Tôhoku Math. Journ.
37 (1985), 367-384.

SPECIAL VALUES OF ZETA FUNCTIONS ASSOCIATED TO CUSP SINGULARITIES

SHOETSU OGATA

(Received May 9, 1984)

0. Introduction. Hirzebruch defined in [5] several geometric invariants for normal isolated singularities, in particular the ϕ-invariants for Hilbert modular cusp singularities. The invariant ϕ is the difference between the L-polynomial and the signature on the desingularization of a compact neighborhood of the singular point. He conjectured that the ϕ-invariant coincides with the invariant w, which was defined by Shimizu [11] as the special value of an L-function. This conjecture was recently proved by Atiyah, Donnelly and Singer [1]. Ehlers [4] defined and computed the ψ-invariant for Hilbert modular cusp singularities, and Satake [9], [10] generalized these to the cusp contributions for certain locally symmetric varieties, i.e., arithmetic varieties of $\boldsymbol{Q}$-rank one. From the dimension formula of Hilbert modular cusp forms, it is conjectured in [6] that the invariants ψ and ϕ coincide.

Here we consider generalized cusp singularities of Tsuchihashi [13]. Generalizing the work of Satake [8], we associate a zeta function to a pair of a nondegenerate open convex cone and a discrete group appearing in the definition of Tsuchihashi's cusp singularity. We show among other things that the special value of the zeta function gives information on the topology of the singularity, namely, the cusp contribution in odd-dimensional cases.

Let N be a free $\boldsymbol{Z}$-module of rank n (>1) and $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Let C be a nondegenerate open convex cone in $N_{\boldsymbol{R}}$ and Γ a subgroup in the group $GL(N) := \mathrm{Aut}_{\boldsymbol{Z}}(N)$ of $\boldsymbol{Z}$-linear automorphisms of N such that C is Γ-invariant, Γ acts on $D := C/\boldsymbol{R}_{>0}$ properly discontinuously and freely, and that D/Γ is compact. Then the semi-direct product $N \cdot \Gamma$ acts on the tube domain $N_{\boldsymbol{R}} + \sqrt{-1}\,C$ in $N_{\boldsymbol{C}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{C}$ properly discontinuously and freely. We get a complex manifold $(N_{\boldsymbol{R}} + \sqrt{-1}\,C)/N \cdot \Gamma$. By adding a point ∞, we can make $\{(N_{\boldsymbol{R}} + \sqrt{-1}\,C)/N \cdot \Gamma\} \cup \{\infty\}$ a complex analytic space. Tsuchihashi's cusp singularity is this point ∞. Let X be the exceptional set of a resolution of this singularity. Then $X = X_1 + \cdots + X_l$ is a toric divisor, that is, X has only normal crossings as singularities, each irreducible component X_j of X is isomorphic to an $(n-1)$-dimensional compact torus embedding and the union $\cup_{k\neq j} X_j \cap X_k$ of the double locus $X_j \cap X_k$ on X_j coincides with the closure of the union of all the codimension one orbits on X_j.

On the other hand, let N^* be the dual $\boldsymbol{Z}$-module of N with the pairing $\langle\ ,\ \rangle: N \times N^* \to \boldsymbol{Z}$. Let dx and dx^* be the Lebesgue measures on $N_{\boldsymbol{R}}$ and $N_{\boldsymbol{R}}^*$, respectively, so normalized that the volume of the parallelotope spanned by a basis of N and N^* is one. Let C^* be the dual cone of C defined by $C^* := \{x^* \text{ in } N_{\boldsymbol{R}}^*; \langle x, x^*\rangle > 0 \text{ for all } x \text{ in } \bar{C}\setminus\{0\}\}$. The characteristic function of C is

$$\phi_C(x) := \int_{C^*} \exp(-\langle x, x^*\rangle) dx^*$$

defined by Vinberg [14]. Then we define the zeta function associated to (C, Γ) by

$$Z(C, \Gamma; s) := \sum_{x \in (N\cap C)/\Gamma} \phi_C(x)^s \quad \text{for} \quad \operatorname{Re} s > 1 .$$

We show in Theorem 2.1 that the function $Z(C, \Gamma; s)$ can be continued meromorphically to the whole complex plane. As we briefly mention below, it is expected that the value of $Z(C, \Gamma; s)$ at $s = 0$ gives topological information on Tsuchihashi's cusp singularity corresponding to (C, Γ). It is indeed the case when the dimension n is either two or odd.

The zeta function associated to a *self-dual homogeneous* cone was studied by Shintani [12] in the case of simplicial cones, by Zagier [15] in the case of two dimensional cones and by Satake [8] in general. We generalize the method employed by Zagier [15] and suggested to the author by Zagier himself, but in a way different from that in Cassou-Noguès [3], since (1) C may not be self-dual homogeneous (cf. Tsuchihashi [13, §5]) and (2) $1/\phi_C(x)$ may not be a polynomial.

We now explain which topological information $Z(C, \Gamma; s)$ is expected to give. We consider the following situation: Let $\mathscr{D}$ be a tube domain $\boldsymbol{R}^n + \sqrt{-1}\, C$ such that C is a self-dual homogeneous open convex cone. Let Γ_0 be an arithmetic group acting on $\mathscr{D}$. Assume, for simplicity, that Γ_0 is torsion-free, that the quotient space $\mathscr{D}/\Gamma_0$ is smooth and that there exists a compactification $Y := (\mathscr{D}/\Gamma_0) \cup \{p_1, \cdots, p_h\}$ of $\mathscr{D}/\Gamma_0$ by addition of a finite number of points $p_1, \cdots, p_h$ called cusps. Mumford gave a method to construct a smooth compactification $Y' := (\mathscr{D}/\Gamma_0) \cup D^{(1)} \cup \cdots \cup D^{(h)}$ by using toroidal embeddings. Here $D := \cup D^{(i)}$ is a divisor with only simple normal crossings on Y' and $D^{(i)}$ are connected components of D. Let $D^{(i)} = \cup_{j\in I^{(i)}} D_j$ be the decomposition of $D^{(i)}$ into the union of irreducible components. Let δ_j in $H^2(Y'; \boldsymbol{Z})$ be the cohomology class determined by D_j. Then the difference $\chi_\infty := \chi(Y') - \bar{\chi}(Y', D)$ of the arithmetic genus of Y' and logarithmic arithmetic genus of (Y', D) depends only on $\mathscr{D}$ and Γ_0 and is called the cusp contribution. We can calculate $\bar{\chi}(Y', D)$ by using the proportionality theorem of Mumford.

By Satake [9], [10], the cusp contribution can be computed as follows:

$$\chi_\infty = \kappa_n\left(\prod_{j\in I}\frac{\delta_j}{1-e^{-\delta_j}}\right),$$

where $I := I^{(1)} \cup \cdots \cup I^{(h)}$ and κ_n denotes $\kappa_n(b) = b_n[Y']$ for the degree $2n$ part b_n of b in $\bigoplus_{i=0}^n H^{2i}(Y'; \boldsymbol{Q})$. It is natural to define the contribution of each cusp p_i by

$$\chi_\infty(p_i) := \kappa_n\left(\prod_{j\in I^{(i)}}\frac{\delta_j}{1-e^{-\delta_j}}\right).$$

Then we have $\chi_\infty = \chi_\infty(p_1) + \cdots + \chi_\infty(p_h)$.

