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ON THE LITTLEWOOD-PALEY AND MARCINKIEWICZ FUNCTIONS IN HIGHER DIMENSIONS

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1. Introduction. In this paper we deal with the generalized Littlewood-Paley, Marcinkiewicz and related square functions of spherical sense in the n-dimensional space. So our functions are different from Stein's $g_\lambda^*(\boldsymbol{x}; f)$ [14. p. 99] and $\mathscr{D}_\alpha(f)(\boldsymbol{x})$ [15, p. 102].

In what follows, we shall use the following notations. $\boldsymbol{x}, \boldsymbol{\xi}, \cdots$ will denote points in the Euclidean n-space $\boldsymbol{R}^n$ ($n \geqq 2$). In coordinate notation we write $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$; $|\boldsymbol{x}|$ denotes the length of the vector $\boldsymbol{x}$, i.e., $|\boldsymbol{x}|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$; $\boldsymbol{x}' = (x_1', x_2', \cdots, x_n')$ denotes the unit vector in the direction of $\boldsymbol{x}$, i.e., $\boldsymbol{x}' = \boldsymbol{x}/|\boldsymbol{x}|$; Σ is the unit sphere, $|\boldsymbol{x}| = 1$; and $d\sigma$ is the Euclidean element of measure on Σ, hence $\int_\Sigma d\sigma = 2\pi^{n/2}/\Gamma(n/2)$.

For $f \in \mathscr{S}(\boldsymbol{R}^n)$, the Schwartz space of rapidly decreasing C^∞-functions, the Fourier transform of f is defined by

$$\hat{f}(\boldsymbol{\xi}) = \int_{\boldsymbol{R}^n} f(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x} ,$$

where $\boldsymbol{x}\cdot\boldsymbol{\xi} = x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n$. Throughout this paper, we assume $f \in \mathscr{S}(\boldsymbol{R}^n)$ unless otherwise specified.

If $K(\boldsymbol{x}) = \Omega(\boldsymbol{x}')/|\boldsymbol{x}|^n$ is the Calderón-Zygmund kernel, then

$$\tilde{f}_\Omega(\boldsymbol{x}) = \lim_{\varepsilon\to 0} \int_{|\boldsymbol{y}|>\varepsilon} K(\boldsymbol{y}) f(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y}$$

exists almost everywhere and

$$\|\tilde{f}_\Omega\|_p \leqq A_p \|f\|_p \quad (1 < p < \infty) .$$

$\tilde{f}_\Omega$ is a conjugate integral in n-dimensions.

The spherical mean of order $\alpha > 0$ of f is

$$(1.1) \qquad (M_t^\alpha f)(\boldsymbol{x}) = c_\alpha t^{-n} \int_{|\boldsymbol{y}|<t} (1 - |\boldsymbol{y}|^2/t^2)^{\alpha-1} f(\boldsymbol{x} - \boldsymbol{y}) d\boldsymbol{y} ,$$

where $c_\alpha = \Gamma(\alpha + n/2)/\pi^{n/2}\Gamma(\alpha)$. Also we define

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(1.2) $$(M^{\alpha}_{\Omega,t}f)(x) = c_{\alpha}t^{-n}\int_{|y|<t}(1 - |y|^2/t^2)^{\alpha-1}\Omega(y')f(x - y)dy$$

for $\alpha > 0$. We need $(M^{\alpha}_t f)(x)$ and $(M^{\alpha}_{\Omega,t}f)(x)$ with negative order α. More generally, $M^{\alpha}_t f$ and $M^{\alpha}_{\Omega,t}f$ can be defined for complex α as distributions (the finite part in the sense of Hadamard or the canonical regularization of Gel'fand-Shilov [6, vol. 1, §3.7]). Then this $M^{\alpha}_t f$ is identical with Stein-Wainger's [20, p. 1270] which was defined by the analytic continuation of its Fourier transform (cf. [6, vol. 1, Ch. II]).

$M^{\alpha}_t f$ was studied in Chandrasekharan [2]. See also Stein [17] and Stein-Wainger [20].

Corresponding to these, let the Riesz-Bochner means of order $\beta > -1$ of the Fourier integral and the conjugate Fourier integral of f be

(1.3) $$(S^{\beta}_R f)(x) = \int_{|\xi|<R}(1 - |\xi|^2/R^2)^{\beta}\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi$$

and

(1.4) $$(\tilde{S}^{\beta}_{\Omega,R}f)(x) = \int_{|\xi|<R}(1 - |\xi|^2/R^2)^{\beta}\hat{K}(\xi)\hat{f}(\xi)e^{2\pi i x\cdot\xi}d\xi ,$$

respectively. From these means, we can define several square functions, see Stein [18]. For example,

(1.5) $$(h^{\beta}f)(x) = \left\{\int_0^{\infty}\left|\frac{\partial}{\partial R}(S^{\beta}_R f)(x)\right|^2 RdR\right\}^{1/2}$$
$$=\left[\int_0^{\infty}| - 2\beta\{(S^{\beta}_R f)(x) - (S^{\beta-1}_R f)(x)\}|^2 dR/R\right]^{1/2}$$

is the generalized Littlewood-Paley function defined by Stein [12, p. 130] and one of the authors [22, p. 504]. Another example is

(1.6) $$(\mu^{\alpha}_{\Omega}f)(x) = \left\{\int_0^{\infty}|(M^{\alpha}_{\Omega,t}f)(x)|^2 dt/t\right\}^{1/2} .$$

This is a generalized Marcinkiewicz function. In fact, if $\alpha = 1$, then (1.6) is equivalent to $\mu(f)(x)$ defined by Stein [13, p. 435], see § 4.

We give more examples of analogous square functions. For examples, set

(1.7) $$(\tilde{h}^{\beta}_{\Omega}f)(x) = \left\{\int_0^{\infty}\left|\frac{\partial}{\partial R}(\tilde{S}^{\beta}_{\Omega,R}f)(x)\right|^2 RdR\right\}^{1/2}$$
$$= \left[\int_0^{\infty}\left|-2\beta\{(\tilde{S}^{\beta}_{\Omega,R}f)(x) - (\tilde{S}^{\beta-1}_{\Omega,R}f)(x)\}\right|^2 dR/R\right]^{1/2}$$

and

$$(1.8)\qquad (\nu^{\alpha}f)(x) = \left\{\int_0^{\infty}\left|\frac{\partial}{\partial t}(M_t^{\alpha}f)(x)\right| t dt\right\}^{1/2}$$
$$= \left[\int_0^{\infty}\left|-2(\alpha + n/2 - 1)\{(M_t^{\alpha}f)(x) - (M_t^{\alpha-1}f)(x)\}\right|^2 dt/t\right]^{1/2} .$$

One of the objects of this paper is to give pointwise relationship among such square functions. For any two square functions Ff and Gf, we shall write $(Ff)(x) \approx (Gf)(x)$, if there exist two positive constants A and B, independent of x and f, such that, for all $x \in R^n$, $(Gf)(x) \leqq A(Ff)(x)$, provided that $(Ff)(x)$ is finite, and $(Ff)(x) \leqq B(Gf)(x)$, provided that $(Gf)(x)$ is finite. If F and G have some parameters, then A and B may depend on them. When both A and B are independent of some of the parameters, we say that the relation $(Ff)(x) \approx (Gf)(x)$ holds uniformly in them. Our typical theorems are as follows.

THEOREM 1. *If $\beta = \alpha + n/2 > 0$ and Y_k is any surface spherical harmonic with degree $k \geqq 1$, then*

$$(\mu_{Y_k}^{\alpha}f)(x) \approx (\tilde{h}_{Y_k}^{\beta}f)(x)/|\gamma_{k,0}|$$

for $f \in \mathscr{S}(R^n)$, where $\gamma_{k,0} = i^{-k}\pi^{n/2}\Gamma(k/2)/\Gamma((k + n)/2)$. This relation holds uniformly in Y_k and k.

