4.4) and (4.2), it follows that

$$\|w_i\|_{s_{l+i-1}} = \|\hat{u}_i - v_{i-1} - \tilde{k}_{l+i}h_k^{-1}E\|_{s_{l+i-1}} \leqq C_1\varepsilon_k\sum_{m=1}^{i-1}\tilde{k}_{l+m}|||\tilde{k}_{l+i}h_k^{-1}E|||_{t_{k-1}}$$
$$\leqq C_1\varepsilon_k\tilde{k}_{l+i}|||E|||_{t_{k-1}} .$$

Thus we obtain (4.14). q.e.d.

LEMMA 4.2.

(4.15) $$|||\hat{u}_i|||_{s_{l+i}} \leqq C_6 .$$

PROOF. From (4.7), (4.13), (3.9) and (3.ε_k) we have

$$|||\hat{u}_1|||_{s_{l+1}} \leqq |||v^\varepsilon(t_{k-1})|||_{t_{k-1}} + \tilde{k}_{l+1}h_k^{-1}|||E|||_{t_{k-1}} \leqq C_4|||u_0|||_{t_0} + \tilde{k}_{l+1}h_k^{-1}|||E|||_{t_{k-1}} .$$

We prove

(4.16) $$|||\hat{u}_{i-1}|||_{s_{l+i-1}} \leqq C_4|||u_0|||_{t_0}\exp\left[C(k)\sum_{m=0}^{i-2}\tilde{k}_{l+m}\right] + [C(k) + h_k^{-1}]\sum_{m=1}^{i-1}\tilde{k}_{l+m}\varepsilon$$

by induction on $i = 1, 2, \cdots$. From (4.10), (4.13), (3.ε_k), (4.8) and (4.16) we have

$$|||\hat{u}_i|||_{s_{l+i}} \leqq |||v_{i-1}|||_{s_{l+i}} + |||w_i|||_{s_{l+i}} + \tilde{k}_{l+i}h_k^{-1}|||E|||_{t_{k-1}}$$
$$\leqq |||v_{i-1} - \hat{u}_{i-1}|||_{s_{l+i}} + |||\hat{u}_{i-1}|||_{s_{l+i-1}} + [C(k)\tilde{k}_{l+i} + \tilde{k}_{l+i}h_k^{-1}]\varepsilon$$
$$\leqq [C(k)\tilde{k}_{l+i-1} + 1]|||\hat{u}_{i-1}|||_{s_{l+i-1}} + [C(k)\tilde{k}_{l+i} + \tilde{k}_{l+i}h_k^{-1}]\varepsilon$$
$$\leqq [C(k)\tilde{k}_{l+i-1} + 1][C_4|||u_0|||_{t_0}]\exp\left[C(k)\sum_{m=0}^{i-2}\tilde{k}_{l+m}\right]$$
$$+ [C(k) + h_k^{-1}]\sum_{m=1}^{i-1}\tilde{k}_{l+m}\varepsilon + [C(k)\tilde{k}_{l+i} + \tilde{k}_{l+i}h_k^{-1}]\varepsilon$$
$$\leqq C_4|||u_0|||_{t_0}\exp\left[C(k)\sum_{m=0}^{i-1}\tilde{k}_{l+m}\right] + [C(k) + h_k^{-1}]\sum_{m=1}^{i}\tilde{k}_{l+m}\varepsilon .$$

By induction we obtain

$$|||\hat{u}_i|||_{sl+i} \leqq C_4|||u_0|||_{t_0}e^\varepsilon + [\varepsilon + 1]\varepsilon \leqq C_6$$

since $\varepsilon \leqq 1$.

LEMMA 4.3. *For $i, l = 1, 2, \cdots,$ we have*

$$|||v_i - u^\varepsilon(t_k)|||_{sl+i} \leqq C_7\varepsilon \tag{4.17}$$

$$|||\hat{u}_i - u^\varepsilon(t_k)|||_{sl+i} \leqq C_8\varepsilon\ . \tag{4.18}$$

PROOF OF (4.17). From (4.10) and (4.11), it follows that

$$v_i - v_{i-1} = w'_i + w_i + \tilde{k}_{l+i}h_k^{-1}E\ .$$

Then from (4.5) we get

$$v_i = v^\varepsilon(t_{k-1}) + \sum_{m=1}^{i} w'_m + \sum_{m=1}^{i} w_m + \Big(\sum_{m=1}^{i} \tilde{k}_{l+m}\Big)h_k^{-1}E\ . \tag{4.19}$$

Therefore by (4.13), (4.14), (4.15) and ($3.\varepsilon_k$), we have

$$\begin{aligned} |||v_i - u^\varepsilon(t_k)|||_{sl+i} &\leqq |||v^\varepsilon(t_{k-1}) - u^\varepsilon(t_k)|||_{t_{k-1}} \\ &\quad + \sum_{m=1}^{i} [|||w'_m|||_{sl+m} + |||w_m|||_{sl+m}] + |||E|||_{t_{k-1}} \\ &\leqq \varepsilon + \sum_{m=1}^{i} [C(k-1)\tilde{k}_{l+m}|||E|||_{t_{k+1}} + C(k-1)\tilde{k}_{l+m}|||\hat{u}_m|||_{sl+m}] + \varepsilon \\ &\leqq 2\varepsilon + C(k-1)h_{k-1}(\max_{1\leqq m\leqq i} |||\hat{u}_m|||_{sl+m} + \varepsilon) \\ &\leqq C_7\varepsilon\ , \end{aligned}$$

showing (4.17).

PROOF OF (4.18). From (4.11) and (4.17), it follows that

$$\begin{aligned} |||\hat{u}_i - u^\varepsilon(t_k)|||_{sl+i} &\leqq |||\hat{u}_i - v_i|||_{sl+i} + |||v_i - u^\varepsilon(t_k)|||_{sl+i} \\ &\leqq |||w'_i|||_{sl+i} + C_7\varepsilon \\ &\leqq C(k-1)\tilde{k}_{l+i-1}|||\hat{u}_i|||_{sl+i-1} + C_7\varepsilon \\ &\leqq C_8\varepsilon\ . \end{aligned}$$

LEMMA 4.4.

$$\|v_j - u^\varepsilon(t_k)\|_{t_{k+1}} \leqq C_9\varepsilon h_k\ . \tag{4.20}$$

PROOF. From (4.11) and (4.10), we have

$$v_j = \sum_{m=1}^{j} w'_m + \sum_{m=1}^{j} w_m + \sum_{m=1}^{j} \tilde{k}_{l+m}h_k^{-1}E + v_0\ . \tag{4.21}$$

By (4.4), (4.5) and $\sum_{m=1}^{j} \tilde{k}_{l+m} = h_k$, we get

$$\sum_{m=1}^{j} \tilde{k}_{l+m}h_k^{-1}E + v_0 = u^\varepsilon(t_k) - v^\varepsilon(t_{k-1}) + v^\varepsilon(t_{k-1}) = u^\varepsilon(t_k)\ . \tag{4.22}$$

Then from (4.21), (4.22), (4.8), (4.11), (4.14) and (4.15) it follows that

$$\begin{aligned}\| v_j - u^\varepsilon(t_k) \| &\leqq \sum_{m=1}^{j} \| w'_m \| + \sum_{m=1}^{j} \| w_m \| \\ &\leqq \Big(\sum_{m=1}^{j} \varepsilon_k \widetilde{k}_{l+m-1} + C_1 \sum_{m=1}^{j} \widetilde{k}_{l+m} \| \hat{u}_m \|_{s_{l+m}} \Big) \varepsilon \,. \\ &\leqq C_9 \varepsilon h_k \,.\end{aligned}$$

Thus we obtain (4.20).