Satake [9], [10] gave a relation:

$$\chi_\infty(p_i) = 2^{-n}(-1)^{n+1}\sum_{J\in\Phi^{(i)}}(-2)^{n-\#J} + \kappa_n\left(\prod_{j\in I^{(i)}}\frac{\delta^j}{2}\frac{1+e^{-\delta_j}}{1-e^{-\delta_j}}\right),$$

where $\Phi^{(i)} := \{J\subset I^{(i)}; J \neq \emptyset \text{ and } D_J := \bigcap_{j\in J} D_j \neq \emptyset\}$. For J in $\Phi^{(i)}$ with $r = \#J$ we denote by $\mathrm{sgn}(D_J)$ the signature of the $(n-r)$-dimensional manifold D_J, i.e., the signature of the bilinear form on $H^{n-r}(D_J; \boldsymbol{R})$ defined by cup product. Then we also have (see Lemma 3.15 in Section 3)

$$(-1)^{n+1}\sum_{J\in\Phi^{(i)}}(-2)^{n-\#J} = \sum_{J\in\Phi^{(i)}}\mathrm{sgn}(D_J).$$

Moreover, if n is odd, then we have

$$\chi_\infty(p_i) = e(\Phi^{(i)})/2,$$

where $e(\Phi^{(i)})$ is the Euler number of the dual graph of $D^{(i)}$, which is an $(n-1)$-dimensional simplicial complex and which is determined by the exceptional divisor $D^{(i)}$.

Using the same formula we can define the cusp contribution of Tsuchihashi's cusp singularity. Namely let (V, p) be Tsuchihashi's cusp siugularity of dimension n, $\pi: W \to V$ a resolution of the singularity and $\bigcup_{j\in I} X_j$ the decomposition of the exceptional set $\pi^{-1}(p)$ into the union of irreducible components. Let δ_j in $H_c^2(W; \boldsymbol{Z})$ be the cohomology class with compact support determined by X_j. Then we define the cusp contribution $\chi_\infty(p)$ by

$$\chi_\infty(p) := \kappa_n\left(\prod_{j\in I}\frac{\delta_j}{1-e^{-\delta_j}}\right).$$

The equalities of Satake also hold in this case:

$$\chi_\infty(p) = 2^{-n} \sum_{J\in\Phi} \operatorname{sgn}(X_J) + \kappa_n\Big(\prod_{j\in I} \frac{\delta_j}{2}\frac{1+e^{-\delta_j}}{1-e^{-\delta_j}}\Big)\,, \quad \text{and}$$

$$\chi_\infty(p) = e(\Phi)/2 \quad \text{if} \quad n \quad \text{is odd}\,,$$

where $\Phi := \{J\subset I; J \neq \varnothing$ and $X_J := \bigcap_{j\in J} X_j \neq \varnothing\}$ (see Lemma 3.15 in Section 3).

Tsuchihashi's cusp singularity in dimension $n = 2$ is a Hilbert modular cusp singularity. The exceptional divisor of its minimal resolution is a cycle of rational curves. $Z(C, \Gamma; 0)$ is expressed in terms of the cusp contribution as

$$Z(C, \Gamma; 0) = -\frac{1}{12}\sum_{j\in I}(\delta_j^2 + 3) = -\chi_\infty(p)\,.$$

When n is odd, we show in Theorem 2.3 that $Z(C, \Gamma; 0)$ coincides with $-1/2$ times the Euler number of the dual graph, hence we have

$$Z(C, \Gamma; 0) = -\chi_\infty(p)\,.$$

Even if $n \geqq 4$ is even, we can expect

$$Z(C, \Gamma; 0) = -\chi_\infty(p)$$

to hold. Unfortunately, we could not prove this, but can express $Z(C, \Gamma; 0)$ in a form very similar to Satake's formula.

The author would like to thank Professors I. Satake, T. Oda and M.-N. Ishida for their useful advice and encouragement.

1. Tsuchihashi's cusp singularities. Generalizing Hilbert modular cusp singularities, Tsuchihashi [13] introduced new normal isolated singularities. These are the singular points appearing at infinity of the quotients of tube domains. In this section we explain these generalized cusp singularities of Tsuchihashi.

Let N be a free $\boldsymbol{Z}$-module of rank n and N^* the dual $\boldsymbol{Z}$-module of N with the natural pairing $\langle\ ,\ \rangle: N \times N^* \to \boldsymbol{Z}$. Consider a pair (C, Γ) consisting of an open convex cone in $N_{\boldsymbol{R}} := N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$ whose closure $\bar{C}$ contains no line in $N_{\boldsymbol{R}}$ and a subgroup Γ in $GL(N) := \mathrm{Aut}_{\boldsymbol{Z}}(N)$ with the following properties:

(1) C is Γ-invariant.

(2) Γ acts on $D := C/\boldsymbol{R}_{>0}$ properly discontinuously and freely.

(3) The quotient D/Γ is compact.

Then there exists a rational partial polyhedral decomposition (r.p.p. decomposition for short) Σ satisfying the following conditions:

(i) $C = \bigcup_{\sigma\in\Sigma\setminus\{0\}} \mathrm{int}(\sigma)$.

(ii) For any compact set K contained in C, the set $\{\sigma\in\Sigma; \sigma\cap K \neq \varnothing\}$

is finite.

(iii) Σ is Γ-invariant.

(iv) Γ acts on $\Sigma\setminus\{\underline{0}\}$ freely.

(v) $(\Sigma\setminus\{\underline{0}\})/\Gamma$ is a finite set.

Here we denote by $\underline{0}$ the cone $\{0\}$ and by $\mathrm{int}(\sigma)$ the relative interior of σ.

Tsuchihashi [13] gave an r.p.p. decomposition Σ using the convex hull θ of $C\cap N$. Taking a Γ-invariant subdivision of Σ, if necessary, we may also assume the following:

(vi) For every σ and τ in Σ, there exists at most one g in Γ with $(g\sigma)\cap\tau\neq\underline{0}$.

(vii) Σ is nonsingular, i.e., for each σ in Σ there exists a $\boldsymbol{Z}$-basis $\{u_1,\cdots,u_n\}$ of N and $r\leqq n$ such that σ is spanned by $u_1,\cdots,u_r$, namely, $\sigma=\boldsymbol{R}_{\geqq 0}u_1+\cdots+\boldsymbol{R}_{\geqq 0}u_r$. In the following, we assume that Σ satisfies the conditions (i)-(vii).

Tsuchihashi [13] associated to such a pair (C,Γ) a cusp singularity $\mathrm{Cusp}(C,\Gamma)$ as follows: $T_N:=N\otimes_{\boldsymbol{Z}}\boldsymbol{C}^\times$ is an algebraic torus. Since Σ is nonsingular, the corresponding torus embedding $Z:=T_N emb(\Sigma)$ is a nonsingular complex analytic space. Since Σ is Γ-invariant, Γ also acts on Z. We define a homomorphism

$$\mathrm{ord}\colon T_N=N\otimes_{\boldsymbol{Z}}\boldsymbol{C}^\times\to N_{\boldsymbol{R}}=N\otimes_{\boldsymbol{Z}}\boldsymbol{R}$$

by $1_N\otimes(-\log|\ \ |)$. $\tilde{W}:=\mathrm{ord}^{-1}(C)\cup(Z\setminus T_N)$ in Z is a Γ-invariant open set and Γ acts on $\tilde{W}$ properly discontinuously and freely. $\tilde{Y}:=Z\setminus T_N$ in $\tilde{W}$ is also Γ-invariant. We set $W:=\tilde{W}/\Gamma$ and $Y:=\tilde{Y}/\Gamma$. By construction they have the following properties:

(a) $Y=X_1+\cdots+X_l$ is a toric divisor with only simple normal crossings, that is, Y has only simple normal crossings as singularities, each irreducible component X_j of Y is isomorphic to a nonsingular $(n-1)$-dimensional compact torus embedding and the union $\cup_{k\neq j}(X_j\cap X_k)$ of the double locus $X_j\cap X_k$ on X_j coincides with the closure of the union of all the codimension one orbits on X_j. We define Φ to be the set of subsets $J\neq\emptyset$ of $\{1,\cdots,l\}$ such that the intersection $X_J:=\cap_{j\in J}X_j$ is nonempty.