THEOREM 2. *If $\beta = \alpha + n/2 - 1 > 0$, then*

$$(\nu^{\alpha}f)(x) \approx (h^{\beta}f)(x)$$

for $f \in \mathscr{S}(R^n)$.

These theorems arose in connection with the Cesàro-Riesz summation concerning a function and its Fourier transform. In an analogous way, we can define some square functions associated with other summation methods. In particular, the spherical Abel-Poisson summation yields the original Littlewood-Paley function $g(f)(x)$.

The plan of this paper is as follows. In §§2 and 3 we prove Theorems 1 and 2. §4 is concerned with Marcinkiewicz function $\mu(f)$ introduced by Stein [13]. §5 contains some theorems about square functions arising as Riesz-potentials. We shall also give there a relationship between our square functions and $g_{\lambda}^{*}(f)$ of Stein [14]. In §6 we give some theorems on Abel-Poisson and other summations. §7 is devoted to applications of our theorems. In particular, we can deduce new and known results on the L^p-boundedness of several square functions constructed from L^p-functions. In this case we can give an answer to a problem by Stein-Wainger [20, p. 1289, Problem 6 (a)].

The method of proof comes from the same idea as in the one-dimensional case by one of the authors [23], that is to say, Wiener's transformation method. However, we shall meet several subtle calculations in the higher dimensional case.

2. Square functions arising from spherical Cesàro-Riesz means of functions. $(M_t^\alpha f)(x)$ and $(M_{\Omega,t}^\alpha f)(x)$ are defined by (1.1) and (1.2), respectively. We consider first $\alpha > 0$. For the sake of simplicity we set, for a fixed function f and a point x, the average over sphere

$$\phi(t) = \phi(t; x, f) = \int_\Sigma f(x - ty')d\sigma(y') . \tag{2.1}$$

Then we can get

$$(M_t^\alpha f)(x) = c_\alpha \int_0^1 r^{n-1}(1 - r^2)^{\alpha-1}\phi(tr)dr . \tag{2.2}$$

Analogously, set

$$\psi(t) = \psi(t; x, f, \Omega) = \int_\Sigma \Omega(y')f(x - ty')d\sigma(y') . \tag{2.3}$$

Then

$$(M_{\Omega,t}^\alpha f)(x) = c_\alpha \int_0^1 r^{n-1}(1 - r^2)^{\alpha-1}\psi(tr)dr . \tag{2.4}$$

For the sake of calculation, we set

$$\theta(t) = \theta(t; x, f) = t\frac{\partial}{\partial t}\phi(t; x, f) = -\int_\Sigma ty' \cdot \nabla f(x - ty')d\sigma(y') . \tag{2.5}$$

Then we get

$$\begin{aligned} t\frac{\partial}{\partial t}(M_t^\alpha f)(x) &= -2\left(\alpha + \frac{n}{2} - 1\right)\{(M_t^\alpha f)(x) - (M_t^{\alpha-1} f)(x)\} \\ &= c_\alpha \int_0^1 r^{n-1}(1 - r^2)^{\alpha-1}\theta(tr)dr . \end{aligned} \tag{2.6}$$

If we change variables by $r = e^y$ and $t = e^{-x}$, then the square functions (1.6) and (1.8) become the L^2-norms of convolutions by (2.4) and (2.6) i.e.,

$$(\mu_\Omega^\alpha f)(x) = \left\{\int_{-\infty}^\infty |(K_\alpha * \Psi)(x)|^2 dx\right\}^{1/2} \tag{2.7}$$

and

$$(\nu^\alpha f)(x) = \left\{\int_{-\infty}^\infty |(K_\alpha * \Theta)(x)|^2 dx\right\}^{1/2} , \tag{2.8}$$

where K_α, Ψ and Θ are defined by the following formulae.

$$K_\alpha(x) = \begin{cases} c_\alpha e^{nx}(1 - e^{2x})^{\alpha-1} & (x < 0) \\ 0 & (x \geqq 0) , \end{cases} \tag{2.9}$$

$$\Psi(x) = \Psi(x; \boldsymbol{x}, f, \Omega) = \psi(e^{-x}) \quad \text{and} \quad \Theta(x) = \Theta(x; \boldsymbol{x}, f) = \theta(e^{-x}) . \tag{2.10}$$

When $\alpha \neq -n/2 - \nu$ $(\nu = 0, 1, 2, \cdots)$, the above relations are preserved in distributional sense (see the proof of Proposition 1).

We now take the Fourier transform of K_α as a distribution and prove the following proposition.

PROPOSITION 1. *If $\alpha > -n/2$, then*

$$(\mu_\Omega^\alpha f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty} |\kappa_\alpha(\xi)\{\Psi(\cdot; \boldsymbol{x}, f, \Omega)\}^\wedge(\xi)|^2 d\xi \tag{2.11}$$

and

$$(\nu^\alpha f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty} |\kappa_\alpha(\xi)\{\Theta(\cdot; \boldsymbol{x}, f)\}^\wedge(\xi)|^2 d\xi , \tag{2.12}$$

where

$$\kappa_\alpha(\xi) = \frac{\Gamma(\alpha + n/2)\Gamma(n/2 - i\pi\xi)}{2\pi^{n/2}\Gamma(\alpha + n/2 - i\pi\xi)} \tag{2.13}$$

is the distributional Fourier transform of K_α, and

$$A(|\xi| + 1)^{-\alpha} \leqq |\kappa_\alpha(\xi)| \leqq B(|\xi| + 1)^{-\alpha} \tag{2.14}$$

for $-\infty < \xi < \infty$.

REMARK. In the sequel, we write the formula such as (2.14) as

$$|\kappa_\alpha(\xi)| \sim (|\xi| + 1)^{-\alpha} \quad (-\infty < \xi < \infty) .$$

PROOF. Assume that α is complex. Since $\int_\Sigma \Omega(\boldsymbol{y}')d\sigma(\boldsymbol{y}') = 0$, we evidently have $\Psi \in \mathcal{S}(-\infty, \infty)$, and $\Theta \in \mathcal{S}(-\infty, \infty)$ is evident. We can establish convolutional rule for these convolutions. The distributional Fourier tranform of K_α is gotten by analytic continuation. See Gel'fand-Shilov [6, vol. 1, Chap. 2, §2]. When $\operatorname{Re}\alpha > 0$, the complex Fourier transform of K_α is

$$\begin{aligned} \hat{K}_\alpha(\zeta) &= c_\alpha \int_{-\infty}^0 (1 - e^{2x})^{\alpha-1} e^{(\zeta+n)x} dx = 2^{-1} c_\alpha \int_0^1 (1 - t)^{\alpha-1} t^{(\zeta+n)/2-1} dt \\ &= 2^{-1} c_\alpha \frac{\Gamma(\alpha)\Gamma((\zeta + n)/2)}{\Gamma(\alpha + (\zeta + n)/2)} = \frac{\Gamma(\alpha + n/2)\Gamma((\zeta + n)/2)}{2\pi^{n/2}\Gamma(\alpha + (\zeta + n)/2)} . \end{aligned}$$

For $\operatorname{Re}\alpha > -n/2$, $\hat{K}_\alpha(\zeta)$ is also equal to the last term by analytic continuation, so we get

$$\hat{K}_\alpha(-2\pi i\xi) = \frac{\Gamma(\alpha+n/2)\Gamma(n/2-i\pi\xi)}{2\pi^{n/2}\Gamma(\alpha+n/2-i\pi\xi)} = \kappa_\alpha(\xi)\ .$$

Since $\kappa_\alpha(\xi) \neq 0$ $(-\infty < \xi < \infty)$, the asymptotic formula of the gamma function, i.e.,

$$Ae^{-\pi|y|/2}|y|^{x-1/2} \leqq |\Gamma(x+iy)| \leqq Be^{-\pi|y|/2}|y|^{x-1/2}$$

for sufficiently large $|y|$, gives us the conclusion. q.e.d.