For each l, i we set

$$u_i \equiv u_i(s_{l+i}; s_{l+i-1}) = (I - \widetilde{k}_{l+i-1} A_0(s_{l+i-1}))^{-1} v_{i-1} \tag{4.23}$$

$$\bar{v}_i \equiv \bar{v}_i(s_{l+i}; s_{l+i}) = (I - \widetilde{k}_{l+i} A_0(s_{l+i})) \hat{u}_{i+1} \,. \tag{4.24}$$

LEMMA 4.5.

$$\| u_i - v_i \|_{s_{l+i}} \leqq C_{10}(\varepsilon + \varepsilon_k) \widetilde{k}_{l+i} \,. \tag{4.25}$$

PROOF. From (4.10), (4.4) and (3.7)

$$v_{i-1} = \hat{u}_i - w_i - \widetilde{k}_{l+i} A_0(t_{k-1}) u^\varepsilon(t_k) \,.$$

By (4.24)

$$\bar{v}_{i-1} = \hat{u}_i - \widetilde{k}_{l+i-1} A_0(s_{l+i-1}) \hat{u}_i \,.$$

From $(3.\varepsilon_k)$, $(3.\#_k)$ and (3.8) we obtain

$$\begin{aligned}&\| [A_0(t_{k-1}) - A(s_{l+i})] u^\varepsilon(t_k) \|_{s_{l+i}} \\ &\leqq \| [A_0(t_{k-1}) - A(s_{l+i})][u^\varepsilon(t_k) - v^\varepsilon(t_{k-1})] \|_{s_{l+i}} \\ &\quad + \| [A(t_{k-1}) - A(s_{l+i})][v^\varepsilon(t_{k-1}) - u^\varepsilon(t_{k-1})] \|_{s_{l+i}} \\ &\quad + \| [A(t_{k-1}) - A(s_{l+i})] u^\varepsilon(t_{k-1}) \|_{s_{l+i}} \\ &\leqq 2c ||| h_{k-1} A_0(t_{k-1}) v^\varepsilon(t_{k-1}) |||_{t_{k-1}} + 2c ||| v^\varepsilon(t_{k-1}) - u^\varepsilon(t_{k-1}) |||_{t_{k-1}} + \varepsilon \\ &\leqq 2c\varepsilon + 2cC_2 C(k-1) h_{k-1} ||| u^\varepsilon(t_{k-1}) |||_{t_{k-1}} + \varepsilon \\ &\leqq 2c\varepsilon + 2cC_2 \varepsilon C_4 ||| u |||_{t_0} + \varepsilon \\ &\leqq C' \varepsilon \,.\end{aligned} \tag{4.26}$$

Then from (4.14), (4.26), (4.18) and (3.8) we have

$$\begin{aligned}\| v_{i-1} - \bar{v}_{i-1} \|_{s_{l+i}} &\leqq \| w_i \|_{s_{l+i}} + \widetilde{k}_{l+i} \| A_0(t_{k-1}) u^\varepsilon(t_k) - A_0(s_{l+i}) \hat{u}_{i+1} \|_{s_{l+i}} \\ &\leqq \| w_i \|_{s_{l+i}} + \widetilde{k}_{l+i} \| [A_0(t_{k-1}) - A(s_{l+i})] u^\varepsilon(t_k) \|_{s_{l+i}} \\ &\quad + \widetilde{k}_{l+i} \| A(s_{l+i}) [u^\varepsilon(t_k) - \hat{u}_{i+1}] \|_{s_{l+i}} \\ &\leqq C_1 \varepsilon_k \widetilde{k}_{l+i} ||| E |||_{t_{k-1}} + \widetilde{k}_{l+i} C' \varepsilon + c \widetilde{k}_{l+i} ||| u^\varepsilon(t_k) - \hat{u}_i |||_{s_{l+i}} \\ &\leqq C_{11} \widetilde{k}_{l+i} \varepsilon \,.\end{aligned}$$

Therefore we get from (4.23) and (4.24)

(4.27) $$\|u_i - \hat{u}_i\|_{s_{l+i}} \leqq (1-\omega\tilde{k}_{l+i-1})^{-1}\|v_{i-1} - \bar{v}_{i-1}\|_{s_{l+i-1}} \leqq C'_{11}\varepsilon\tilde{k}_{l+i}\,.$$

From (4.11), (4.8) and (4.15) it follows that

$$\|\hat{u}_i - v_i\|_{s_{l+i}} \leqq \varepsilon_k\tilde{k}_{l+1}|||\tilde{u}_i|||_{s_{l+i}} \leqq C_6\varepsilon_k\tilde{k}_{l+1}\,.$$

Hence from (4.27) we get

$$\|u_i - v_i\|_{s_{l+1}} \leqq \|u_i - \hat{u}_i\|_{s_{l+1}} + \|\hat{u}_i - v_i\|_{s_{l+i}} \leqq C_{10}(\varepsilon + \varepsilon_k)\tilde{k}_{l+i}\,.$$

Thus (4.25) is proved.

LEMMA 4.6.

(4.28) $$\|v^\delta(t_k) - v_j\|_{t_k} \leqq \|v^\delta(t_{k-1}) - v^\varepsilon(t_{k-1})\|_{t_{k-1}} + C_{12}(\varepsilon+\delta)h_k\,.$$

PROOF. Let

$$v_i^\delta = v^\delta(s_{l+i})\,, \qquad u_i^\delta = u^\delta(s_{l+i})\,, \quad i = 0, 1, 2, \cdots, j\,.$$

From (3.7), (4.25), (3.8) and (4.23) we obtain

$$\begin{aligned}\|v_j^\delta - v_j\|_{t_k} &\leqq \|v_j^\delta - u_j^\delta\|_{s_{l+i}} + \|u_j^\delta - u_j\|_{s_{l+i}} + \|u_j - v_j\|_{s_{l+j}}\\ &\leqq \delta_{l+j}\tilde{k}_{l+j}|||u_j^\delta|||_{s_{l+j}} + \|u_j^\delta - u_j\|_{s_{l+j}} + C_{10}(\varepsilon+\varepsilon_k)\tilde{k}_{l+j}\\ &\leqq \|v_{j-1}^\delta - v_{j-1}\|_{s_{l+j-1}} + \delta_{l+j}\tilde{k}_{l+j}C_4|||u_0|||_{t_0} + 2C_{10}\varepsilon\tilde{k}_{l+j}\\ &\leqq \|v_{j-1}^\delta - v_{j-1}\|_{s_{l+j-1}} + C_{12}(\varepsilon+\delta)\tilde{k}_{l+j}\\ &\cdots\\ &\leqq \|v_0^\delta - v_0\|_{s_l} + C_{12}(\varepsilon+\delta)\sum_{m=1}^{j}\tilde{k}_{l+j}\\ &= \|v^\delta(t_{k-1}) - v^\varepsilon(t_{k-1})\|_{t_{k-1}} + C_{12}(\varepsilon+\delta)h_k\,.\end{aligned}$$

PROOF OF PROPOSITION 4.2. From (4.20), (4.28), (3.8) and (3.$\sharp_k$) it follows that

$$\begin{aligned}&\|u^\varepsilon(t_k) - u^\delta(t_k)\|_{t_k}\\ &\quad\leqq \|u^\varepsilon(t_k) - v_j\|_{t_k} + \|v_j - v_j^\delta\|_{t_k} + \|v_j^\delta - u^\delta(t_k)\|_{t_k}\\ &\quad\leqq C_2\varepsilon h_k|||u^\varepsilon(t_k)|||_{t_k} + \|v^\varepsilon(t_{k-1}) - v^\delta(t_{k-1})\|_{t_{k-1}} + C_{12}(\varepsilon+\delta)h_k\\ &\qquad + C_2\delta\tilde{k}_{l+j}|||u^\delta(t_{k-1})\|_{t_k}\\ &\quad\leqq C_2C_4|||u_0|||_{t_0}\varepsilon h_k + \|v^\varepsilon(t_{k-1}) - u^\varepsilon(t_{k-1})\|_{t_{k-1}} + \|u^\varepsilon(t_{k-1}) - u^\delta(t_{k-1})\|_{t_{k-1}}\\ &\qquad + \|u^\delta(t_{k-1}) - v^\delta(t_{k-1})\|_{t_{k-1}} + C_{12}(\varepsilon+\delta)h_k + C_2C_4|||u_0|||_{t_0}\delta\tilde{k}_{l+j}\\ &\quad\leqq C_2\varepsilon h_{k-1}|||u^\varepsilon(t_{k-1})|||_{t_{k-1}} + \|u^\varepsilon(t_{k-1}) - u^\delta(t_{k-1})\|_{t_{k-1}}\\ &\qquad + C_2\delta\tilde{k}_{l-1}|||u^\delta(t_{k-1})|||_{t_k} + [C_{12} + C_2C_4|||u_0|||_{t_0}](\varepsilon+\delta)h_k\\ &\quad\leqq \|u^\varepsilon(t_{k-1}) - u^\delta(t_{k-1})\|_{t_{k-1}} + [C_{12} + C_2C_4|||u_0|||_{t_0}](\varepsilon+\delta)(h_k + h_{k-1})\\ &\quad\leqq \|u^\varepsilon(t_{k-1}) - u^\delta(t_{k-1})\|_{t_{k-1}} + C_5(\varepsilon+\delta)(h_k + h_{k-1})\,.\end{aligned}$$

Thus we obtain (4.1).