(b) For each J in Φ, the analytic space X_J is isomorphic to a compact nonsingular torus embedding of dimension $n-{}^\sharp J$. We can choose a complete set of representatives $\{\sigma(J)\in\Sigma;J\in\Phi\}$ of $\Sigma\setminus\{\underline{0}\}$ modulo Γ so that for J in Φ the closure in $\tilde{W}$ of the torus orbit corresponding to $\sigma(J)$ is a compact nonsingular torus embedding isomorphic to X_J. Indeed by the theory of torus embeddings, we have a canonical bijection between Σ and the set of torus orbits in $T_N emb(\Sigma)$. For each σ in Σ, let us denote by $V(\sigma)$ the closure of the orbit corresponding to σ. $\dim V(\sigma)=n-\dim\sigma$. If $\sigma\neq\{0\}$, then $V(\sigma)$ is contained in $\tilde{Y}=T_N emb(\Sigma)\setminus T_N\subset\tilde{W}$.

We have $Y = \tilde{Y}/\Gamma = X_1 \cup \cdots \cup X_l$. Hence for each J in Φ, there exists an $(n - {}^{\#}J)$-dimensional cone $\sigma(J)$ in Σ such that the inverse image of X_J under the projection $\tilde{W} \to W$ is the disjoint union of $V(g\sigma(J))$ with g running through Γ. Obviously, $\Sigma \setminus \{\underline{0}\} = \{g\sigma(J); g \in \Gamma, J \in \Phi\}$.

(c) Φ gives a triangulation of D/Γ by the projection $C \to D = C/\boldsymbol{R}_{>0}$. Namely, each k-dimensional cone σ in Σ gives rise to a $(k-1)$-simplex $(\sigma \setminus \{0\})/\boldsymbol{R}_{>0}$ in $D = C/\boldsymbol{R}_{>0}$ and we get a triangulation $\{(\sigma \setminus \{0\})/\boldsymbol{R}_{>0}; \sigma \in \Sigma \setminus \{\underline{0}\}\}$ of D. Thus by the projection $D \to D/\Gamma$ the complete set of representatives $\{\sigma(J)/\boldsymbol{R}_{>0}; J \in \Phi\}$ of simplices modulo Γ gives rise to a triangulation of D/Γ.

We obtain Tsuchihashi's cusp singularity $\mathrm{Cusp}(C, \Gamma)$ by contracting Y to a point p.

$$\begin{array}{ccc} W & \supset & Y \\ \downarrow & & \downarrow \\ V & \ni & p \end{array}$$

The germ of the analytic space (V, p) depends only on the pair (C, Γ) and is independent of the choice of Σ.

2. Main theorems. For an open convex cone C in $N_{\boldsymbol{R}}$, we define the dual cone C^* in $N_{\boldsymbol{R}}^*$ by

$$C^* := \{x^* \in N_{\boldsymbol{R}}^*; \langle x, x^* \rangle > 0 \text{ for all } x \text{ in } \bar{C} \setminus \{0\}\} .$$

Denote by dx^* the Lebesgue measure on $N_{\boldsymbol{R}}^*$ normalized so that the volume of the parallelotope spanned by a $\boldsymbol{Z}$-basis of N^* is one. Then

$$\phi_C(x) := \int_{C^*} \exp(-\langle x, x^* \rangle) dx^*$$

is the characteristic function of C defined by Vinberg [14]. The value $\phi_C(x)$ is positive for every point x in C and goes to infinity as x approaches the boundary of C. Moreover, if g is a linear transformation of $N_{\boldsymbol{R}}$ preserving C, then we have $\phi_C(gx) = |\det g|^{-1} \phi_C(x)$.

DEFINITION. For a pair (C, Γ) as in Section 1, we define the zeta function associated to (C, Γ) by

$$Z(C, \Gamma; s) := \sum_{x \in (N \cap C)/\Gamma} \phi_C(x)^s \quad \text{for} \quad \mathrm{Re}\, s > 1 .$$

REMARK. When the cone C is homogeneous and self-dual, this function coincides with that defined by Satake [8].

To see that it is well-defined, we need only to prove that for each σ in $\Sigma \setminus \{\underline{0}\}$, the partial zeta function

$$Z(\sigma, s) := \sum_{x \in N \cap \mathrm{int}(\sigma)} \phi_C(x)^s$$

converges absolutely for $\mathrm{Re}\, s > 1$. Indeed,

$$C = \bigcup_{\sigma \in \Sigma \setminus \{\underline{0}\}} \mathrm{int}(\sigma)$$

and $\Sigma \setminus \{\underline{0}\}$ modulo Γ is finite. Since $\{\sigma(J); J \in \Phi\}$ is a complete set of representatives of $\Sigma \setminus \{\underline{0}\}$ modulo Γ, we have a finite sum

$$Z(C, \Gamma; s) = \sum_{J \in \Phi} Z(\sigma(J), s) .$$

Choose an open simplicial cone $\Delta = \boldsymbol{R}_{>0}v_1 + \cdots + \boldsymbol{R}_{>0}v_n$ contained in C and containing σ. $v_1, \cdots, v_n$ are points in $\bar{C}$ and form a basis of $N_{\boldsymbol{R}}$. For any x in $N \cap \mathrm{int}(\sigma)$ we may write $x = a_1v_1 + \cdots + a_nv_n$ with positive real numbers a_i. Since C^* is contained in Δ^*, we have

$$\phi_C(x) \leqq \phi_\Delta(x) = \frac{K}{a_1 \cdots a_n}$$

for a positive constant K. If $\dim \sigma = r$, there exist $u_1, \cdots, u_r$ in $N \cap C$ so that $\sigma = \boldsymbol{R}_{\geqq 0}u_1 + \cdots + \boldsymbol{R}_{\geqq 0}u_r$. Let $u_i = \sum_{j=1}^n u_i^{(j)}v_j$. Then we have

$$\begin{aligned} |Z(\sigma, s)| &\leqq \sum_{x \in N \cap \mathrm{int}(\sigma)} |\phi_C(x)^s| \\ &\leqq \sum_{m_1, \cdots, m_r = 1}^{\infty} K^{\mathrm{Re}\, s} \prod_{j=1}^{n} (m_1u_1^{(j)} + \cdots + m_ru_r^{(j)})^{-\mathrm{Re}\, s} . \end{aligned}$$

The right hand side converges absolutely for $\mathrm{Re}\, s > r/n$, since it has exactly the same form as that appearing in the Hilbert modular case (cf. Shintani [12, Proposition 1]). Therefore $Z(C, \Gamma; s)$ is well-defined.

In the next section, we shall prove the following theorems.