3. Square functions arising from Bochner-Riesz means of Fourier and conjugate Fourier integral. First we define the space of distributions of which test functions are between the space $\mathscr{S}(-\infty, \infty)$ and the space $\mathscr{D}(-\infty, \infty)$ following the method of Zemanian [25, Chap. 3]. We shall prove that in this space the above mentioned functions Ψ and Θ are the convolutes in the sense of Gel'fand-Shilov [6, vol. II, p. 137 and p. 148]. f is the convolute in the space $\mathscr{F}$ of test functions, if the distribution $f \in \mathscr{F}'$ has the property that $(\check{f}*\phi)(x) = \langle f(y), \phi(x+y)\rangle \in \mathscr{F}$ for any $\phi \in \mathscr{F}$ and that the relation $\phi_\nu \to 0$ implies $\check{f}*\phi_\nu \to 0$ in the topology of $\mathscr{F}$.

Let m be a large positive number defined in a moment. $\{a_p\}$ and $\{b_p\}$ are positive decreasing sequences such that

$$m < a_p < m+1\ , \quad 1/2 < b_p < 1\ , \tag{3.1}$$

$\lim a_p = m$ and $\lim b_p = 1/2$. Set

$$k_p(x) = \begin{cases} \exp(a_p x) & (x \geqq 0) \\ \exp(-b_p x) & (x < 0)\ . \end{cases} \tag{3.2}$$

For any $\phi \in C^\infty(-\infty, \infty)$, set

$$\gamma_{p,q}(\phi) = \sup\{k_p(x)|D^q\phi(x)|;\ -\infty < x < \infty\} \tag{3.3}$$

$(q = 0, 1, 2, \cdots)$. The class of functions $\phi \in C^\infty(-\infty, \infty)$ such that

$$\gamma_{p,q}(\phi) < \infty \quad (q = 0, 1, 2, \cdots) \tag{3.4}$$

is denoted by $\mathscr{L}_p = \mathscr{L}_{a_p, b_p}$ and its topology is defined by the method of Zemanian [25, p. 50]. Set $\mathscr{F}_m = \cup_{p=1}^\infty \mathscr{L}_p$. Then the fundamental space $\mathscr{F}_m$ of test functions is contained in $\mathscr{S}(-\infty, \infty)$ and the convergence of $\mathscr{F}_m$ implies that of $\mathscr{S}(-\infty, \infty)$; see [25, p. 55]. In $\mathscr{F}_m'$, the distributional space defined on $\mathscr{F}_m$, we have the following lemma.

LEMMA 1. *The function Φ such that*

$$|\Phi(x)| \leqq \begin{cases} Ce^{-x} & (x \geqq 0) \\ Ce^{(m+1)x} & (x < 0) \end{cases} \tag{3.5}$$

is a convolute in the space $\mathscr{F}_m$.

PROOF. For any $\phi \in \mathscr{L}_p$, set $\psi(x) = \int_{-\infty}^{\infty} \Phi(y)\phi(x+y)\,dy$. We must estimate $I(x) = (D^q\psi)(x)$, where

$$I(x) = \int_{-\infty}^{\infty} \Phi(y-x)(D^q\phi)(y)dy \; . \tag{3.6}$$

Since $\gamma_{p,q}(\phi) < \infty$ by (3.3) and (3.4), we have

$$|I(x)| \leqq \gamma_{p,q}(\phi)\left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right)\{|\Phi(y-x)|/k_p(y)\}dy = \gamma_{p,q}(\phi)(I_1 + I_2) \; ,$$

say. Then, by (3.2),

$$I_1 = \int_0^{\infty} |\Phi(-x-y)| \exp(-b_p y)dy$$

and

$$I_2 = \int_0^{\infty} |\Phi(-x+y)| \exp(-a_p y)dy \; .$$

If $x \geqq 0$, then by (3.5) and (3.1)

$$I_1 \leqq Ce^{-(m+1)x}\int_0^{\infty} \exp\{-(m+1+b_p)y\}dy \leqq C' \exp(-a_p x)$$

and

$$\begin{aligned} I_2 &\leqq C\left[e^{-(m+1)x}\int_0^{x} \exp\{(\mathrm{m}+1-a_p)y\}dy + e^{x}\int_x^{\infty} \exp\{-(1+a_p)y\}dy\right] \\ &\leqq C' \exp(-a_p x) \; , \end{aligned}$$

because $m + 1 - a_p > 0$.

If $x < 0$, then

$$\begin{aligned} I_1 &\leqq C\left[e^{x}\int_0^{-x} \exp\{(1-b_p)y\}dy + e^{-(m+1)x}\int_{-x}^{\infty} \exp\{-(m+1+b_p)y\}dy\right] \\ &\leqq C' \exp(b_p x) \end{aligned}$$

by $1 - b_p > 0$, and

$$I_2 \leqq Ce^{x}\int_0^{\infty} \exp\{-(1+a_p)y\}dy \leqq C' \exp(b_p x) \; .$$

Hence by (3.6) $\gamma_{p,q}(\psi) = \sup\{k_p(x)|I(x)|; -\infty < x < \infty\} \leqq C'\gamma_{p,q}(\phi)$.

q.e.d.

In (1.4) we set $K(\boldsymbol{x}) = Y_k(\boldsymbol{x}')/|\boldsymbol{x}|^n$, where Y_k is the surface spherical harmonic of degree $k(\geqq 1)$. Then by Stein-Weiss [21, p. 164],

$$\hat{K}(\boldsymbol{\xi}) = \gamma_{k,0} Y_k(\boldsymbol{\xi}') \; ,$$

where $\gamma_{k,0} = i^{-k}\pi^{n/2}\Gamma(k/2)/\Gamma((k+n)/2)$. Hence (1.4) becomes

$$(3.7)\qquad \begin{aligned}(\tilde{S}^{\beta}_{Y_k,R}f)(\boldsymbol{x}) &= \gamma_{k,0}\int_{|\xi|<R}(1-|\boldsymbol{\xi}|^2/R^2)^{\beta}Y_k(\boldsymbol{\xi}')\hat{f}(\boldsymbol{\xi})e^{2\pi i\boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}\\ &= \gamma_{k,0}\int_{R^n}f(\boldsymbol{x}-R^{-1}\boldsymbol{y})d\boldsymbol{y}\int_{|\xi|<1}(1-|\boldsymbol{\xi}|^2)^{\beta}Y_k(\boldsymbol{\xi}')e^{2\pi i\boldsymbol{y}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}\\ &= \gamma_{k,0}\int_{R^n}f(\boldsymbol{x}-R^{-1}\boldsymbol{y})|\boldsymbol{y}|^{-n}Y_k(\boldsymbol{y}')\tilde{\gamma}_{\beta,k}(|\boldsymbol{y}|)d\boldsymbol{y}\ ,\end{aligned}$$

where

$$(3.8)\qquad \tilde{\gamma}_{\beta,k}(t) = (2\pi)^{n/2}t^n\int_0^1 u^{n-1}(1-u^2)^{\beta}(2\pi itu)^k V_{k+(n/2)-1}(2\pi tu)du\ ,$$

$V_{\mu}(t) = J_{\mu}(t)/t^{\mu}$ and J_{μ} is the Bessel function of order μ; see Stein-Weiss [21, p. 158].

Now we set as in (2.3)

$$(3.9)\qquad \psi(t) = \psi(t;\boldsymbol{x},f,Y_k) = \int_{\Sigma}Y_k(\boldsymbol{y}')f(\boldsymbol{x}-t\boldsymbol{y}')d\sigma(\boldsymbol{y}')\ .$$

Then

$$(3.10)\qquad (\tilde{S}^{\beta}_{Y_k,R}f)(\boldsymbol{x}) = \gamma_{k,0}\int_0^{\infty}\tilde{\gamma}_{\beta,k}(r)\psi(r/R)dr/r\ .$$

For $k = 1, 2, \cdots$, if $\beta < (n-1)/2$, then

$$(3.11)\qquad \tilde{\gamma}_{\beta,k}(t) \sim t^{-\beta+(n-1)/2}$$

for large t; see Chang [3, p.p. 17-18, Lemma 7].