5. Proof of theorems. Let

$$(5.1)\qquad \begin{cases} u^\varepsilon(t) = u^\varepsilon(t_k)\,, & u^\varepsilon(0) = u_0 \\ v^\varepsilon(t) = v^\varepsilon(t_{k-1})\,, & v^\varepsilon(0) = u_0 \\ A^\varepsilon(t) = A(t_{k-1}) & \text{for } \ t_{k-1} < t \leqq t_k\,. \end{cases}$$

Then we have

$$(5.2)\qquad \begin{cases} h_k^{-1}[u^\varepsilon(t) - u^\varepsilon(t-h_k)] - A^\varepsilon(t)v^\varepsilon(t+0) \\ \qquad = h_k^{-1}[v^\varepsilon(t+0) - u^\varepsilon(t-h_k)]\,, & t_{k-1} < t \leqq t_k \\ u^\varepsilon(t) = u_0\,, & -1 \leqq t \leqq 0\,. \end{cases}$$

Therefore

$$\begin{aligned}(5.3)\qquad & u^\varepsilon(t) - u^\varepsilon(0) \\ &= u^\varepsilon(t) - u^\varepsilon(t-h_k) + \sum_{l=1}^{k-1}\left\{u^\varepsilon\left(t - \sum_{m=l}^{k} h_m\right) - u^\varepsilon\left(t - \sum_{m=1}^{k-1} h_m\right)\right\} \\ &\quad + u^\varepsilon\left(t - \sum_{n=1}^{k} h_n\right) - u^\varepsilon(0) \\ &= h_k A^\varepsilon(t) v^\varepsilon(t+0) + v^\varepsilon(t+0) - u^\varepsilon(t-h_k) \\ &\quad + \sum_{l=1}^{k-1}\left[h_l A^\varepsilon\left(t - \sum_{m=1}^{l} h_m\right) v^\varepsilon\left(t - \sum_{m=1}^{l} h_m + 0\right)\right. \\ &\quad \left. + v^\varepsilon\left(t - \sum_{m=1}^{l} h_m + 0\right) - u^\varepsilon\left(t - \sum_{m=1}^{l+1} h_m\right)\right] \\ &= \int_0^t A^\varepsilon(s) v^\varepsilon(s)\,ds + \sum_{l=2}^{k}\{v^\varepsilon(t_{l-1}) - u^\varepsilon(t_{l-1})\} \quad \text{for} \quad t \in (t_{k-1}, t_k]\,. \end{aligned}$$

From $(3.\#_k)$ and (3.8) we get

$$\begin{aligned} \left\|\sum_{l=2}^{k} v^\varepsilon(t_{l-1}) - u^\varepsilon(t_{l-1})\right\|_{t_k} &\leqq \sum_{l=2}^{k} \| v^\varepsilon(t_{l-1}) - u^\varepsilon(t_{l-1}) \|_{t_l} \\ &\leqq \sum_{l=2}^{k} \varepsilon_l h_l ||| u^\varepsilon(t_{l-1}) |||_{t_1} \leqq \varepsilon C_0 ||| u_0 |||_{t_0} T\,. \end{aligned}$$

This means

$$(5.4)\qquad \sum_{l=2}^{k}\{v^\varepsilon(t_{l-1}) - u^\varepsilon(t_{l-1})\} \rightrightarrows 0 \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for} \quad \forall\, t \in [0, T]$$

in X with respect to the norm $\|\cdot\|$, where the notation $\rightrightarrows$ means the uniform convergence.

From Propositions 3.2 and 4.1 there exists $u(t) \in X$ such that

$$(5.5)\qquad u^\varepsilon(t) \rightrightarrows u(t) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for} \quad t \in (0, T] \quad \text{in } X \text{ w.r.t. } \|\cdot\|$$

$$(5.6)\qquad v^\varepsilon(t) \rightrightarrows u(t) \quad \text{as} \quad \varepsilon \downarrow 0 \quad \text{for} \quad t \in (0, T] \quad \text{in } X \text{ w.r.t. } \|\cdot\|\,.$$

We now prove Theorem 1.

LEMMA 5.1.

(5.7) $A^{\varepsilon}(t)v^{\varepsilon}(t) \to A(t)u(t)$ *for almost every* t *as* $\varepsilon \downarrow 0$ *in* (X, τ) .

PROOF. There exists $k > 0$ such that $t_{k-1} < t \leqq t_k$. From (5.1) we have

$$\begin{aligned} A^{\varepsilon}(t)v^{\varepsilon}(t) - A(t)u(t) &= A(t_{k-1})v^{\varepsilon}(t_{k-1}) - A(t)u(t) \\ &= [A(t_{k-1}) - A(t)]v^{\varepsilon}(t_{k-1}) + A(t)[v^{\varepsilon}(t_{k-1}) - u(t)] . \end{aligned}$$

Then from (2.2), (A.4) and (5.6) we obtain (5.7). The closedness of $\{y \in Y(t); |||y|||_t \leqq 1\}$ in $(X, \|\cdot\|)$ implies $u(s) \in D(A_0(s))$. Thus from (5.3), (5.4), (5.5) and (5.7) it follows that

$$u(t) = u_0 + \tau\text{-}\int_0^t A_0(s)u(s)ds .$$

LEMMA 5.2.

$$u(t) = u_0 + \int_0^t A_0(s)u(s)ds . \tag{5.8}$$

PROOF. If $A_0(\cdot)u(\cdot)$ is a weakly measurable function from $[0, T]$ to $(X, \|\cdot\|)$, then it is strongly measurable by the separability of X and Pettis's theorem. From Proposition 3.1 it follows that

$$\int_0^T \|A_0(s)u(s)\|ds \leqq \int_0^T C'|||u(s)|||ds \leqq CT < +\infty .$$

Therefore $A_0(\cdot)u(\cdot)$ is integrable on $[0, T]$ with respect to $(X, \|\cdot\|)$.

To complete the proof of Lemma 5.2, it is sufficient to show the following:

LEMMA 5.3. *If* $f: [0, t] \to (X, \tau)$ *is weakly measurable then* $f: [0, T] \to (X, \|\cdot\|)$ *is weakly measurable.*

PROOF. Let X' and X'_τ be the respective dual space of $(X, \|\cdot\|)$ and (X, τ) and B' a closed unit ball of X'. Since X is separable, B' is a metrizable compact set with respect to the w^*-topology.

We define $X'_\tau(\alpha)$ inductively as follows.

$$X'_\tau(0) = X'_\tau$$

$$X'_\tau(1) = \{x \in X'; \exists\{x_n\} \subset X'_\tau(0) \text{ such that } x_n \xrightarrow{w^*} x \text{ as } n \to \infty\}$$

$$X'_\tau(\alpha) = \{x \in X': \exists\{x_n\} \subset \bigcup_{\beta<\alpha} X'_\tau(\beta) \text{ such that } x_n \xrightarrow{w^*} x \text{ as } n \to \infty\}$$

i.e. $X'_\tau(\alpha)$ is the set of limit points of w^*-sequentially convergent sequences of $\cup_{\beta<\alpha} X'_\tau(\beta)$.

Set $\tilde{X}'_\tau = \cup_{\alpha<\Omega} X'_\tau(\alpha)$, where Ω is the first uncountable ordinal number.

Then $\tilde{X}'_\tau$ is closed with respect to the w^*-sequential convergence. Since B' is metrizable $\tilde{X}'_\tau \cap nB'$ is w^*-closed for all $n \in N_+$. Thus by Krein-Šmulian's theorem $\tilde{X}'_\tau$ is w^*-closed. On the other hand, $\tilde{X}'_\tau$ is w^*-dense in X' since X'_τ, is w^*-dense in X'. Hence $\tilde{X}'_\tau$ coincides with X'.

For any $x \in X'_\tau(1)$, there exists $\{x_n\} \subset X'_\tau(0)$ such that $x_n \xrightarrow{w^*} x$ and that $\langle f(t), x_n \rangle$ is measurable for all x_n. Hence $\langle f(t), x \rangle$ is measurable. In the same manner, we see that $\langle f(t), x \rangle$ is measurable for all $x \in X'_\tau(\alpha)$, if $\langle f(t), x \rangle$ is measurable for all $x \in \cup_{\beta<\alpha} X'_\tau(\beta)$.