THEOREM 2.1. *The zeta function $Z(C, \Gamma; s)$ associated to (C, Γ) can be continued meromorphically to the whole complex plane.*

For any positive integer k, let $\Sigma(k) := \{\sigma \in \Sigma; \dim \sigma = k\}$ and $\sigma(1) := \{\tau; \tau \prec \sigma \text{ and } \dim \tau = 1\}$ for σ in Σ. For any ρ in $\Sigma(1)$, we denote by ∂_ρ the derivation in the direction ρ, that is, for a function $F(x)$ on C,

$$\partial_\rho F(x) := \lim_{h \to 0} \{F(x + hu(\rho)) - F(x)\}/h ,$$

where $u(\rho)$ is the unique primitive element in $\rho \cap N$. For any compact complex manifold M of dimension r, we denote by $\mathrm{sgn}(M)$ the signature of the bilinear form on $H^r(M; \boldsymbol{R})$ defined by cup product $H^r(M; \boldsymbol{R}) \times H^r(M; \boldsymbol{R}) \to H^{2r}(M; \boldsymbol{R}) \cong \boldsymbol{R}$.

THEOREM 2.2. *For any integer $\nu \geqq 2$, we have*

$$
\begin{aligned}
Z(C, \Gamma; 0) &= \sum_{\tau \in (\Sigma \setminus \{\underline{0}\})/\Gamma} \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau \\
&= -2^{-n} \sum_{J \in \Phi} \operatorname{sgn}(X_J) + \sum_{J \in \Phi} \left(-\frac{1}{2} \right)^{\dim(X_J)} \operatorname{sgn}(X_J) \\
&\quad \times \int_{\sigma(J)} \left[\prod_{\rho \in \sigma(J)(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \sigma(J)} G_\nu(x) dx_{\sigma(J)} ,
\end{aligned}
$$

where $G_\nu(x) := \exp(-\phi_C(x)^{-\nu})$, dx_τ is the Lebesgue measure on the linear subspace $\tau + (-\tau)$ of $N_{\boldsymbol{R}}$ normalized so that for a $\boldsymbol{Z}$-basis $\{u_1, \cdots, u_n\}$ of N with $\tau = \boldsymbol{R}_{\geqq 0} u_1 + \cdots + \boldsymbol{R}_{\geqq 0} u_r$ the volume of the parallelotope spanned by $\{u_1, \cdots, u_r\}$ is one, and $\{\sigma(J); J \in \Phi\}$ is a complete set of representative of $(\Sigma \setminus \{\underline{0}\})$ modulo Γ as in Section 1. We denote by

$$
\left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{1 - e^{-\partial_\rho}} \right]_k
$$

the total degree k part of the formal power series expansion of

$$
\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{1 - e^{-\partial_\rho}} ,
$$

regarded as a differential operator of order k acting on the function $G_\nu(x)$.

THEOREM 2.3. *When n is odd, we have*

$$
Z(C, \Gamma; 0) = -2^{-1} e(D/\Gamma) ,
$$

where $e(D/\Gamma)$ is the Euler number of D/Γ.

3. Proof of the theorems. For the proof of Theorem 2.1, it is enough to show that $Z(\sigma, s) = \sum_{x \in N \cap \operatorname{int}(\sigma)} \phi_C(x)^s$ can be continued meromorphically to the whole complex plane for each σ in $\Sigma \setminus \{\underline{0}\}$. In fact, we show the following in Proposition 3.7:

For any positive integer ν, the function $Z_\nu(\sigma, s) := Z(\sigma, \nu s)$ can be continued meromorphically to the half plane

$$
\operatorname{Re} s > -1 + 1/\nu .
$$

Thus $Z(\sigma, s)$ can be continued meromorphically to the half plane $\operatorname{Re} s > -\nu + 1$ for any positive integer ν, hence to the whole complex plane. Moreover, using the complete set of representatives $\{\sigma(J); J \in \Phi\}$ of $\Sigma \setminus \{\underline{0}\}$ modulo Γ, we have a finite sum

$$
Z(C, \Gamma; 0) = \sum_{J \in \Phi} Z(\sigma(J), 0) .
$$

To prove Proposition 3.7, we use a general result on the special values of Dirichlet series. The following proposition is a slight generalization of that in Zagier [15] to the case where $f(t)$ is expanded asymptotically in terms of fractional powers of t.

PROPOSITION 3.1. *Suppose a Dirichlet series*

$$\psi(s) := \sum_{k=0}^{\infty} a_k \lambda_k^{-s}$$

converges absolutely for $\mathrm{Re}\, s > 1$, *where* $\{\lambda_k\}_{k=0}^{\infty}$ *is a sequence of positive real numbers which diverges to infinity, and let*

$$f(t) := \sum_{k=0}^{\infty} a_k \exp(-\lambda_k t)$$

be the corresponding exponential series defined for $t > 0$. *If* $f(t)$ *has an asymptotic expansion at* $t = 0$ *of the form*

$$f(t) = \sum_{k=-l}^{K-1} b_k t^{k/l} + O(t^{K/l}) \quad \text{as} \quad t \to 0 \, ,$$

for positive integers K *and* l, *then* $\psi(s)$ *admits a meromorphic continuation to the half plane* $\mathrm{Re}\, s > -K/l$ *and is holomorphic at* $s = 0$ *with* $\psi(0) = b_0$.

PROOF. For each $t > 0$, there exists a positive number y_0 with $e^{ty} > y^{1+t}$ for all $y \geqq y_0$. Thus using the gamma function $\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt$, we can write

$$\Gamma(s)\psi(s) = \int_0^\infty f(t) t^{s-1} dt \quad \text{for} \quad \mathrm{Re}\, s > 1 \, .$$

Set

$$I_1(s) = \int_0^1 f(t) t^{s-1} dt \quad \text{and} \quad I_2(s) = \int_1^\infty f(t) t^{s-1} dt \, .$$

Since the absolute value of $f(t) = O(\exp(-\lambda_0))$ decreases exponentially as $t \to +\infty$, $I_2(s)$ converges absolutely for all s and uniformly on compact sets. Thus $I_2(s)$ is an entire function in s.

On the other hand, for $\mathrm{Re}\, s > 1$, we have

$$\int_0^1 \Big(\sum_{k<K} b_k t^{k/l}\Big) t^{s-1} dt = \Big[\sum_{k<K} b_k t^{s+k/l} (s + k/l)^{-1}\Big]_0^1 = \sum_{k<K} b_k (s + k/l)^{-1}$$

and

$$I_1(s) = \sum_{k<K} b_k (s + k/l)^{-1} + \int_0^1 \Big(f(t) - \sum_{k<K} b_k t^{k/l}\Big) t^{s-1} dt \, .$$

Here the integral on the right hand side converges absolutely for the half plane $\operatorname{Re} s > -K/l$ and uniformly on compact sets in the half plane by assumption, hence is holomorphic there. Therefore the function $\Gamma(s)\psi(s) - \sum_{k<K} b_k(s + k/l)^{-1}$ has a holomorphic continuation to the half plane $\operatorname{Re} s > -K/l$. Since $1/\Gamma(s)$ is an entire function, $\psi(s)$ can be continued meromorphically to the half plane $\operatorname{Re} s > -K/l$.

Finally $\Gamma(s)\psi(s)$ has a simple pole at $s = 0$ with the residue b_0 and $\Gamma(s)$ has a simple pole at $s = 0$ with the residue 1. Therefore $\psi(s)$ is holomorphic at $s = 0$ with $\psi(0) = b_0$. q.e.d.