If we change variables by $r = e^y$ and $R = e^x$, and set

$$(3.12)\qquad \begin{aligned}\Psi(x) = \Psi(x;\boldsymbol{x},f,Y_k) &= \psi(e^{-x})\quad\text{and}\\ \tilde{K}^*_{\beta,k}(x) &= -2\beta\gamma_{k,0}\{\tilde{\gamma}_{\beta,k}(e^x)-\tilde{\gamma}_{\beta-1,k}(e^x)\}\ ,\end{aligned}$$

then the square function $(\tilde{h}^{\beta}_{Y_k}f)(\boldsymbol{x})$ becomes

$$(3.13)\qquad (\tilde{h}^{\beta}_{Y_k}f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty}|(\tilde{K}^*_{\beta,k}*\Psi)(x)|^2dx$$

by (1.7), (3.10) and (3.12). Now we can prove the following.

PROPOSITION 2. *For* $\beta > 0$,

$$(3.14)\qquad (\tilde{h}^{\beta}_{Y_k}f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty}|\lambda^*_{\beta,k}(\xi)\{\Psi(\cdot;\boldsymbol{x},f,Y_k)\}^{\wedge}(\xi)|^2d\xi\ ,$$

where $\lambda^*_{\beta,k}$ *is the distributional Fourier transform of* $\tilde{K}^*_{\beta,k}$,

$$(3.15)\qquad \lambda^*_{\beta,k}(\xi) = \frac{\Gamma(\beta+1)\Gamma(k/2)}{\Gamma((k+n)/2)}\,\frac{\pi^{2\pi i\xi}\Gamma(1+i\pi\xi)\Gamma((k+n)/2-i\pi\xi)}{\Gamma(\beta+1+i\pi\xi)\Gamma(k/2+i\pi\xi)}$$

and

$$|\lambda^*_{\beta,k}(\xi)| \sim (|\xi| + 1)^{-\beta+(n/2)} \quad (-\infty < \xi < \infty) . \tag{3.16}$$

PROOF. By the formulas (3.8), (3.11) and (3.12), we have

$$|\tilde{K}^*_{\beta,k}(x)| \leqq C \max\{1, \exp([-\beta + (n+1)/2]x)\}$$

for $x \geqq 0$. If we take a positive number m such that $m > (n+1)/2 - \beta$ in $\mathscr{F}_m$ of Lemma 1, then $\tilde{K}^*_{\beta,k} \in \mathscr{F}'_m$ and the convolution rule is established, because Ψ satisfies the condition (3.5). Hence

$$\int_{-\infty}^{\infty} |(\Psi * \tilde{K}^*_{\beta,k})(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{\Psi}(\xi)(\tilde{K}^*_{\beta,k})\hat{}(\xi)|^2 d\xi ,$$

where $(\tilde{K}^*_{\beta,k})\hat{}$ is the distributional Fourier transform of $\tilde{K}^*_{\beta,k}$. However,

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{\zeta x} \tilde{K}^*_{\beta,k}(x) dx \\
&= -2\beta\gamma_{k,0} \int_0^{\infty} t^{\zeta} \{\tilde{\gamma}_{\beta,k}(t) - \tilde{\gamma}_{\beta-1,k}(t)\} dt/t \\
&= (2\pi)^{n/2} 2\beta\gamma_{k,0} \int_0^{\infty} t^{\zeta+n-1} dt \int_0^1 u^{n+1}(1-u^2)^{\beta-1}(2\pi i t u)^k V_{k+(n/2)-1}(2\pi t u) du \\
&= 2\beta\gamma_{k,0} i^k (2\pi)^{-\zeta-(n/2)+1} \int_0^1 u^{-\zeta+2}(1-u^2)^{\beta-1} du \int_0^{\infty} (2\pi u t)^{\zeta+k+n-1} V_{k+(n/2)-1}(2\pi u t) dt \\
&= \frac{\Gamma(\beta+1)\Gamma(k/2)}{\Gamma((k+n)/2)} \, \frac{\Gamma(-(\zeta/2)+1)\Gamma((\zeta+k+n)/2)}{\pi^{\zeta}\Gamma(-(\zeta/2)+\beta+1)\Gamma((-\zeta+k)/2)}
\end{aligned}$$

for $-(k+n) < \operatorname{Re}\zeta < -(n+1)/2$ by Watson [24, p. 391, (1)]. The last formula is analytic in a broader domain which contains the imaginary axis. Hence by the argument of Gel'fand-Shilov, we get $(\tilde{K}^*_{\beta,k})\hat{}(\xi)$ by letting $\zeta = -2\pi i \xi$ in the last formula. We denote this by $\lambda^*_{\beta,k}(\xi)$ as in (3.15). Since $\lambda^*_{\beta,k}$ has no zero and the asymptotic formula for Γ-function is applicable, we get (3.16). q.e.d.

For $(h^{\beta}f)(\boldsymbol{x})$, we can proceed in an analogous way. Since

$$(S^{\beta}_R f)(\boldsymbol{x}) = \int_{|\boldsymbol{\xi}|<R} (1 - |\boldsymbol{\xi}|^2/R^2)^{\beta} \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$$

by (1.3), we get

$$(S^{\beta}_R f)(\boldsymbol{x}) = \int_0^{\infty} \gamma_{\beta}(r)\phi(r/R; \boldsymbol{x}, f) dr/r , \tag{3.17}$$

where $\phi(t; \boldsymbol{x}, f)$ is the same as in (2.1) and

$$\gamma_{\beta}(t) = 2^{\beta}(2\pi)^{n/2}\Gamma(\beta+1) t^n V_{\beta+(n/2)}(2\pi t) ,$$

see Stein-Weiss [21, p. 171]. Differentiating (3.17) with respect to R, we get

$$R\frac{\partial}{\partial R}(S_R^\beta f)(x) = -\int_0^\infty \gamma_\beta(r)\theta(r/R;\, x, f)dr/r\ ,$$

where θ is defined by (2.5). If we set $r = e^y$ and $R = e^x$, then the square function $(h^\beta f)(x)$ defined by (1.5) becomes

$$(h^\beta f)^2(x) = \int_{-\infty}^{\infty} |(K_\beta^* * \Theta)(x)|^2 dx\ , \tag{3.18}$$

where

$$K_\beta^*(x) = \gamma_\beta(e^x) = 2^\beta(2\pi)^{n/2}\Gamma(\beta+1)e^{nx}V_{\beta+(n/2)}(2\pi e^x) \tag{3.19}$$

and $\Theta(x) = \theta(e^{-x})$ as in (2.10). Since the order of $\gamma_\beta(t)$ is $t^{-\beta+(n-1)/2}$ as t tends to infinity, $K_\beta^* \notin \mathscr{S}'(-\infty, \infty)$, if $\beta < (n-1)/2$. Now we take $m > (n-1)/2 - \beta$ in Lemma 1 and consider the test function space $\mathscr{T}_m$. Then $K_\beta^* \in \mathscr{T}_m'$. Evidently $|\Theta(x)| \leqq Ce^{-2x} \leqq Ce^{-x}$ $(x \geqq 0)$, $\leqq Ce^{(m+1)x}$ $(x < 0)$. Therefore, Θ is a convolute of this space. Hence the convolution rule is true for $K_\beta^* * \Theta$. The complex Fourier transform of K_β^* is

$$\int_{-\infty}^{\infty} e^{\zeta x}K_\beta^*(x)dx = 2^\beta(2\pi)^{n/2}\Gamma(\beta+1)\int_0^\infty t^{\zeta+n-1}V_{\beta+(n/2)}(2\pi t)dt$$
$$= \frac{\Gamma(\beta+1)\Gamma((\zeta+n)/2)}{2\pi^{\zeta+(n/2)}\Gamma(-\zeta/2+\beta+1)}\ ,$$

and is analytic in $-n < \operatorname{Re}\zeta < m - \{(n-1)/2 - \beta\}$. Hence we get the following.