Consequently, f: $[0, T] \to (X, \|\cdot\|)$ is weakly measurable. q.e.d.

Therefore we get

$$u(t) = u_0 + \int_0^t A_0(s)u(s)ds .$$

LEMMA 5.4 (Kato [4]). *Let S be the set of those $s \geqq 0$ at which the strong solution $u(s)$ of $du(t)/dt \in Au(t)$ is strongly differentiable. Then, we have,*

$$2^{-1}\frac{d}{ds}\|u(s)\|^2 = \|u(s)\|\frac{d\|u(s)\|}{ds} = \mathrm{Re}\left\langle \frac{du(s)}{ds}, f \right\rangle , \quad \text{at almost every } s \in S$$

whenever $f \in F(u(s))$, where F is the duality map.

The uniqueness of the solution to (2.5) now follows.

Indeed, let $u_1(t)$ and $u_2(t)$ be two solutions to the equation (2.5). From the above lemma, we obtain, for a certain $f \in F(u_1(t) - u_2(t))$,

$$\begin{aligned} 2^{-1}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 &= \mathrm{Re}\left\langle \frac{du_1(t)}{dt} - \frac{du_2(t)}{dt}, f \right\rangle \\ &= \mathrm{Re}\langle A_0(t)u_1(t) - A_0(t)u_2(t), f \rangle \leqq 0 . \end{aligned}$$

Therefore we get

$$\|u_1(t) - u_2(t)\|^2 \leqq \|u_1(0) - u_2(0)\|^2 = \|u_0 - u_0\|^2 = 0 .$$

This means that $u_1 \equiv u_2$.

This finishes the proof of Theorem 1.

We now prove Theorem 2.

LEMMA 5.5. *For any $\varepsilon > 0$, $2\omega^{-1} > \lambda > 0$ and $t \in [0, T]$ there exists $\delta > 0$ such that*

(5.9) $\quad \|(I - \lambda A_0(t))^{-1} - (I - \lambda A_0(s))^{-1}\| < \varepsilon$ *for* $|t - s| < \delta$

(5.10) $\quad \|(I - \lambda A_0^*(t))^{-1} - (I - \lambda A_0^*(s))^{-1}\| < \varepsilon$ *for* $|t - s| < \delta$,

where $A_0^*(t)$ is the adjoint operator of $A_0(t)$.

PROOF. Let $x \in X$ be fixed. Set

$$\begin{cases} y_t \equiv (I - \lambda A_0(t))^{-1}x \\ y_s \equiv (I - \lambda A_0(s))^{-1}x \,, \end{cases} \tag{5.11}$$

where $t < s$. From (A.5) for any $\eta > 0$ there exists $z_s \in Y(s)$ such that

$$\begin{cases} |||y_t - z_s|||_s \leqq C_2 C(\eta, t)|s - t|\cdot|||y_t|||_t \\ \|y_t - z_s\|_s \leqq C_1\eta|s - t|\cdot|||y_t|||_t \,. \end{cases} \tag{5.12}$$

Let

$$z \equiv (I - \lambda A_0(s))z_s \,. \tag{5.13}$$

From (5.11) and (5.13)

$$\begin{aligned} \|z_s - y_s\|_s &= \|(I - \lambda A_0(s))^{-1}z - (I - \lambda A_0(s))^{-1}x\|_s \\ &\leqq (1 - \lambda\omega)^{-1}\|z - x\|_s \\ &= C\|(I - \lambda A_0(s))z_s - (I - \lambda A(t))z_s + (I - \lambda A(t))z_s \\ &\quad - (I - \lambda A_0(t))y_t\|_s \\ &\leqq C[\lambda\|[A(s) - A(t)]z_s\|_s + \|(I - \lambda A(t))(z_s - y_t)\|_s] \,. \end{aligned} \tag{5.14}$$

Since $A(\cdot)$ is norm-continuous, for any $\eta' > 0$ there exists $\zeta = \zeta(t) > 0$ such that

$$\|[A(s) - A(t)]z_s\|_s \leqq \eta'|||z_s|||_s \quad \text{for} \quad |s - t| < \zeta \,. \tag{5.15}$$

From (5.12) we have

$$|||z|||_s \leqq (1 + C_2C(\eta, t))|||y_t|||_t \,. \tag{5.16}$$

Note that $D(A(t)) = Y$ and the norm $|||\cdot|||_t$ is equivalent to the graph norm $\|\cdot\|_t + \|A(t)\cdot\|_t$. Then from (5.14), (5.15) and (5.16) it follows that

$$\|z_s - y_s\|_s \leqq C\{\lambda\eta'(1 + C_2C(\eta, t)) + c_0C_2C(\eta, t)|s - t|\}|||y_t|||_t \,. \tag{5.17}$$

From (5.11) we obtain

$$\begin{aligned} |||y_t|||_t &\leqq c_0(\|y_t\|_t + \|A(t)y_t\|_t) \\ &= c_0(\|(I - \lambda A_0(t))^{-1}x\|_t + \|A_0(t)(I - \lambda A_0(t))^{-1}x\|_t) \\ &\leqq c_0(\|x\|_t + \lambda^{-1}\|[I - (I - \lambda A_0(t))^{-1}]x\|_t) \\ &\leqq c_0(1 + 2\lambda^{-1})\|x\|_t \,. \end{aligned} \tag{5.18}$$

Thus from (5.17) and (5.18) we have

$$\|z_s - y_s\|_s \leqq Cc_0(1 + 2\lambda^{-1})\{\lambda\eta' + C_2C(\eta, t)[\lambda\eta' + c_0|s - t|]\}\|x\|_t \,. \tag{5.19}$$

From (5.11), (5.12), (5.18) and (5.19)

$$
\begin{aligned}
&\|[(I-\lambda A_0(t))^{-1}-(I-\lambda A_0(s))^{-1}]x\|_s \\
&\quad = \|y_t - y_s\|_s \leqq \|y_t - z_s\|_s + \|z_s - y_s\|_s \leqq C_1\eta|s-t|c_0(1+2^{-1}\lambda)\|x\|_t + \|z_s - y_s\|_s \\
&\quad \leqq c_0(1+2^{-1}\lambda)\{\lambda[1+C_2C(\eta, t)]\eta' + [c_0C_2C(\eta, t) + C_1\eta]|s-t|\}\|x\|_t .
\end{aligned}
$$

Then for $\eta' > 0$ and $|s-t|$ sufficiently small we obtain (5.9).

Let $\sigma(t) = \sigma(X, D(A_0^*(t)))$ be the weak topology on X with respect to $D(A_0^*(t)) = (I - \lambda A_0^*(t))^{-1}X'$. Then we have

(5.20) $\qquad A(t)\colon (Y; \|\cdot\|) \to (X; \sigma(t))$ is continuous for $t \in [0, T]$.

Let $x' \in X'$ be fixed. Set

$$
(5.21) \qquad \begin{cases}
\{u_n(t)\} = \{u^{1/n}(t); t_l\}\ , & t \in [0, T] \\
w_n(t) = A_0(t_k)u_n(t_k)\ , & t_{k-1} \leqq t < t_k \\
f_\lambda(t) = (I - \lambda A_0^*(t))^{-1}x'\ , & t \in [0, T] \\
f_{\lambda,n}(t) = (I - \lambda A_0^*(t_k))^{-1}x'\ , & t_{k-1} \leqq t < t_k .
\end{cases}
$$

LEMMA 5.6. *For some positive sequence $\varepsilon_n \to 0$ we have*

$$
(5.22) \qquad \int_t^{t'} |\langle w_n(s), f_\lambda(s) - f_{\lambda,n}(s)\rangle| ds \leqq C\varepsilon_n|t - t'| .
$$

PROOF. From (5.10) for any $s \in [0, T]$ and $m \in N_+$ there exists $n \in N$, or a partition $\{t_{l(n)}\}$ with respect to $\{u^{1/n}\}$, such that $s \in [t_{(k-1)(n)}, t_{k(n)})$ and

$$
\|(I - \lambda A_0^*(s))^{-1}x' - (I - \lambda A_0^*(t_{k(n')}))^{-1}x'\| \leqq 1/m \quad \text{for} \quad n' \geqq n .
$$

n depends on s, but we get $n_0(m) \in N_+$ such that

$$
\|(I - \lambda A_0^*(s))^{-1}x' - (I - \lambda A_0^*(t_{k(n)}))^{-1}x'\| \leqq 1/m \quad \text{for} \quad n \geqq n_0(m) .
$$

for any $s \in [0, T]$, since $[0, T]$ is compact. Let $\varepsilon_n = 1/m$ for $n_0(m) \leqq n < n_0(m+1)$. Then for $k = k(n)$ and $s \in [t_{k-1}, t_k)$ it follows from Lemma 5.5 and Proposition 3.1 that

$$
\begin{aligned}
|\langle w_n(s), f_\lambda(s) - f_{\lambda,n}(s)\rangle| &= |\langle A_0(t_k)u_n(t_k), (I - \lambda A_0^*(s))^{-1}x' - (I - \lambda A_0^*(t_k))^{-1}x'\rangle| \\
&\leqq \|A_0(t_k)u_n(t_k)\|C\varepsilon_n\|x'\| \leqq C'|||u_n(t_k)|||\cdot\|x'\|\varepsilon_n \\
&\leqq C'C_4|||u_0|||\cdot\|x'\|\varepsilon_n .
\end{aligned}
$$

Integrating both sides of the above inequality we get (5.22).