In order to apply Proposition 3.1 to the proof of Theorems 2.1 and 2.2, we need an asymptotic expansion of $\sum_{x \in N \cap \operatorname{int}(\sigma)} \exp(-\phi_C(x)^{-\nu} t)$ at $t = 0$ for each σ in $\Sigma \setminus \{\underline{0}\}$ and any positive integer ν. We use the Bernoulli polynomials $B_k(x)$ defined as follows:

DEFINITION.

$$\sum_{k=0}^{\infty} B_k(x) t^k / k! = \frac{t e^{tx}}{e^t - 1} .$$

$B_k := B_k(0)$ are ordinary Bernoulli numbers. These polynomials satisfy the following properties:

$$(d/dx) B_k(x) = k B_{k-1}(x) ,$$

$$B_k(x + 1) = B_k(x) + k x^{k-1} \quad \text{for} \quad k \geqq 1 .$$

For a real number x, denote by $[x]$ the Gauss symbol.

LEMMA 3.2 (Euler-Maclaurin summation formula, see, for instance, Bourbaki [2]). *For any positive integers L, K and any C^K-function $g(x)$ on $[0, L]$, we have*

$$\sum_{l=1}^{L} g(l) = \int_0^L g(x) dx + \sum_{k=0}^{K-1} \frac{(-1)^k B_{k+1}}{(k+1)!} (g^{(k)}(0) - g^{(k)}(L))$$
$$- \frac{(-1)^K}{K!} \int_0^L B_K(x - [x]) g^{(K)}(x) dx ,$$

where $g^{(k)}(x)$ is the k-th derivative of $g(x)$.

REMARK. Lemma 3.2 is also true for a function $g(x)$ in $C^K(0, L]$ such that the derivatives $g^{(k)}(x)$ of orders up to $K - 1$ have limits at $x = 0$ and that $g^{(K)}(x)$ is bounded.

Set

$$\beta_k := (-1)^k B_k / k!$$

for every positive integer k. Thus $\beta_0 = 1$ and

$$\frac{t}{1-e^{-t}} = \sum_{k=0}^{\infty} \beta_k t^k .$$

We say that a continuous function $f(x)$ on $[0, \infty)^n$ tends to 0 rapidly at infinity if for any positive integer m the function $(1 + \|x\|)^m f(x)$ is bounded for $\|x\|$ sufficiently large, where $\|x\|$ denotes a fixed Euclidean norm of x in $\boldsymbol{R}^n$. Using Lemma 3.2, we easily have the following:

COROLLARY 3.3 (Zagier [15]). *Let $g(x)$ be a C^{K+1}-function on $(0, \infty)$ such that its derivatives of orders up to $K+1$ tend to 0 rapidly at infinity, while its derivatives of orders up to K have limits at $x = 0$ and the $(K+1)$-th derivative is bounded. Then the function*

$$f(t) := \sum_{l=1}^{\infty} g(tl) \quad \text{for} \quad t > 0$$

has an asymptotic expansion at $t = 0$ of the form

$$\begin{aligned} f(t) &= t^{-1} \int_0^{\infty} g(x)dx - \sum_{k=1}^{K} \beta_k g^{(k-1)}(0) t^{k-1} + O(t^K) \\ &= \sum_{k=0}^{K} t^{k-1} \int_0^{\infty} \beta_k g^{(k)}(x)dx + O(t^K) \quad \text{as} \quad t \to 0 . \end{aligned}$$

Applying Corollary 3.3 repeatedly to each variable, we have the following for functions $G(x) = G(x_1, \cdots, x_r)$ of r variables:

PROPOSITION 3.4. *Let K be a positive integer and assume that $G(x)$ is a C^{K+1}-function on $[0, \infty)^r \setminus \{(0, \cdots, 0)\}$ such that its partial derivatives of total orders up to $K+1$ tend to 0 rapidly at infinity, while its partial derivatives of total orders up to K have limits as x goes to the origin and its partial derivatives of total order $K+1$ are bounded. Then we have the following asymptotic expansion at $t = 0$:*

$$\begin{aligned} &\sum_{m_1, \cdots, m_r = 1}^{\infty} G(tm_1, \cdots, tm_r) \\ &\quad = \sum_{0 \leq |k| \leq K} t^{|k|-r} \int_{(\boldsymbol{R}_{\geq 0})^r} \beta_k (\partial/\partial x)^k G(x)dx + O(t^{K+1-r}) \quad \text{as} \quad t \to 0 , \end{aligned}$$

where $k := (k_1, \cdots, k_r) \in (\boldsymbol{Z}_{\geq 0})^r$, $|k| := k_1 + \cdots + k_r$, $\beta_k := \beta_{k_1} \cdots \beta_{k_r}$ and $(\partial/\partial x)^k := (\partial/\partial x_1)^{k_1} \cdots (\partial/\partial x_r)^{k_r}$.

For each r-dimensional cone σ in Σ, we have primitive elements $u_1, \cdots, u_n$ in $N \cap C$ such that $\sigma = \boldsymbol{R}_{\geq 0} u_1 + \cdots + \boldsymbol{R}_{\geq 0} u_r$. Then the function $\phi_C(x_1 u_1 + \cdots + x_r u_r)$ is a C^∞-function on $[0, \infty)^r \setminus \{(0, \cdots, 0)\}$.

LEMMA 3.5. *For each positive integer ν and r-dimensional cone σ in Σ, the function*

$$G_{\sigma,\nu}(x) = G_{\sigma,\nu}(x_1, \cdots, x_r) := \exp(-\phi_C(x_1u_1 + \cdots + x_ru_r)^{-\nu})$$

and its partial derivatives of all orders tend to 0 *rapidly at infinity.*

PROOF. We identify the linear subspace $\sigma + (-\sigma)$ of $N_{\boldsymbol{R}}$ with the Euclidean space $\boldsymbol{R}^r$ by regarding $(u_1, \cdots, u_r)$ as an orthonormal basis. It is clear that $\phi_C(x)^{-m}\exp(-\phi_C(x)^{-\nu})$ is bounded for x in σ with sufficiently large $\|x\|$, since $\phi_C(x)$ goes to 0 at infinity.

Now we choose as in Section 2 an open simplicial cone $\Delta = \boldsymbol{R}_{>0}v_1 + \cdots + \boldsymbol{R}_{>0}v_n$ contained in C and containing σ. Then we have

$$\phi_\Delta(x_1u_1 + \cdots + x_ru_r) = K_1 \prod_{j=1}^{n} (x_1u_1^{(j)} + \cdots + x_ru_r^{(j)})^{-1},$$

for a positive constant K_1. Hence for any $(m_1, \cdots, m_r)$ in $(\boldsymbol{Z}_{\geqq 0})^r$, we have

$$\begin{aligned}
&|(\partial/\partial x_1)^{m_1} \cdots (\partial/\partial x_r)^{m_r}\phi_C(x_1u_1 + \cdots + x_ru_r)| \\
&\quad = \int_{C^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} \exp(-\langle x_1u_1 + \cdots + x_ru_r, x^*\rangle)dx^* \\
&\quad \leqq \int_{\Delta^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} \exp(-\langle x_1u_1 + \cdots + x_ru_r, x^*\rangle)dx^* \\
&\quad = \left|(\partial/\partial x_1)^{m_1} \cdots (\partial/\partial x_r)^{m_r} K_1 \prod_{j=1}^{n} (x_1u_1^{(j)} + \cdots + x_ru_r^{(j)})^{-1}\right|.
\end{aligned}$$

This is obviously bounded at infinity. Note that every partial derivative of $G_{\rho,\nu}(x)$ is represented as the product of $G_{\sigma,\nu}$ with a polynomial in ϕ_C^{-1} and partial derivatives of ϕ_C.