PROPOSITION 3. *For* $\beta > 0$,

$$(h^\beta f)^2(x) = \int_{-\infty}^{\infty} |\kappa_\beta^*(\xi)\{\Theta(\cdot\,;\, x, f)\}^\wedge(\xi)|^2 d\xi\ , \tag{3.20}$$

where κ_β^* *is the distributional Fourier transform of* K_β^*,

$$\kappa_\beta^*(\xi) = \frac{\Gamma(\beta+1)}{2\pi^{n/2}}\,\frac{\pi^{2\pi i\xi}\Gamma(n/2 - i\pi\xi)}{\Gamma(\beta+1+i\pi\xi)} \tag{3.21}$$

and

$$|\kappa_\beta^*(\xi)| \sim (|\xi|+1)^{-\beta+(n/2)-1} \quad (-\infty < \xi < \infty)\ . \tag{3.22}$$

From Propositions 1, 2 and 3, we get Theorems 1 and 2, because any bounded function is an L^2-multiplier. To prove the uniformity in Theorem 1, it is sufficient to note that (3.16) holds uniformly in k, if $\lambda_{\beta,k}^*(\xi)$ is replaced by $\lambda_{\beta,k}^*(\xi)/\gamma_{k,0}$.

4. Other square functions associated with the Marcinkiewicz function.

Stein [13] introduced the square function $\mu(f)$:

$$(4.1)\qquad \mu(f)(x) = \left\{\int_0^\infty \left| t^{-1}\int_{|\,|<t} |y|^{-n+1}\Omega(y')f(x-y)dy\right|^2 dt/t\right\}^{1/2} .$$

This is a generalization of the classical Marcinkiewicz function to the higher dimensional case. Hörmander [8, p. 136] generalized this. We consider now more general square function $\mu_\Omega^{*\alpha,\delta}f$. We set first

$$(4.2)\qquad (\tilde{M}^{\alpha,\delta}_{\Omega,t}f)(x) = c'_{\alpha,\delta}t^{-\delta}\int_{|y|<t}(1-|y|^2/t^2)^{\alpha-1}|y|^{-n+\delta}\Omega(y')f(x-y)dy$$

for $\delta > 0$, where $c'_{\alpha,\delta} = \Gamma(n/2)\Gamma(\alpha+\delta/2)/\pi^{n/2}\Gamma(\alpha)\Gamma(\delta/2)$ and define $\mu_\Omega^{*\alpha,\delta}f$ by

$$(4.3)\qquad (\mu_\Omega^{*\alpha,\delta}f)(x) = \left\{\int_0^\infty |(\tilde{M}^{\alpha,\delta}_{\Omega,t}f)(x)|^2 dt/t\right\}^{1/2} .$$

Obviously $(\mu_\Omega^{*1,\delta}f)(x)$ coincides with the one defined by Hörmander and $(\mu_\Omega^{*1,1}f)(x)=\{\Gamma(n/2)/2\pi^{n/2}\}\mu(f)(x)$. Furthermore, $(\mu_\Omega^{\alpha}f)(x) = (\mu_\Omega^{*\alpha,n}f)(x)$. Tracing the proof of Proposition 1, we have the following.

PROPOSITION 4. *Let*

$$\tilde{\kappa}_{\alpha,\delta}(\xi) = \frac{\Gamma(n/2)\Gamma(\alpha+\delta/2)}{2\pi^{n/2}\Gamma(\delta/2)}\,\frac{\Gamma(\delta/2-i\pi\xi)}{\Gamma(\alpha+\delta/2-i\pi\xi)} .$$

If $\alpha > -\delta/2$ *and* $\delta > 0$, *then*

$$(4.4)\qquad (\mu_\Omega^{*\alpha,\delta}f)^2(x) = \int_{-\infty}^\infty |\tilde{\kappa}_{\alpha,\delta}(\xi)\{\Psi(\cdot\,; x, f, \Omega)\}^\wedge(\xi)|^2 d\xi$$

and

$$(4.5)\qquad |\tilde{\kappa}_{\alpha,\delta}(\xi)| \sim (|\xi|+1)^{-\alpha} \quad (-\infty < \xi < \infty) .$$

Taking (2.14) and (4.5) into account, we have the following from (2.11) and (4.4).

THEOREM 3. *If* $\alpha > -\delta/2$ *and* $\delta > 0$, *then*

$$(\mu_\Omega^{*\alpha,\delta}f)(x) \approx (\mu_\Omega^{\alpha}f)(x)$$

for $f \in \mathscr{S}(\boldsymbol{R}^n)$, *and the relation holds uniformly in* Ω.

We set further

$$(T^\beta_{\Omega,t}f)(x) = c'_\beta\int_{\boldsymbol{R}^n} t^{-n}V_{\beta+(n/2)}(2\pi|y|/t)\Omega(y')f(x-y)dy ,$$

where $c'_\beta = 2^\beta(2\pi)^{n/2}\Gamma(\beta+1)$, and

$$(\tau^\beta_\Omega f)(x) = \left\{\int_0^\infty |(T^\beta_{\Omega,t}f)(x)|^2 dt/t\right\}^{1/2} .$$

Then

$$(T^{\beta}_{\Omega,t}f)(\boldsymbol{x}) = \{K^*_\beta * \Psi(\cdot\,;\boldsymbol{x}, f, \Omega)\}(x) \quad (t = e^{-x}) ,$$

where K^*_β is defined by (3.19). As shown in §3,

$$|\hat{K}^*_\beta(\xi)| = |\kappa^*_\beta(\xi)| \sim (|\xi| + 1)^{-\beta+(n/2)-1} \quad (-\infty < \xi < \infty) .$$

Comparing this with Proposition 1, we have the following.

THEOREM 4. *If* $\beta = \alpha + n/2 - 1 > 0$, *then*

$$(\boldsymbol{\tau}^{\beta}_{\Omega}f)(\boldsymbol{x}) \approx (\mu^{\alpha}_{\Omega}f)(\boldsymbol{x})$$

for $f \in \mathscr{S}(\boldsymbol{R}^n)$, *and the relation holds uniformly in* Ω.

5. Spherical square functions arising as Riesz potentials. In this section we assume $\hat{f}(\boldsymbol{\xi}) = 0$ near the origin for $f \in \mathscr{S}(\boldsymbol{R}^n)$ and denote the class of all such f by $\mathscr{S}_0(\boldsymbol{R}^n)$. The Riesz potential of f is defined by

$$(I_\alpha f)(\boldsymbol{x}) = \int_{\boldsymbol{R}^n} |\boldsymbol{\xi}|^{-\alpha}\hat{f}(\boldsymbol{\xi})e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} . \tag{5.1}$$

Set

$$(I^\alpha f)(\boldsymbol{x}) = \int_{\boldsymbol{R}^n} |\boldsymbol{\xi}|^{\alpha}\hat{f}(\boldsymbol{\xi})e^{2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} . \tag{5.2}$$

Now we will define such a spherical square function as

$$(D^\alpha f)(\boldsymbol{x}) = \left[\int_0^\infty \left| t^{-\alpha}\int_\Sigma \{f(\boldsymbol{x} - t\boldsymbol{y}') - f(\boldsymbol{x})\}d\sigma(\boldsymbol{y}')\right|^2 dt/t\right]^{1/2} . \tag{5.3}$$

Then $(D^\alpha f)(\boldsymbol{x})$ is essentially smaller than

$$\mathscr{D}_\alpha(f)(\boldsymbol{x}) = \left\{\int_{\boldsymbol{R}^n} |f(\boldsymbol{x} - \boldsymbol{y}) - f(\boldsymbol{x})|^2 |\boldsymbol{y}|^{-n-2\alpha} d\boldsymbol{y}\right\}^{1/2} \tag{5.4}$$

of Stein [15, p. 102], because

$$\mathscr{D}_\alpha(f)(\boldsymbol{x}) = \left\{\int_0^\infty\int_\Sigma |f(\boldsymbol{x} - t\boldsymbol{y}') - f(\boldsymbol{x})|^2 d\sigma(\boldsymbol{y}')t^{-1-2\alpha}dt\right\}^{1/2} .$$

We will prove the following.