LEMMA 5.7.

$$
\begin{aligned}
(5.23) \qquad & \int_t^{t'} \langle w_n(s), f_{\lambda,n}(s)\rangle ds \\
& \qquad \to \int_t^{t'} \langle A_0(s)u(s), (I - \lambda A_0^*(s))^{-1}x'\rangle ds\ , \quad \text{as} \quad n \to \infty .
\end{aligned}
$$

PROOF. Since $\|(A(t_k) - A(t))u_n(t_k)\| \leqq \|A(t_k) - A(t)\|_{Y\to X}|||u_n(t_k)||| \to 0$

(where $\|\cdot\|_{Y\to X}$ is the norm of $B(Y, X)$) and since $u_n(t_k)\to u(t)$ as $n\to\infty$, it follows from (5.20) that

$$(5.24)\qquad w_n(t) = (A(t_k) - A(t))u_n(t_k) + A(t)u_n(t_k) \to A(t)u(t) = w(t)\ ,$$
$$\text{with respect to } \sigma(t) \text{ for any fixed } t\in[0, T]\ .$$

By virtue of (5.24), $\langle w_n(s), f_\lambda(s)\rangle$ converges to $\langle w(s), f_\lambda(s)\rangle$ for each $s\in[0, T]$ as $n\to\infty$, and so the function $\langle w(s), f_\lambda(s)\rangle$ is measurable in s. On the other hand, we have

$$(5.25)\qquad \left|\int_t^{t'}\langle w_n(s), f_{\lambda,n}(s)\rangle ds - \int_t^{t'}\langle w(s), f_\lambda(s)\rangle ds\right|$$
$$\leqq \left|\int_t^{t'}\langle w_n(s) - w(s), f_\lambda(s)\rangle ds\right| + \int_t^{t'}|\langle w_n(s), f_\lambda(s) - f_{\lambda,n}(s)\rangle| ds\ .$$

From $\|f_\lambda(s)\|\leqq\|x'\|$, (5.25), Lebesgue's convergence theorem and Lemma 5.6 it follows that

$$\int_t^{t'}\langle w_n(s), f_{\lambda,n}(s)\rangle ds \to \int_t^{t'}\langle w(s), f_\lambda(s)\rangle ds \quad\text{as}\quad n\to\infty\ .$$

LEMMA 5.8.

$A_0(\cdot)u(\cdot)$: $[0, T]\to(X, \|\cdot\|)$ is weakly measurable.

PROOF. $\langle w(s), f_\lambda(s)\rangle = \langle A_0(s)u(s), (I - \lambda A_0^*(s))^{-1}x'\rangle$ converges to $\langle A_0(s)u(s), x\rangle$ as $\lambda\to 0$ for each $s\in[0, T]$. Thus for any $x'\in X'$ the function $\langle A_0(s)u(s), x'\rangle$ is measurable in $s\in[0, T]$, since $\langle \mathrm{w}(s), f_\lambda(s)\rangle$ is measurable.

From the separability of X and Pettis's theorem we get the strong measurability of $A_0(s)u(s)$. Therefore the equality

$$\langle u(t) - u(0), x'\rangle = \int_0^t\langle A_0(s)u(s), x'\rangle ds \quad\text{for any}\quad x'\in X'$$

implies

$$u(t) = u_0 + \int_0^t A_0(s)u(s)ds\ .$$

REMARK. Suppose X, Y are reflexive Banach spaces. From Proposition 3.1 there exist subsequences $\{u^{\varepsilon_n}(t_k);\ \varepsilon_n\downarrow 0\}\subset\{u^\varepsilon(t_k)\}$ and $\{v^{\varepsilon_n}(t_k);\ \varepsilon_n\downarrow 0\}\subset\{v^\varepsilon(t_k)\}$ such that they are weakly convergent sequences in Y. From $Y\subset D(A(t))$ and the closed graph theorem $A(t)$ is continuous from Y to X. Therefore we can prove our theorem under weaker assumptions without Proposition 4.2.

6. Application to wave equations. As a simple application of preceding results, we consider the mixed problem for hyperbolic equations of second order:

$$
\text{(M.P)} \quad \begin{cases} \dfrac{\partial^2}{\partial t^2}u(t, x) = \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(t, x)\dfrac{\partial}{\partial x_j}u(t, x)\Big) , & (t, x) \in [0, T] \times \Omega \\ u(0, x) = u_0(x) , \quad u_t(0, x) = u_1(x) , & x \in \Omega \\ \sum_{i,j=1}^{n} a_{ij}(t, x)\nu_i \dfrac{\partial}{\partial x_j} u(t, x)\Big|_{x \in \partial\Omega} = 0 , & 0 \leqq t \leqq T , \end{cases}
$$

where Ω is a domain such that $\partial\Omega$ is a C^1-manifold and satisfies the following:

Condition (C). For some $r_0 > 0$, the projection $P_{\partial\Omega}$: $\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < r_0\} \to \partial\Omega$ is a well-defined differentiable map.

$(\nu_1, \cdots, \nu_n)$ is the outer unit normal of $\partial\Omega$ at $x \in \partial\Omega$. $\sum_{i,j=1}^{n} (\partial/\partial x_i)(a_{ij}(t, x)\, \partial/\partial x_j)$ is an elliptic operator satisfying

$$
(6.1) \quad \begin{cases} a_{ij},\ \partial a_{ij}/\partial t,\ \partial a_{ij}/\partial x_i,\ \partial^2 a_{ij}/\partial x_i \partial x_j,\ \partial^2 a_{ij}/\partial t \partial x_i \in \mathscr{B}([0, T] \times \Omega) \\ \sup_{i,j,t,x} \Big\{ |a_{ij}(t, x)|, \ \Big|\dfrac{\partial}{\partial t} a_{ij}(t, x)\Big|, \ \Big|\dfrac{\partial}{\partial x_i} a_{ij}(t, x)\Big| \Big\} \leqq M , \\ \sum_{i,j=1}^{n} a_{ij}(t, x)\xi_i\xi_j \geqq d \sum_{i=1}^{n} \xi_i^2 , \quad d > 0 \quad \forall (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n \\ a_{ij}(t, x) = a_{ji}(t, x) \quad \text{for} \quad \forall (t, x) \in [0, T] \times \Omega . \end{cases}
$$