Let k be any positive integer. Then there exist positive constants K_2, K_3 and K_4 such that

$$\begin{aligned}
\|x\|^k &\leqq K_2(x_1 + \cdots + x_r)^k \leqq K_3 \prod_{j=1}^{n} (x_1u_1^{(j)} + \cdots + x_ru_r^{(j)})^k \\
&\leqq K_4\phi_C(x_1u_1 + \cdots + x_ru_r)^{-k}
\end{aligned}$$

for $\|x\|$ sufficiently large. Therefore $G_{\sigma,\nu}$ and its partial derivatives of all orders tend to 0 rapidly at infinity. q.e.d.

To apply Proposition 3.4 to this $G_{\sigma,\nu}$, we need to investigate the behavior of $G_{\sigma,\nu}$ and its partial derivatives near the origin.

LEMMA 3.6. *For each positive integer* ν *and r-dimensional cone* σ *in* Σ, *let* $G_{\sigma,\nu}$ *be as in Lemma* 3.5. *Then* $G_{\sigma,\nu}$ *and its partial derivatives of total orders up to* $n\nu - 1$ *have limits at the origin and the partial derivatives of total order* $n\nu$ *are bounded.*

PROOF. We identify $N_{\boldsymbol{R}}$ with the Euclidean space $\boldsymbol{R}^n$ by regarding $(u_1, \cdots, u_n)$ as an orthonormal basis. Since $\phi_C(v)$ diverges to infinity as v

in C goes to the origin, $G_{\sigma,\nu}(x)$ goes to 1 as $\|x\|$ tends to 0.

Let x' be any point in σ with $\|x'\| = 1$. Then tx' is a point in σ for $t > 0$. For each $m = (m_1, \cdots, m_r)$ in $(\boldsymbol{Z}_{\geq 0})^r$, we write the value of partial derivatives of $\phi_C(x_1u_1 + \cdots + x_ru_r)$ at tx' as

$$((\partial/\partial x)^m \phi_C)(tx') .$$

Then we have

$$\begin{aligned}|((\partial/\partial x)^m \phi_C)(tx')| &= \int_{C^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} e^{-\langle tx', x^*\rangle} dx^* \\ &= t^{-|m|-n} \int_{C^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} e^{-\langle x', x^*\rangle} dx^* .\end{aligned}$$

Let dx_1^* be the Haar measure on the hyperplane $H(x', \alpha) := \{x^* \in N_{\boldsymbol{R}}^*; \langle x', x^*\rangle = \alpha\}$ defined by the following condition: For any continuous function f with compact support on $N_{\boldsymbol{R}}^*$, we have

$$\int_{N_{\boldsymbol{R}}^*} f(x^*) dx^* = \int_{-\infty}^{+\infty} d\alpha \int_{H(x',\alpha)} f(x^*) dx_1^* .$$

Then we have

$$\int_{C^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} e^{-\langle x', x^*\rangle} dx^* = \int_0^\infty e^{-\alpha} d\alpha \int_{H(x',\alpha)\cap C^*} \prod_{j=1}^{r} \langle u_j, x^*\rangle^{m_j} dx_1^* .$$

Since $H(x', \alpha) \cap C^*$ is bounded, the volume $V(x', \alpha)$ of $H(x', \alpha) \cap C^*$ and $M_j(x', \alpha) := \sup\{\langle u_j, x^*\rangle; x^* \in H(x', \alpha) \cap C^*\}$ are fininite for $j = 1, \cdots, r$. Since $H(x', \alpha)$ is obtained from $H(x', 1)$ by the homothety with respect to α, we get $V(x', \nu) = \alpha^{n-1} V(x', 1)$ and $M_j(x', \alpha) = \alpha M_j(x', 1)$. Therefore we have

$$\begin{aligned}|((\partial/\partial x)^m \phi_C)(tx')| &\leqq t^{-|m|-n} V(x', 1) \prod_{j=1}^{r} M_j(x', 1)^{m_j} \int_0^\infty e^{-\alpha} \alpha^{|m|+n-1} d\alpha \\ &= t^{-|m|-n} V(x', 1) \prod_{j=1}^{r} M_j(x', 1)^{m_j} \Gamma(|m| + n) .\end{aligned}$$

Similarly we have

$$\phi_C(tx') = t^{-n} V(x', 1) \int_0^\infty e^{-\alpha} \alpha^{n-1} d\alpha = t^{-n} V(x', 1) \Gamma(n) .$$

Since $\sigma \cap \{x \in N_{\boldsymbol{R}}; \|x\| = 1\}$ is compact, $\sup\{\prod_{j=1}^{r} M(x', 1)^{m_j}; x' \in \sigma \text{ and } \|x'\| = 1\}$ and $\sup\{V(x', 1)^{-1}; x' \in \sigma \text{ and } \|x'\| = 1\}$ are finite. Thus

$$|((\partial/\partial x)^m \phi_C^{-\nu})(tx')| = O(t^{n\nu-|m|}) \quad \text{as} \quad t \to 0 .$$

Therefore the partial derivatives of $G_{\sigma,\nu}(x)$ of total orders up to $n\nu - 1$ have limits 0 at the origin and those of total order $n\nu$ are bounded at the origin. q.e.d.

By Lemmas 3.5 and 3.6 and Proposition 3.4, we have for

$$G_{\sigma,\nu}(x)=\exp(-\phi_C(x_1u_1+\cdots+x_ru_r)^{-\nu})$$

the following asymptotic expansion at $t=0$:

$$\sum_{m_1,\cdots,m_r=1}^{\infty} G_{\sigma,\nu}(tm_1,\cdots,tm_r)$$
$$=\sum_{0\leqq|k|\leqq n\nu-1} t^{|k|-r}\int_{(\boldsymbol{R}_{\geqq 0})^r}\beta_k(\partial/\partial x)^k G_{\sigma,\nu}(x)dx+O(t^{n\nu-r}) \quad \text{as} \quad t\to 0\,.$$

Note that $G_{\sigma,\nu}(tx)=t^{n\nu}G_{\sigma,\nu}(x)$. Hence replacing t by $t^{1/n\nu}$ above and applying Proposition 3.1, we have the following:

PROPOSITION 3.7. *Let $\sigma=\boldsymbol{R}_{\geq 0}u_1+\cdots+\boldsymbol{R}_{\geq 0}u_r$ be an r-dimensional cone in Σ. Then for any positive integer ν, the function $Z_\nu(\sigma,s)=Z(\sigma,\nu s)$ can be continued meromorphically to the half plane* $\operatorname{Re}s>-1+r/n\nu$. *Hence the partial zeta function $Z(\sigma,s)$ can be continued meromorphically to the whole complex plane. For any integer $\nu\geqq 2$, we have*

$$Z(\sigma,0)=\sum_{|k|=r}\int_{(\boldsymbol{R}_{\geqq 0})^r}\beta_k(\partial/\partial x)^k G_{\sigma,\nu}(x)dx\,, \tag{3.8}$$

where $G_{\sigma,\nu}(x)=\exp(-\phi_C(x_1u_1+\cdots+x_ru_r)^{-\nu})$.

This completes the proof of Theorem 2.1.