THEOREM 5. *If* $\beta = \alpha + n/2$ *and* $0 < \alpha < 1$, *then*

$$(h^\beta f)(\boldsymbol{x}) \approx D^\alpha(I_\alpha f)(\boldsymbol{x}) \tag{5.5}$$

for any $f \in \mathscr{S}_0(\boldsymbol{R}^n)$.

For the proof of Theorem 5, we give the following two propositions. First we consider

$$(\tau^\beta_R f)(\boldsymbol{x}) = (S^\beta_R f)(\boldsymbol{x}) - (S^{\beta-1}_R f)(\boldsymbol{x}) .$$

Then elementary calculation yields

(5.6)
$$\tau^{\beta}_{1/t}(I^{\alpha}f)(\boldsymbol{x}) = -(2\pi)^{n/2}t^{-\alpha}\int_0^\infty \left\{r^{n-1}\int_0^1 u^{\alpha+n+1}(1-u^2)^{\beta-1}V_{(n/2)-1}(2\pi ru)du\right\} \times\phi(tr;\, \boldsymbol{x},\, f)dr\ ,$$

where $\phi(t) = \phi(t; \boldsymbol{x}, f)$ is given by (2.1). Set

(5.7)
$$\Gamma_0(t) = \int_0^t r^{n-1}dr \int_0^1 u^{\alpha+n+1}(1-u^2)^{\beta-1}V_{(n/2)-1}(2\pi ru)du = t^n\int_0^1 u^{\alpha+n+1}(1-u^2)^{\beta-1}V_{n/2}(2\pi tu)du\ .$$

Then by integration by parts we have

(5.8)
$$\tau^{\beta}_{1/t}(I^{\alpha}f)(\boldsymbol{x}) = (2\pi)^{n/2}t^{-\alpha}\int_0^\infty \Gamma_0(r)\theta(tr;\, \boldsymbol{x}, f)dr/r\ ,$$

where $\theta(t) = \theta(t; \boldsymbol{x}, f)$ is given by (2.5). Moreover, we set

(5.9)
$$\theta_{-\alpha}(t) = \theta_{-\alpha}(t;\, \boldsymbol{x}, f) = t^{-\alpha}\theta(t;\, \boldsymbol{x}, f)\ .$$

As in the preceding sections, putting $K^*(x) = K^*_{\alpha,\beta}(x) = (2\pi)^{n/2}e^{\alpha x}\Gamma_0(e^x)$ and $\Theta_{-\alpha}(x; \boldsymbol{x}, f) = \theta_{-\alpha}(e^{-x})$, (5.8) becomes

$$\tau^{\beta}_{1/t}(I^{\alpha}f)(\boldsymbol{x}) = \{K^* * \Theta_{-\alpha}(\cdot;\, \boldsymbol{x}, f)\}(x) \quad (t = e^{-x})\ .$$

The complex Fourier transform of K^* is

$$\begin{aligned}\int_{-\infty}^\infty & e^{\zeta x}K^*(x)dx \\ &= (2\pi)^{n/2}\int_0^\infty t^{\zeta+\alpha+n-1}dt\int_0^1 u^{\alpha+n+1}(1-u^2)^{\beta-1}V_{n/2}(2\pi tu)du \\ &= (2\pi)^{n/2}\int_0^1 u^{\alpha+n+1}(1-u^2)^{\beta-1}du\int_0^\infty t^{\zeta+\alpha+n-1}V_{n/2}(2\pi ut)dt \\ &= \frac{\Gamma(\beta)\Gamma(-\zeta/2+1)\Gamma((\zeta+\alpha+n)/2)}{4\pi^{\zeta+\alpha+(n/2)}\Gamma(-\zeta/2+\beta+1)\Gamma(-\zeta/2-\alpha/2+1)}\end{aligned}$$

for $-(\alpha+n) < \operatorname{Re}\zeta < -\alpha-(n-1)/2$. By an argument analogous to that in Proposition 2, we have:

PROPOSITION 5. *For* $-n < \alpha \leqq 1$ *and* $\beta > 0$,

(5.10)
$$\{h^{\beta}(I^{\alpha}f)(\boldsymbol{x})\}^2 = \int_{-\infty}^\infty |\eta^*_{\alpha,\beta}(\xi)\{\Theta_{-\alpha}(\cdot;\, \boldsymbol{x}, f)\}^\wedge(\xi)|^2 d\xi\ ,$$

where $\eta^*_{\alpha,\beta}$ *is the distributional Fourier transform of* $K^*_{\alpha,\beta}$, *that is to say,*

$$\eta^*_{\alpha,\beta}(\xi) = \frac{\Gamma(\beta+1)}{2\pi^{\alpha+(n/2)}}\,\frac{\pi^{2\pi i\xi}\Gamma(1+i\pi\xi)\Gamma((\alpha+n)/2-i\pi\xi)}{\Gamma(\beta+1+i\pi\xi)\Gamma(-\alpha/2+1+i\pi\xi)}$$

and

$$|\eta^{*}_{\alpha,\beta}(\xi)| \sim (|\xi|+1)^{\alpha-\beta+(n/2)-1} \quad (-\infty<\xi<\infty)\,. \tag{5.11}$$

Concerning $(D^\alpha f)(\boldsymbol{x})$ defined by (5.3) we proceed analogously. By (2.1), (2.5) and (5.9), we have

$$\phi(t;\boldsymbol{x},f)-\phi(0;\boldsymbol{x},f)=\int_0^1\theta(tr;\boldsymbol{x},f)dr/r$$

and

$$t^{-\alpha}\{\phi(t;\boldsymbol{x},f)-\phi(0;\boldsymbol{x},f)\}=\int_0^1 r^\alpha\theta_{-\alpha}(tr;\boldsymbol{x},f)dr/r\,. \tag{5.12}$$

Hence, if we set $K(x)=e^{\alpha x}$ $(x\leqq 0)$ and $=0$ $(x>0)$, then (5.12) becomes

$$\{K*\Theta_{-\alpha}(\cdot;\boldsymbol{x},f)\}(x)$$

with $t=e^{-x}$. Hence we get:

PROPOSITION 6. *If* $0<\alpha<1$, then

$$(D^\alpha f)^2(\boldsymbol{x})=\int_{-\infty}^{\infty}|\kappa(\xi)\{\Theta_{-\alpha}(\cdot;\boldsymbol{x},f)\}\hat{}(\xi)|^2d\xi\,,$$

where $\kappa(\xi)=(\alpha-2\pi i\xi)^{-1}$ *and*

$$|\kappa(\xi)|\sim(|\xi|+1)^{-1}\quad(-\infty<\xi<\infty)\,.$$

Theorem 5 follows, if we take $I_\alpha f$ as f in Propositions 5 and 6.

For $(\tilde{h}^\beta_{Y_k}f)(\boldsymbol{x})$, we get analogous one. For a surface spherical harmonic Y_k of degree $k(\geqq 1)$, set

$$(D^\alpha_{Y_k}f)(\boldsymbol{x})=\left\{\int_0^\infty\left|t^{-\alpha}\int_\Sigma f(\boldsymbol{x}-t\boldsymbol{y}')Y_k(\boldsymbol{y}')d\sigma(\boldsymbol{y}')\right|^2dt/t\right\}^{1/2}. \tag{5.13}$$

THEOREM 6. *If* $\beta=\alpha+n/2$ *and* $0<\alpha<1$, *then the relation*

$$(\tilde{h}^\beta_{Y_k}f)(\boldsymbol{x})/|\gamma_{k,0}|\approx D^\alpha_{Y_k}(I_\alpha f)(\boldsymbol{x})$$

holds uniformly in Y_k *and* k *for any* $f\in\mathscr{S}_0(\boldsymbol{R}^n)$, *where the constant* $\gamma_{k,0}$ *is the same as in Theorem* 1.