We treat this problem (M.P) as the following evolution equation (A.M.P):

$$
\text{(A.M.P)} \quad \begin{cases} \dfrac{d}{dt} U(t) = A_0(t) U(t) , \quad 0 \leqq t \leqq T \\ U(0) = U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \end{cases}
$$

where

$$
(6.2) \quad X = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} ; u \in H^1(\Omega), \ v \in L^2(\Omega) \right\} , \quad Y = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} ; u \in H^2(\Omega), \ v \in H^1(\Omega) \right\} ,
$$

$$
(6.3) \quad A(t) = \begin{bmatrix} 0 & 1 \\ \sum_{i,j=1}^{n} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(t, x)\dfrac{\partial}{\partial x_j}\Big) & 0 \end{bmatrix} : Y \to X
$$

$$
(6.4) \quad B(t) = \Big[\sum_{i,j=1}^{n} a_{ij}(t, x)\nu_i \dfrac{\partial}{\partial x_j} \quad 0 \Big] : Y \to H^{1/2}(\partial\Omega)
$$

$$
(6.5) \quad Y(t) = \{ U \in Y; B(t) U|_{\partial\Omega} = 0 \}
$$

$$
(6.6) \quad A_0(t) = A(t)|_{Y(t)} .
$$

We introduce norms in the spaces X and Y as follows:

$$\| U\|^2 = \| u\|^2_{H^1(\Omega)} + \| v\|^2_{L^2(\Omega)} \quad \text{for} \quad U = \begin{pmatrix} u \\ v \end{pmatrix} \in X \tag{6.7}$$

$$||| U|||^2 = \| u\|^2_{H^2(\Omega)} + \| v\|^2_{H^1(\Omega)} \quad \text{for} \quad U = \begin{pmatrix} u \\ v \end{pmatrix} \in Y \tag{6.8}$$

$$\begin{aligned} \| U\|^2_t = \exp(-Mtd^{-1})\Big[\sum_{i,j=1}^{n} \Big(a_{ij}(t, x)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big)_{L^2(\Omega)} \\ + (u, u)_{L^2(\Omega)} + (v, v)_{L(\Omega)}\Big] \quad \text{for} \quad U = \begin{pmatrix} u \\ v \end{pmatrix} \in X \end{aligned} \tag{6.9}$$

$$||| U|||^2_t = \exp(-Mtd^{-2})[\| A(t)U\|^2_t + \| U\|^2_t] \quad \text{for} \quad U = \begin{pmatrix} u \\ v \end{pmatrix} \in Y\,. \tag{6.10}$$

By (6.1), a_{ij} and $(\partial/\partial x_i)a_{ij}$ are Lipschitz continuous in $t \in [0, T]$ in the following sense:

$$\begin{cases} \| a_{ij}(t, x)u - a_{ij}(s, x)u\|_{L^2(\Omega)} \leqq C|t-s|\cdot\| u\|_{L^2(\Omega)} \\ \Big\| \dfrac{\partial}{\partial x_i}a_{ij}(t, x)u - \dfrac{\partial}{\partial x_i}a_{ij}(s, x)u\Big\|_{L^2(\Omega)} \leqq C|t-s|\cdot\| u\|_{L^2(\Omega)}\,. \end{cases} \tag{6.11}$$

As is easily seen, our norms in (6.8) and (6.9) satisfy the assumptions (A.2) and (A.3).

In order to apply our theorem, it is suffices to show that the assumption (A.5) holds.

Let any $t \in [0, T)$ be fixed. There exist families of open sets $\{O_\beta\}_{\beta=1}^\infty$ and homeomorphic mappings $\{\Psi_\beta^{-1}\}_{\beta=1}^\infty$ of C^2-class such that

$$\bigcup_{\beta=1}^{\infty} O_\beta \supset \partial\Omega\,, \quad \begin{aligned} &\Psi_\beta^{-1}: O_\beta \to \{y \in \boldsymbol{R}^n;\ |y| < 1\} \\ &\Psi_\beta^{-1}: O_\beta \cap \Omega \to \{y \in \boldsymbol{R}^n;\ |y| < 1,\ y_1 > 0\} \\ &\Psi_\beta^{-1}: O_\beta \cap \partial\Omega \to \{y \in \boldsymbol{R}^n;\ |y| < 1,\ y_1 = 0\} \end{aligned}$$

$$\Big|\frac{\partial \Psi_\beta}{\partial y}\Big| \leqq \mu \qquad \text{for all} \quad \beta \quad \text{and some} \quad \mu > 0\,.$$

For any local coordinate $y_0' = (y_2^0, \cdots, y_n^0)$ of $O_\beta \cap \partial\Omega$.

We consider the following ordinary differential equation:

$$\begin{cases} \dfrac{d}{dy}X(y, y_0') = -A(t, X)\cdot\vec{\nu} \\ X(0, y_0') = x_0 \equiv \Psi_\beta(0, y_0')\,, \end{cases} \tag{6.12}$$

where $A(t, X) = (a_{ij}(t, X))$ and $-\vec{\nu} = (X - P_{\partial\Omega}X)/\| X - P_{\partial\Omega}X\|$. By the condition C, $\vec{\nu}$ is a differentiable map, and (6.12) has a unique solution

$X(\tilde{y}_1, y_0')$. Then the mapping $X(\tilde{y}_1, y_0') = (x_1, \cdots, x_n) \to y_0 = (y_0, y_0')$ defines another local coorditate of $O_\beta \cap \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < r_0/2\}$. From $a_{ij} \in \mathscr{B}^2(\Omega)$ there exists $\kappa > 0$ such that

$$\frac{1}{\kappa} \geqq \frac{\partial(\tilde{y}_1, y_2^0, \cdots, y_n^0)}{\partial(x_1, x_2, \cdots, x_n)} \geqq \kappa \quad \text{for} \quad \operatorname{dist}(\tilde{y}_1, \partial\Omega) \leqq 2^{-1} r_0 . \tag{6.13}$$

Note that κ does not depend on O_β and t.

We define $e(s, y) = (e_1(s, y), \cdots, e_n(s, y))$ by

$$\sum a_{ik} \nu_k \frac{\partial}{\partial x_j} = \sum e_j \frac{\partial}{\partial y_j} . \tag{6.14}$$

Then we get

$$\begin{cases} e_1(t, y) = 1 \\ e_i(t, y) = 0 , \quad i \geqq 2 \quad \text{for} \quad \operatorname{dist}(y_1, \partial\Omega) \leqq r_0/2 . \end{cases} \tag{6.14$'$}$$

Let $\zeta(y) \in C_0^m(\boldsymbol{R}^1)$ be a function such that

$$0 \leqq \zeta(y_1) \leqq 1 , \quad \zeta(0) = 1 , \quad \zeta(y_1) = 0 \quad \text{for} \quad y_1 \geqq r_0/2 , \quad \zeta'(0) = 0 .$$

Let $\{\xi_\beta\}_{\beta=0}^\infty$ be a partition of unity on $\partial\Omega$ such that

$$\xi_\beta \in C^2(\partial\Omega) , \quad \operatorname{supp} \xi_\beta \subset O_\beta \cap \partial\Omega , \quad \sum_{\beta=0}^\infty \xi_\beta(x) \equiv 1 \quad \text{on} \quad \partial\Omega .$$

For $u \in Y(t)$ set $u_\beta(x) = \zeta(\tilde{y}_1)\xi_\beta(\Psi_\beta(0, y_0'))u(x)$ for $x = X(\tilde{y}_1, y_0')$ and

$$f(y) = \begin{cases} u_\beta(X(y)) & |y| < 1 , \quad y_1 > 0 \\ 0 & |y| \geqq 1 , \quad y_1 > 0 . \end{cases}$$

For simplicity, we denote $u = u_\beta$, $w = w_\beta$ etc. We consider the following matrix

$$Q(s) = \begin{bmatrix} q_1 & 0 & & \\ q_2 & 1 & & 0 \\ \cdots & & \ddots & \\ q_n & 0 & & 1 \end{bmatrix} \tag{6.15}$$

where for positive constats λ, h we set

$$\begin{aligned} q_i &= (e_1(t, y))^{-1} p_i \quad i = 1, 2, \cdots, n \\ p_1 &= e_1(s, y) \\ p_j &= -e_j(t, y) + (h^{-1} + \lambda) e_j(s, y) \quad \text{for} \quad j \geqq 2 . \end{aligned}$$

Then the inverse matrix is given by

$$Q^{-1}(s)=\begin{bmatrix}\tilde q_1 & 0 & & \\ -\tilde q_2 & 1 & & 0\\ \cdots & & \ddots & \\ -\tilde q_n & 0 & & 1\end{bmatrix} \tag{6.16}$$

where $\tilde q_i=(e_1(s,y))^{-1}p_i,\ i=1,2,\cdots,n$.