We now calculate the value $Z(C,\Gamma;0)$. We reformulate (3.8) as follows: Since

$$\beta_k(\partial/\partial x)^k=\prod_{i=1}^{r}\beta_{k_i}(\partial/\partial x_i)^{k_i}$$

and

$$t(1-e^{-t})^{-1}=\sum_{m=0}^{\infty}\beta_m t^m\,,$$

we can write

$$\sum_{|k|=r}\beta_k(\partial/\partial x)^k=\left[\prod_{i=1}^{r}(\partial/\partial x_i)(1-e^{-(\partial/\partial x_i)})^{-1}\right]_r\,,$$

where $[\ \ ]_r$ denotes the total degree r part of the formal power series expansion. Recall that $\Sigma(1)$ is the set of one-dimensional cones in Σ. For any ρ in $\Sigma(1)$, we denote by ∂_ρ the derivation in the direction ρ, that is, for a function $F(x)$ on C,

$$\partial_\rho F(x):=\lim_{h\to 0}\{F(x+hu(\rho))-F(x)\}/h\,,$$

where $u(\rho)$ is the unique primitive element in $\rho\cap N$. Then

$$Z(\sigma,0)=\int_\sigma\left[\prod_{\rho\in\sigma(1)}\frac{\partial_\rho}{1-e^{-\partial_\rho}}\right]_{\dim\sigma}G_\nu(x)dx_\sigma\,, \tag{3.9}$$

where $G_\nu(x) := \exp(-\phi_C(x)^{-\nu})$ and dx_σ is the Lebesgue measure on the linear subspace $\sigma + (-\sigma)$ of $N_{\boldsymbol{R}}$ normalized so that for a $\boldsymbol{Z}$-basis $\{u_1, \cdots, u_n\}$ of N with $\sigma = \boldsymbol{R}_{\geq 0}u_1 + \cdots + \boldsymbol{R}_{\geq 0}u_r$ the volume of the parallelotope spanned by $\{u_1, \cdots, u_r\}$ is one. Therefore we have the following:

PROPOSITION 3.10.

$$Z(C, \Gamma; 0) = \sum_{\sigma \in (\Sigma \setminus \{0\})/\Gamma} \int_\sigma \left[\prod_{\rho \in \sigma(1)} \frac{\partial_\rho}{1 - e^{-\partial_\rho}} \right]_{\dim \sigma} G_\nu(x) dx_\sigma \, .$$

Since $t(1 - e^{-t})^{-1} = \sum_{m=0}^\infty \beta_m t^m$, $\beta_0 = 1$, $\beta_1 = 1/2$ and $\beta_{2m+1} = 0$ for $m \geqq 1$, we see that

$$\frac{t}{1 - e^{-t}} - \frac{t}{2} = \frac{1}{2} \frac{1 + e^{-t}}{1 - e^{-t}}$$

is a power series in t^2, Thus

$$\prod_{\rho \in \sigma(1)} \frac{\partial_\rho}{1 - e^{-\partial_\rho}} = \sum_{\tau < \sigma} \left(\prod_{\rho \in \sigma(1) \setminus \tau(1)} \frac{\partial_\rho}{2} \right) \left(\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right) .$$

Performing integration in the directions $\rho \in \sigma(1) \setminus \tau(1)$, we thus have

$$(3.11) \qquad Z(\sigma, 0) = \sum_{\tau < \sigma} \left(-\frac{1}{2} \right)^{\dim \sigma - \dim \tau} \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau \, .$$

PROPOSITION 3.12.

$$Z(C, \Gamma; 0) = \sum_{j=1}^n \left(-\frac{1}{2} \right)^j \sharp(\Sigma(j)/\Gamma) + \sum_{\tau \in (\Sigma \setminus \{0\})/\Gamma} \left(-\frac{1}{2} \right)^{\dim V(\sigma)} \operatorname{sgn} V(\sigma)$$
$$\times \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau \, ,$$

where $V(\sigma)$ is the closure of the T_N-orbit in Z corresponding to the polyhedral cone σ in Σ.

PROOF. By (3.11) we have

$$Z(C, \Gamma; 0) = \sum_{\sigma \in (\Sigma \setminus \{0\})/\Gamma} \sum_{\tau < \sigma} \left(-\frac{1}{2} \right)^{\dim \sigma - \dim \tau} \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau \, .$$

Since the integral is independent of the choice of representatives τ, we have

$$Z(C, \Gamma; 0) = \sum_{\tau \in \Sigma/\Gamma} \sum_{\sigma \in (\Sigma \setminus \{0\})/\Gamma, \sigma > \tau} \left(-\frac{1}{2} \right)^{\dim \sigma - \dim \tau} \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau \, .$$

Separating the summation into the term for $\tau = \{0\}$ and the summation over $\tau \in (\Sigma \setminus \{0\})$ modulo Γ, we have

$$Z(C, \Gamma; 0) = \sum_{\sigma \in (\Sigma \setminus \{\underline{0}\})/\Gamma} \left(-\frac{1}{2}\right)^{\dim \sigma} + \sum_{\tau \in (\Sigma \setminus \{\underline{0}\})/\Gamma} \int_\tau \left[\prod_{\rho \in \tau(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \tau} G_\nu(x) dx_\tau$$
$$\times \sum_{\sigma \in \Sigma, \sigma \succ \tau} \left(-\frac{1}{2}\right)^{\dim \sigma - \dim \tau} .$$

Obviously we have

$$\sum_{\sigma \in (\Sigma \setminus \{\underline{0}\})/\Gamma} \left(-\frac{1}{2}\right)^{\dim \sigma} = \sum_{j=1}^{n} \left(-\frac{1}{2}\right)^{j} {}^\#(\Sigma(j)/\Gamma) .$$

Since the r.p.p. decomposition $\{\sigma \in \Sigma; \sigma \succ \tau\}$ gives a torus embedding $V(\tau)$ for τ in $\Sigma \setminus \{0\}$, the following lemma shows that

$$\sum_{\sigma \in \Sigma, \sigma \succ \tau} (-1/2)^{\dim \sigma - \dim \tau} = (-1/2)^{\dim V(\tau)} \operatorname{sgn} V(\tau) .$$

LEMMA 3.13 (Ehlers [4]). *Let $Z' = T_{N'}emb(\Xi)$ be a compact nonsingular torus embedding of dimension r'. Then we have*

$$\begin{aligned} \operatorname{sgn}(Z') &= \sum_{j=0}^{r'} (-2)^j \, {}^\#\Xi(r' - j) = \sum_{\xi \in \Xi} (-2)^{r' - \dim \xi} \\ &= (-2)^{\dim Z'} \sum_{\xi \in \Xi} (-1/2)^{\dim \xi} . \end{aligned}$$

When r' is odd, both sides vanish.

Since $t(1 + e^{-t})/2(1 - e^{-t})$ is a power series in t^2, we see that

$$\left[\prod_{\rho \in \sigma(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_o}} \right]_{\dim \sigma}$$

vanishes if $\dim \sigma$ is odd, while $\operatorname{sgn} V(\sigma) = 0$ if $\dim V(\sigma) = n - \dim \sigma$ is odd. Thus when n is odd,

$$\left(-\frac{1}{2}\right)^{\dim V(\sigma)} \operatorname{sgn} V(\sigma) \int_\sigma \left[\prod_{\rho \in \sigma(1)} \frac{\partial_\rho}{2} \frac{1 + e^{-\partial_\rho}}{1 - e^{-\partial_\rho}} \right]_{\dim \sigma} G_\nu(x) dx_\sigma$$

vanishes for all σ in $\Sigma \setminus \{\underline{0}\}$. Hence in the notations in Section 1, we have the following:

PROPOSITION 3.14. *When n is odd,*

$$Z(C, \Gamma; 0) = \sum_{j=1}^{n} (-1/2)^j \, {}^\#(\Sigma(j)/\Gamma) = \sum_{j=1}^{n} (-1/2)^j \, {}^\#\Phi(j) .$$

LEMMA 3.15 (Satake [9], [10]).