The method of proof is the same as that for Theorem 1 and the one above. If we set

$$\psi_{-\alpha}(t;\boldsymbol{x},f,Y_k)=t^{-\alpha}\psi(t;\boldsymbol{x},f,Y_k)$$

and

$$\Psi_{-\alpha}(x;\boldsymbol{x},f,Y_k)=\psi_{-\alpha}(e^{-x};\boldsymbol{x},f,Y_k)\,,$$

then we have

$$(D^{\alpha}_{Y_k}f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty}|\Psi_{-\alpha}(x;\ \boldsymbol{x}, f,\ Y_k)|^2 dx \tag{5.14}$$

by definition. On the other hand, by an argument parallel to that in the proof of Proposition 2, we have

$$\tilde{S}^{\beta}_{Y_k,R}(I^{\alpha}f)(\boldsymbol{x}) = \gamma_{k,0}\int_0^{\infty}\tilde{\gamma}_{\alpha,\beta,k}(r)\psi_{-\alpha}(r/R;\ \boldsymbol{x}, f,\ Y_k)dr/r$$

and

$$\{\tilde{h}^{\beta}_{Y_k}(I^{\alpha}f)(\boldsymbol{x})\}^2 = \int_{-\infty}^{\infty}|\{\tilde{K}^{*}_{\alpha,\beta,k}*\Psi_{-\alpha}(\cdot;\ \boldsymbol{x}, f,\ Y_k)\}(x)|^2 dx\ ,$$

where $\tilde{K}^{*}_{\alpha,\beta,k}(x) = -2\beta\gamma_{k,0}\{\tilde{\gamma}_{\alpha,\beta,k}(e^x) - \tilde{\gamma}_{\alpha,\beta-1,k}(e^x)\}$ and

$$\tilde{\gamma}_{\alpha,\beta,k}(t) = (2\pi)^{n/2}t^{\alpha+n}\int_0^1 u^{\alpha+n-1}(1-u^2)^{\beta}(2\pi i t u)^k V_{k+(n/2)-1}(2\pi t u)du\ .$$

Furthermore, the same calculation as in the proof of Proposition 2 yields that the complex Fourier transform of $\tilde{K}^{*}_{\alpha,\beta,k}$ is equal to

$$\pi^{-\alpha}\frac{\Gamma(\beta+1)\Gamma(k/2)}{\Gamma((k+n)/2)}\ \frac{\pi^{-\zeta}\Gamma(-\zeta/2+1)\Gamma(\zeta/2+(\alpha+k+n)/2)}{\Gamma(-\zeta/2+\beta+1)\Gamma(-\zeta/2-\alpha/2+k/2)}\ . \tag{5.15}$$

Let $\lambda^{*}_{\alpha,\beta,k}(\xi)$ be in the form which we obtain by exchanging ζ by $-2\pi i\xi$ in (5.15). Then

$$\{\tilde{h}^{\beta}_{Y_k}(I^{\alpha}f)(\boldsymbol{x})\}^2 = \int_{-\infty}^{\infty}|\lambda^{*}_{\alpha,\beta,k}(\xi)\{\Psi_{-\alpha}(\cdot;\ \boldsymbol{x}, f,\ Y_k)\}^{\wedge}(\xi)|^2 d\xi\ . \tag{5.16}$$

By the asymptotic estimate of Γ-function, we have

$$|\lambda^{*}_{\alpha,\beta,k}(\xi)|/|\gamma_{k,0}| \sim (|\xi|+1)^{\alpha-\beta+(n/2)} \quad (-\infty<\xi<\infty) \tag{5.17}$$

uniformly in k. Replacement of f by $I_{\alpha}f$ in (5.14) and (5.16), and the relation (5.17) prove Theorem 6.

Now, we give a relation between $h^{\beta}f$ defined by (1.5) and the Littlewood-Paley g^*-function $g^{*}_{\lambda}(f)$:

$$g^{*}_{\lambda}(f)(\boldsymbol{x}) = \left\{\int_0^{\infty}\int_{R^n}\frac{t^{\lambda+1}}{(|\boldsymbol{x}-\boldsymbol{y}|^2+t^2)^{(\lambda+n)/2}}|\nabla u(\boldsymbol{y}, t)|^2 d\boldsymbol{y}dt\right\}^{1/2}, \tag{5.18}$$

defined by Stein [14], where u is the Poisson integral of f.

As remarked in the definition of $\mathscr{D}_{\alpha}(f)$ in (5.4), we have $(D^{\alpha}f)(\boldsymbol{x}) \leqq C_n\mathscr{D}_{\alpha}(f)(\boldsymbol{x})$. Theorem 5 shows $(h^{\beta}f)(\boldsymbol{x}) \leqq C_{\beta}D^{\alpha}(I_{\alpha}f)(\boldsymbol{x})$ $(\beta=\alpha+n/2, 0<\alpha<1)$. Stein [15] showed that $\mathscr{D}_{\alpha}(I_{\alpha}f)(\boldsymbol{x}) \leqq C_{\alpha,\lambda}g^{*}_{\lambda}(f)(\boldsymbol{x})$ $(0<\alpha<1, 0<\lambda<2\alpha)$. Therefore we have

$$(h^{\beta}f)(\boldsymbol{x}) \leqq C_{\beta,\lambda}g^{*}_{\lambda}(f)(\boldsymbol{x}) \quad (0<\lambda<2, \lambda+n<2\beta)\ . \tag{5.19}$$

Next we consider the relation between $\mu_{\Omega}^{*\alpha} f = \mu_{\Omega}^{*\alpha,1} f$ and $g_{\lambda}^{*}(f)$. By Theorems 3, 1 and 6, we have

$$(\mu_{Y_k}^{*\alpha} f)(\boldsymbol{x}) \approx D_{Y_k}^{\alpha}(I_\alpha f)(\boldsymbol{x}) \quad (0 < \alpha < 1)$$

uniformly in Y_k. Hence, by the Schwarz inequality and the above result of Stein,

$$D_{Y_k}^{\alpha}(I_\alpha f)(\boldsymbol{x}) \leqq \| Y_k \|_{L^2(\Sigma)} \mathscr{D}_\alpha(I_\alpha f)(\boldsymbol{x}) \leqq C_{\alpha,\lambda} \| Y_k \|_{L^2(\Sigma)} g_{\lambda}^{*}(f)(\boldsymbol{x})$$
$$(0 < \alpha < 1, 0 < \lambda < 2\alpha) .$$

Therefore we have

$$(\mu_{Y_k}^{*\alpha} f)(\boldsymbol{x}) \leqq C_{\alpha,\lambda} \| Y_k \|_{L^2(\Sigma)} g_{\lambda}^{*}(f)(\boldsymbol{x}) \tag{5.20}$$

for $0 < \alpha < 1$ and $0 < \lambda < 2\alpha$. If we have any good condition for the expansion $\Omega = \sum Y_k$, we shall be able to get

$$(\mu_{\Omega}^{*\alpha} f)(\boldsymbol{x}) \leqq C_{\alpha,\lambda,\Omega} g_{\lambda}^{*}(f)(\boldsymbol{x}) \quad (0 < \alpha < 1, 0 < \lambda < 2\alpha) .$$