Set

$$z\equiv\begin{pmatrix}z_1\\ \vdots\\ z_n\end{pmatrix}=Q(s)\begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}, \tag{6.17}$$

$$g(z)=f(Q^{-1}(s)z)\ \ (=f(y))\ , \tag{6.18}$$

$$w_{\lambda,h}(z)=\lambda hg\Big(\frac{z_1}{\lambda},\ z_2,\ \cdots,\ z_n\Big)\quad \lambda>0\ ,\quad h>0\ . \tag{6.19}$$

LEMMA 6.1. *For $u\in Y(t)$ and $w_{\lambda,h}$ as above, we have*

$$\begin{pmatrix}u+w_{\lambda,h}\\ 0\end{pmatrix}\in Y(s)\ .$$

PROOF. We simply denote $w_{\lambda,h}$ by w.

$$\begin{cases} w(y)=\lambda hf\Big(\dfrac{e_1(t,y)}{e_1(s,y)}\dfrac{y_1}{\lambda},\ \tilde y_2+y_2,\ \cdots,\ \tilde y_i+y_i,\ \cdots,\ \tilde y_n+y_n\Big)\\ \text{where}\ \ \tilde y_i=\Big[\dfrac{e_i(t,y)}{e_1(s,y)}-\dfrac{(h^{-1}+\lambda)e_i(s,y)}{e_1(s,y)}\Big]\dfrac{y_1}{\lambda}\ ,\quad i=2,\cdots,n\ .\end{cases} \tag{6.20}$$

So with $f_j=\partial f/\partial y_j$ we have

$$\begin{cases} \dfrac{\partial}{\partial y_1}w=\lambda hf_1\dfrac{\partial}{\partial y_1}\Big[\dfrac{e_1(t,y)}{e_1(s,y)}\dfrac{y_1}{\lambda}\Big]+\sum\limits_{j=2}^{n}\lambda hf_j\dfrac{\partial\tilde y_j}{\partial y_1}\\ \dfrac{\partial}{\partial y_j}w=\lambda hf_1\dfrac{\partial}{\partial y_j}\Big[\dfrac{e_1(t,y)}{e_1(s,y)}\dfrac{y_1}{\lambda}\Big]+\sum\limits_{i=2}^{n}\lambda hf_i\dfrac{\partial\tilde y_i}{\partial y_j}+\lambda hf_j\ ,\quad j=2,\cdots,n\ .\end{cases} \tag{6.21}$$

Using

$$e_1(s,y)\frac{\partial}{\partial y_1}w\Big|_{y_1=0}=\sum_{j=1}^{n}e_j(t,y)hf_j\Big|_{y_1=0}-\sum_{j=2}^{n}e_j(s,y)(f_j+\lambda hf_j)\Big|_{y_1=0}$$

and

$$e_j(s,y)\frac{\partial}{\partial y_j}w\Big|_{y_1=0}=e_j(s,y)\lambda hf_j\Big|_{y_1=0}$$

we get

$$\sum_{l,j=1}^{n}a_{lj}(s,x)\nu_l\frac{\partial}{\partial x_j}[u(x)+w(x)]\Big|_{x\in\partial\Omega} \tag{6.22}$$

$$= \sum_{j=1}^{n} e_i(s, y) f_j(y)\Big|_{y_1=0} + \sum_{j=1}^{n} e_j(s, y) W_j(y)\Big|_{y_1=0}$$

$$= \sum_{j=1}^{n} e_j(s, y) f_j(y)\Big|_{y_1=0} + h\sum_{j=1}^{n} e_j(t, y) f_j(y)\Big|_{y_1=0} - \sum_{j=1}^{n} e_j(s, y) f_j(y)\Big|_{y_1=0}$$

$$= 0 ,$$

by (6.14). Therefore $\begin{pmatrix} u + w_{\lambda,h} \\ 0 \end{pmatrix} \in Y(s)$.

LEMMA 6.2. *For any $\varepsilon > 0$ and $h = s - t > 0$ we obtain*

$$\| w_{\varepsilon^2, s-t} \|_s < \varepsilon h ||| u |||_t \tag{6.23}$$

$$||| w_{\varepsilon^2, s-t} |||_s \leqq \varepsilon^{-1} C_2 h ||| u |||_t \tag{6.24}$$

where $\| w \|_s = \left\| \begin{pmatrix} w \\ 0 \end{pmatrix} \right\|_s , \quad ||| w |||_s = \left|\left|\left| \begin{pmatrix} w \\ 0 \end{pmatrix} \right|\right|\right|_s .$

PROOF. We simply denote $w_{\varepsilon^2, s-t}$ by w. Then with $z_1 = \lambda^{-1} y$, we have

$$\int_{O_\beta} | w(y) |^2 dy = (\lambda h)^2 \int_{O_\beta} | g(z_1, y') |^2 \lambda dz_1 dy' \leqq \lambda^3 h^2 \| g \|^2 \tag{6.25}$$

and

$$\int_{O_\beta} \left| \frac{\partial}{\partial y_1} w(y) \right|^2 dy = \lambda h^2 \int_{O_\beta} \left| \frac{\partial g}{\partial z_1} (z_1, y') \right|^2 dz_1 dy' . \tag{6.26}$$

Set

$$g_1 = \frac{\partial g}{\partial z_1} \qquad g_{11} = \frac{\partial^2 g}{\partial z_1^2} .$$

Then it is easy to see that

$$| g_1(z_1, y') | \leqq \left[\int_0^{z_1} | g_{11}(x, y') |^2 dx \right]^{2^{-1}} + | g_1(0, y') | .$$

Hence, by the Shwarz's inequality

$$\begin{aligned} &\int_{O_\beta} | g_1(z, y') |^2 dz dy' \\ &\leqq 2 \int_0^1 z \int_{y'} \int_0^z | g_{11}(x, y') |^2 dx dy' dz + 2 \int_0^1 \int_{y'} g_1(0, y')^2 dy' dz \\ &\leqq 2 ||| g ||| \int_0^1 z dz + 2 \| g \|_{\partial O_\beta}^2 \int_0^1 1 dz \leqq C ||| g |||^2 , \end{aligned} \tag{6.27}$$

since $\| g \|_{\partial O_\beta} \leqq \| g \|_{H^{1+1/2}(O_\beta)} \leqq ||| g |||$ and $\operatorname{supp} g \subset \{(x, y') \colon |x| \leqq 1\}$. From (6.26) and (6.27) we obtain

$$\int_{O_\beta} \left| \frac{\partial}{\partial y_1} w(y) \right|^2 dy \leqq \lambda h^2 C ||| g |||^2 . \tag{6.28}$$

Combining (6.25) and (6.28) we have (6.23).

$$\int_{O_\beta}\left|\frac{\partial^2}{\partial y_1^2}w(y)\right|^2 dy = \frac{h^2}{\lambda}\int_{O_\beta}|g_{11}(z)|^2 dz dy' = \frac{h^2}{\lambda}|||g|||^2 . \tag{6.29}$$

By (6.13) and $C' \leqq \det Q \leqq C''$, we have $|||g||| \leqq C|||u|||$. Hence from (6.28) and (6.29) follows (6.24).

LEMMA 6.3. *If $h = \varepsilon^2$ and $U \in D(A_0(t))_{Y(t)\to Y(t)}$ then we have*

$$|||U + W_{\lambda,h}|||_{t+h} \leqq (1 + C_3 h)|||[I - hA_0(t)]U|||_t \tag{6.30}$$

where $W_{\lambda,h} = \begin{pmatrix} w_{\lambda,h} \\ 0 \end{pmatrix}$ with $w_{\lambda,h}$ as in (6.19).

PROOF. Set

$$V = (I - hA_0(t))U . \tag{6.31}$$

We simply denote $W_{\lambda,h}$ by W.

We first show that for $s = t + h$ we get

$$|\langle A(s)U, A(s)W\rangle_x| \leqq 2^{-1}|||V - U|||^2 + \hat{C}h|||U|||^2 . \tag{6.32}$$

From (6.3) we have

$$\begin{aligned}\langle A(s)U, A(s)W\rangle_x &= \iint_\Omega \sum \frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum\frac{\partial}{\partial x_l}\left(a_{lm}(s, x)\frac{\partial}{\partial x_m}w\right)dx \\ &= \int_{\partial\Omega}\sum\frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum a_{lm}(s, x)\nu_l\frac{\partial}{\partial x_m}w ds \\ &\quad - \iint_\Omega a_{lm}(s, x)\frac{\partial}{\partial x_l}\left(\frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\frac{\partial}{\partial x_m}w\right)ds .\end{aligned} \tag{6.33}$$

Let I and II be the first and the second terms. From (6.1), (6.23) and (6.31) it follows that

$$\begin{aligned}|\mathrm{II}| &\leqq M|||A(s)U|||\cdot\|W\| \leqq c_0M|||A(t)U|||\cdot\|W\| \\ &\leqq c_0M|||hA_0(t)U|||\sqrt{h}|||U||| = c_0M|||V - U|||\sqrt{h}|||U||| \\ &\leqq 4^{-1}|||V - U|||^2 + M^2c_0^2h|||U|||^2 .\end{aligned} \tag{6.34}$$