$$\begin{aligned} \sum_{J \in \Phi} \operatorname{sgn}(X_J) &= (-1)^{n+1} \sum_{j=1}^{n} (-2)^{n-j} \, {}^\#\Phi(j) \\ &= -\sum_{j=1} (-2)^{n-j} \, {}^\#\Phi(j) + 2^n \sum_{j=1}^{n} (-1)^{n-j} \, {}^\#\Phi(j) . \end{aligned}$$

Consequently, when n is odd,

$$\sum_{J\in\Phi}\operatorname{sgn}(X_J) = 2^{n-1}\sum_{j=1}^{n}(-1)^{n-j}\,{}^{\#}\Phi(j)\ .$$

PROOF. By our definition in Section 1, we have $X_J \cong V(\sigma(J))$ for $J\in\Phi$. By Lemma 3.13, we have

$$\operatorname{sgn}(X_J) = \sum_{\tau\in\Sigma,\tau>\sigma(J)}(-2)^{n-\dim\tau}\ ,$$

hence

$$\sum_{J\in\Phi}\operatorname{sgn}(X_J) = \sum_{J\in\Phi}\sum_{\tau\in\Sigma,\tau>\sigma(J)}(-2)^{n-\dim\tau}\ .$$

Since for τ in Σ with $\tau \succ \sigma(J)$ there exists J' in Φ such that $\tau = \sigma(J')$ modulo Γ, we may change the order of the summation to get

$$\begin{aligned}\sum_{J\in\Phi}\operatorname{sgn}(X_J) &= \sum_{J'\in\Phi}(-2)^{n-\dim\sigma(J')}\,{}^{\#}\{\tau\in\Sigma\setminus\{\underline{0}\};\,\tau\prec\sigma(J')\}\\ &= \sum_{J\in\Phi}(-2)^{n-\dim\sigma(J)}\,(2^{\dim\sigma(J)}-1)\\ &= 2^n\sum_{J\in\Phi}(-1)^{n-\dim\sigma(J)} - \sum_{J\in\Phi}(-2)^{n-\dim\sigma(J)}\\ &= 2^n\sum_{j=1}^{n}(-1)^{n-j}\,{}^{\#}\Phi(j) - \sum_{j=1}^{n}(-2)^{n-j}\,{}^{\#}\Phi(j)\ .\end{aligned}$$

On the other hand, we may write

$$\operatorname{sgn}(X_J) = (-1)^{n-\dim\sigma(J)}\sum_{\tau\in\Sigma,\tau>\sigma(J)}(-2)^{n-\dim\tau}\ ,$$

because both sides vanish when $\dim X_J = n-\dim\sigma(J)$ is odd. Thus we have

$$\begin{aligned}\sum_{J\in\Phi}\operatorname{sgn}(X_J) &= \sum_{J\in\Phi}(-1)^{n-\dim\sigma(J)}\sum_{\tau\in\Sigma,\tau>\sigma(J)}(-2)^{n-\dim\tau}\\ &= (-1)^n\sum_{J'\in\Phi}(-2)^{n-\dim\sigma(J')}\sum_{\tau\in\Sigma\setminus\{\underline{0}\},\tau\prec\sigma(J')}(-1)^{\dim\tau}\\ &= (-1)^n\sum_{J'\in\Phi}(-2)^{n-\dim\sigma(J')}\{(1-1)^{\dim\sigma(J')}-1\}\\ &= (-1)^{n+1}\sum_{J'\in\Phi}(-2)^{n-\dim\sigma(J')}\\ &= (-1)^{n+1}\sum_{j=1}^{n}(-2)^{n-j}\,{}^{\#}\Phi(j)\ .\end{aligned}$$

q.e.d.

By Lemma 3.15, we have

$$\sum_{J\in\Phi}\operatorname{sgn}(X_J) = -2^n\sum_{j=1}^{n}(-2)^{-j}\,{}^{\#}\Phi(j) = -2^n\sum_{j=1}^{n}(-2)^{-j}\,{}^{\#}(\Sigma(j)/\Gamma)\ .$$

By this and Proposition 3.12, we complete the proof of Theorem 2.2. By Proposition 3.14, we have

$$Z(C, \Gamma; 0) = -e(D/\Gamma)/2$$

when n is odd, since Φ gives a triangulation of the $(n-1)$-dimensional compact manifold D/Γ and since $\Sigma(k)/\Gamma$ is in bijective correspondence with the set of $(k-1)$-simplices in Φ for each positive integer k. Thus we conclude the proof of Theorem 2.3.

REFERENCES

[1] M. F. ATIYAH, H. DONNELLY AND I. M. SINGER, Eta invariants, signature defects of cusps, and values of L-functions, Ann. of Math. 118 (1983), 131–177.

[2] N. BOURBAKI, Fonctions d'une Variable réelle (Théorie élémentaire), Chapitres 4, 5, 6 et 7, Élements de Mathématique XII, Deuxième édition, Hermann, Paris, 1961.

[3] P. CASSOU-NOGUÈS, Valeurs aux entiers négatifs des séries de Dirichlet associées à un polynôme, I, J. Number Theory 14 (1982), 32–64.

[4] F. EHLERS, Eine Klasse komplexer Mannigfaltigkeiten und die Auflösung einiger isolierter Singularitäten, Math. Ann. 218 (1975), 127–156.

[5] F. HIRZEBRUCH, Hilbert modular surfaces, Enseign, Math. 19 (1973), 183–281.

[6] F. HIRZEBRUCH AND G. VAN DER GEER, Lectures on Hilbert modular surfaces, Presses de l'Université de Montréal, Modtréal, 1981.

[7] T. ODA, Lectures on Torus Embeddings and Applications (Based on joint work with K. Miyake), Tata Inst. of Fund. Res., Bombay, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[8] I. SATAKE, Special values of zeta functions associated with self-dual homogeneous cones, in Manifolds and Lie groups (Notre Dame Ind., 1980), Progress in Math. 14, Birkhäuser, Boston, 1981, 359–384.

[9] I. SATAKE, On numerical invariants of arithmetic varieties (the case of $\boldsymbol{Q}$-rank one) (in Japanese), Sugaku 35 (1983), 210–220.

[10] I. SATAKE, On numerical invariants of arithmetic varietis of $\boldsymbol{Q}$-rank one, Automorphic Forms of Several Variables, Taniguchi Symposium, Katata, 1983 (I. Satake and Y. Morita, eds.), Progress in Math. 46, Birkhäuser, Basel, Boston, Stuttgart, 1984, 353–369.

[11] H. SHIMIZU, On discontinuous groups operating on the product of upper half planes, Ann. of Math. 77 (1963), 33–71.

[12] T. SHINTANI, On evaluation of zeta functions of totally real algebraic number fields at non-positive integers, J. Fac. Sci. Univ. Tokyo Sect. IA, Math. 23 (1976), 393–417.

[13] H. TSUCHIHASHI, Higher dimensional analogues of periodic continued fractions and cusp singularities, Tôhoku Math. J. 35 (1983), 607–639.

[14] E. B. VINBERG, Theory of homogeneous convex cones, Trans. Moscow Math. Soc. 12 (1967), 303–368.

[15] D. ZAGIER, Valeurs des fonctions zêta des corps quadratiques réels aux entiers négatifs, Journées Arithmétiques de Caen (Univ. Caen, Caen, 1976), Astérisque Nos. 41–43, Soc. Math. France, Paris, 1977, 135–151.

[16] D. ZAGIER, Zetafunktionen und quadratische Körper, Springer-Verlag, Berlin, Heidelberg, New York, 1981.

MATHEMATICAL INSTITUTE
TÔHOKU UNIVERSITY
SENDAI, 980
JAPAN