From $U + W \in Y(s)$ we have

$$\begin{aligned}\mathrm{I} &= \int_{\partial\Omega}\sum\frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum a_{lm}(s, x)\nu_l\frac{\partial}{\partial x_m}(w + u - u)ds \\ &= -\int_{\partial\Omega}\sum\frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum a_{lm}(s, x)\nu_l\frac{\partial}{\partial x_m}uds .\end{aligned} \tag{6.35}$$

From (6.13) we get

$$\sum_{l,m=1}^n a_{lm}(s, x)\nu_l\frac{\partial}{\partial x_m}u = \sum_{k=1}^n e_k(s, y)\frac{\partial}{\partial y_k}f$$

and

$$\sum_{k=1}^{n} e_k(s, y)\frac{\partial}{\partial y_k}f = a_1 \sum_{k=1}^{n} e_k(t, y)\frac{\partial}{\partial y_k}f + \sum_{j=2}^{n} b_j(y)\frac{\partial}{\partial y_j}f$$

where $a_1 = e_1(s, y)/e_1(t, y)$, $b_j(y) = (-e_1(s, y)e_j(t, y) + e_1(t, y)e_j(s, y))/e_1(t, y)$ for $j = 2, \cdots, n$.
Note that (6.14) and $U \in Y(t)$ imply

$$a_1 = e_1(s, y)\ , \quad b_j(y) = e_j(s, y)\ , \quad \sum_{k=1}^{n} e_k(t, y)\frac{\partial}{\partial y_k}f = 0 \quad \text{for} \quad y \in \partial\Omega\ . \tag{6.36}$$

From (6.13) we get

$$\sum_{k=1}^{n} e_k(s, y)\frac{\partial}{\partial y_k}f - \sum_{k=1}^{n} e_k(t, y)\frac{\partial}{\partial y_k}f = \sum_{l,m=1}^{n} [a_{lm}(s, x) - a_{lm}(t, x)]\nu_l\frac{\partial}{\partial x_m}u. \tag{6.37}$$

Therefore from (6.36), (6.37) and (6.1) it follows that

$$|b_j(y)| \leqq Ch \quad \text{for} \quad y \in \partial\Omega\ . \tag{6.38}$$

Let

$$\tilde{f}(y) = \int_{y_1^0}^{y_1} \tilde{\zeta}(y_1)\frac{\partial}{\partial y_1}f dy_1\ , \tag{6.39}$$

where $\operatorname{dist}((y_1^0, y'), \partial\Omega) \geqq r_0$, $\tilde{\zeta}(y_1) = \zeta(r_0 h^{-1} y_1)$.
From (6.35), (6.38) and (6.39) we get

$$\begin{aligned} |I| &\leqq \left|\int_{\partial\Omega} \sum \frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum_{k=1}^{n} e_k(s, y)\frac{\partial}{\partial y_k}fds\right| \\ &= \left|\int_{\partial\Omega} \sum \frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum_{j=2}^{n} b_j(y)\frac{\partial}{\partial y_j}fds\right| \\ &= \left|\int_{\partial\Omega} \sum \frac{\partial}{\partial x_i}\left(a_{ij}(s, x)\frac{\partial}{\partial x_j}u\right)\sum_{j=2}^{n} b_j(y)\frac{\partial}{\partial y_j}\tilde{f}ds\right| \\ &\leqq C\|hA(s)U\|_{H^{1/2}(\partial\Omega)}\|\tilde{f}\|_{H^{1/2}(\partial\Omega)} \leqq C'|||hA(s)U|||\cdot\|\tilde{f}\|_{H^1(\Omega)}\ . \end{aligned} \tag{6.40}$$

We now estimate $\|\tilde{f}\|_{H^1(\Omega)}$.

$$\begin{aligned} \iint \left(\frac{\partial}{\partial y_1}\tilde{f}\right)^2 dy &= \iint \left(\tilde{\zeta}(y)\frac{\partial}{\partial y_1}f\right)^2 dy \leqq \iint \tilde{\zeta}^2(y)\left[\frac{\partial}{\partial y_1}f\right]^2 dy \\ &\leqq \iint \tilde{\zeta}^2(y) \sup_{0 \leq y_1 \leq h}\left[\frac{\partial}{\partial y_1}f\right]^2 dy\ . \end{aligned} \tag{6.41}$$

$$\left[\frac{\partial}{\partial x_m}u(x_1, x')\right]^2 \leqq 2\int_0^{x_1}\left[\frac{\partial}{\partial x_1}\cdot\frac{\partial}{\partial x_m}u(x_1, x')\right]^2 dx_1 + 2\left[\frac{\partial}{\partial x_m}u(0, x')\right]^2 . \tag{6.42}$$

Therefore from (6.1) we have

$$\sup_{0 \leq y_1 \leq h}\left[\frac{\partial}{\partial y_1}\tilde{f}\right]^2 = \sup_{0 \leq x_1 \leq h'}\left[\sum a_{lm}(s, x_1, x')\nu_l\frac{\partial}{\partial x_m}u(x_1, x')\right]^2 \tag{6.43}$$

$$\leqq 2M\left\{\int_0^{h'}\left[\frac{\partial}{\partial x_m}u(x_1, x')\right]^2 dx_1 + \left[\frac{\partial}{\partial x_m}u(0, x')\right]^2\right\} .$$

From (6.41) and (6.43) we have

$$\iint\left(\frac{\partial}{\partial y_1}\tilde{f}\right)^2 dy \leqq \int\tilde{\zeta}^2(y_1)dy_1\int\sup_{0\leqq y_1\leqq h}\left[\frac{\partial}{\partial y_1}f\right]^2 dy'$$
$$\leqq 2Ch[|||U|||^2 + \|U\|^2_{H^{1+1/2}(\Omega)}] \leqq 4Ch|||U|||^2 .$$

From (6.39) and (6.13) we have

$$\iint\left(\frac{\partial}{\partial y_k}\tilde{f}\right)^2 dy \leqq \iint\left(\int\tilde{\zeta}^2(y_1)dy_1\int_{y_1^0}^{y_1}\left(\frac{\partial}{\partial y_k}\frac{\partial}{\partial y_1}f\right)^2 dy_1\right)dy$$
$$\leqq Ch|||U|||^2 \quad \text{for} \quad k = 2, \cdots, n .$$

Therefore

$$\|\tilde{f}\| \leqq Ch^{1/2}|||U||| . \tag{6.44}$$

From (6.40), (6.44) and (6.31) it follows that

$$|\mathrm{I}| \leqq 4^{-1}|||V - U|||^2 + Ch|||U|||^2 . \tag{6.45}$$

From (6.34) and (6.45) we get (6.32). By definition we have

$$\langle\!\langle U, W\rangle\!\rangle_s = [\langle U, W\rangle_s + \langle A(s)U, A(s)W\rangle_s]\exp(-Msd^{-2}) . \tag{6.46}$$

From $U\in D(A_0(t))_{Y(t)\to Y(t)}$ (6.23) and (6.32) it follows that

$$2\langle\!\langle U, W\rangle\!\rangle_s \leqq 2\|U\|_s\cdot\|W\|_s + 2^{-1}|||V - U|||_s^2 + \hat{C}h|||U|||_s^2 \tag{6.47}$$
$$\leqq C''h|||U|||_t^2 + 2^{-1}|||V - U|||_s^2 .$$

Since $(Au, u) \leqq \omega\|u\|^2$ for a large enough constant C, we have $|||U|||_t^2 + |||U - V|||_t^2 \leqq (1 + Ch)^2|||V|||_t^2$. Then it follows from (6.47) and (6.24) that

$$|||U + W|||_s^2 = |||U|||_s^2 + 2\langle\!\langle U, W\rangle\!\rangle_s + |||W|||_s^2$$
$$\leqq |||U|||_s^2 + C''h|||U|||_t^2 + |||U - V|||_s^2 + C_2^2h^2\varepsilon^{-2}|||U|||_t^2$$
$$= [1 + Ch^2\varepsilon^{-2}]|||U|||_t^2 + C''h|||U|||_s^2 + |||U - V|||_s^2$$
$$\leqq (1 + C_h)^2|||V|||_t^2 + [Ch\varepsilon^{-2} + C'']h|||U|||_t^2$$
$$\leqq (1 + C_h)^2|||V|||_t^2 + 2C_3h|||U|||_t^2$$
$$\leqq [1 + C_3h]^2|||[I - hA_0(t)]U|||_t^2 .$$

q.e.d.

From Lemmas 6.1, 6.2 and 6.3, we can now see that the condition (A.5) holds.

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