6. Square functions arising from the Abel-Poisson summation. We define the spherical Abel-Poisson means of a function f by

$$(A_t^{m,\alpha} f)(\boldsymbol{x}) = c_{m,\alpha} \int_{R^n} |\boldsymbol{y}|^{\alpha} \exp(-|\boldsymbol{y}|^{m+1}) f(\boldsymbol{x} - t\boldsymbol{y}) d\boldsymbol{y} , \tag{6.1}$$

where $c_{m,\alpha} = (m+1)\Gamma(n/2)/2\pi^{n/2}\Gamma((\alpha + n)/(m+1))$, $m > -1$ and $\alpha > -n$, following Levinson [11]. The corresponding square function is

$$(\delta^{m,\alpha} f)(\boldsymbol{x}) = \left\{ \int_0^\infty \left| \frac{\partial}{\partial t} (A_t^{m,\alpha} f)(\boldsymbol{x}) \right|^2 t dt \right\}^{1/2} \tag{6.2}$$
$$= \left[\int_0^\infty | -(\alpha + n)\{(A_t^{m,\alpha} f)(\boldsymbol{x}) - (A_t^{m,\alpha+m+1} f)(\boldsymbol{x})\} |^2 dt/t \right]^{1/2} .$$

We also define the square function from the spherical means of Abel-Poisson type of Fourier transform. Let

$$u_m(\boldsymbol{x}, t) = c_m'' \int_{R^n} (|\boldsymbol{y}|^{2(m+1)} + 1)^{-(n+1)/2} f(\boldsymbol{x} - t\boldsymbol{y}) d\boldsymbol{y} , \tag{6.3}$$

where $m > -1/(n+1)$ and the constant c_m'' is taken so that $u_m(\boldsymbol{x}, 0) = f(\boldsymbol{x})$. Set

$$g_{m+1}(f)(\boldsymbol{x}) = \left\{ \int_0^\infty \left| \frac{\partial}{\partial t} u_m(\boldsymbol{x}, t) \right|^2 t dt \right\}^{1/2} . \tag{6.4}$$

When $m = 1$ and $\alpha = 0$, (6.1) agrees with the Gauss-Weierstrass integral of f, and when $m = 0$, (6.3) is the Poisson integral of f and (6.4) is the "real part" of the original Littlewood-Paley function $g(f)(\boldsymbol{x})$. See Stein [16, p. 83], where it is denoted by $g_1(f)(\boldsymbol{x})$.

We can prove the following:

THEOREM 7. *If $m > -1/(n+1)$ and $\alpha = (m-1)n/2$, then*

$$(\delta^{m,\alpha}f)(\boldsymbol{x}) \approx g_{m+1}(f)(\boldsymbol{x})$$

for $f \in \mathscr{S}(\boldsymbol{R}^n)$.

The proof uses the same idea as that in the preceding sections.

PROPOSITION 7. *For $m > -1$ and $\alpha > -n$,*

$$(\delta^{m,\alpha}f)^2(\boldsymbol{x}) = \int_{-\infty}^{\infty} |\hat{\mathscr{A}}_{m,\alpha}(\xi)\{\Theta(\cdot\,;\boldsymbol{x}, f)\}^{\wedge}(\xi)|^2 d\xi\ ,$$

where Θ is defined by (2.10),

$$\hat{\mathscr{A}}_{m,\alpha}(\xi) = \frac{\Gamma(n/2)}{2\pi^{n/2}\Gamma((\alpha+n)/(m+1))}\Gamma\Big(\frac{\alpha+n-i2\pi\xi}{m+1}\Big)$$

and

$$|\hat{\mathscr{A}}_{m,\alpha}(\xi)| \sim (|\xi|+1)^{(\alpha+n)/(m+1)-(1/2)}\exp(-\pi^2|\xi|/(m+1)) \qquad (-\infty < \xi < \infty)\ .$$

PROPOSITION 8. *For $m > -1/(n+1)$,*

$$\{g_{m+1}(f)(\boldsymbol{x})\}^2 = \int_{-\infty}^{\infty} |\hat{P}_m(\xi)\{\Theta(\cdot\,;\boldsymbol{x}, f)\}^{\wedge}(\xi)|^2 d\xi\ , \tag{6.5}$$

where

$$\hat{P}_m(\xi) = c_m'''\Gamma\Big(\frac{n-i2\pi\xi}{2(m+1)}\Big)\Gamma\Big(\frac{m(n+1)+1+i2\pi\xi}{2(m+1)}\Big) \tag{6.6}$$

with $c_m''' = \Gamma(n/2)/2\pi^{n/2}\Gamma(n/2(m+1))\Gamma(\{m(n+1)+1\}/2(m+1))$, *and*

$$|\hat{P}_m(\xi)| \sim (|\xi|+1)^{(n-1)/2}\exp(-\pi^2|\xi|/(m+1)) \quad (-\infty < \xi < \infty)\ . \tag{6.7}$$

PROOF OF PROPOSITION 7. Set

$$\mathscr{A}_{m,\alpha}(x) = c_{m,\alpha}e^{(\alpha+n)x}\exp(-e^{(m+1)x})\ .$$

Then, by the change of variables $t = e^{-x}$, we have

$$t\frac{\partial}{\partial t}(A_t^{m,\alpha}f)(\boldsymbol{x}) = \{\mathscr{A}_{m,\alpha}*\Theta(\cdot\,;\boldsymbol{x}, f)\}(x)$$

as in the proof of Proposition 1. In this case, the convolution is ordinary and we can prove Proposition 7 without the concept of distribution. It is easy to calculate the Fourier transform $\hat{\mathscr{A}}_{m,\alpha}$ of $\mathscr{A}_{m,\alpha}$ and we get Proposition 7. q.e.d.

PROOF OF PROPOSITION 8. We set

$$P_m(x) = c_m'' e^{nx}\{e^{2(m+1)x} + 1\}^{-(n+1)/2} .$$

Moreover, by the change of variables $t = e^{-x}$, then

$$t\frac{\partial}{\partial t} u_m(\boldsymbol{x}, t) = \{P_m * \Theta(\cdot\,; \boldsymbol{x}, f)\}(x) .$$

The Fourier transform of P_m is (6.6). q.e.d.

REMARK. Except when $m = 0$, $u_m(\boldsymbol{x}, t)$ in (6.3) does not represent the exact Abel-Poisson mean of Fourier transform of f. In fact, in the case $m = 1$ and $\alpha = 0$, $(A_t^{m,\alpha} f)(\boldsymbol{x})$ is the Gauss-Weierstrass mean of function f and also that of its Fourier transform coincidentally. However, if we take $m = 1$ and $\alpha = 0$ in Proposition 7 and $m = 0$ in Proposition 8, then we have

$$|\hat{\mathscr{A}}_{1,0}(\xi)| \sim (|\xi| + 1)^{(n-1)/2} \exp(-\pi^2|\xi|/2) \quad \text{and}$$
$$|\hat{P}_0(\xi)| \sim (|\xi| + 1)^{(n-1)/2} \exp(-\pi^2|\xi|) .$$

These show that the square function $(\delta^{1,0} f)(\boldsymbol{x})$ arising from the Gauss-Weierstrass summation is not smaller than the classical Littlewood-Paley function $g_1(f)(\boldsymbol{x})$. Hardy [7, p. 176] already observed that a summable (W) Fourier series is certainly summable (A).

It may be natural to consider the square functions

$$(\tilde{\delta}_{\Omega}^{m,\alpha} f)(\boldsymbol{x}) = \left\{\int_0^\infty |(\tilde{A}_{\Omega,t}^{m,\alpha} f)(\boldsymbol{x})|^2 dt/t\right\}^{1/2}$$

for $m > -1$ and $\alpha > -n$, and

$$\tilde{g}_{\Omega,m+1}(f)(\boldsymbol{x}) = \left\{\int_0^\infty |\tilde{u}_{\Omega,m}(\boldsymbol{x}, t)|^2 dt/t\right\}^{1/2}$$

for $m > -1/(n + 1)$, as the counterparts of $(\delta^{m,\alpha} f)(\boldsymbol{x})$ and $g_{m+1}(f)(\boldsymbol{x})$, where

$$(\tilde{A}_{\Omega,t}^{m,\alpha} f)(\boldsymbol{x}) = c_{m,\alpha}\int_{R^n} \Omega(\boldsymbol{y}')|\boldsymbol{y}|^\alpha \exp(-|\boldsymbol{y}|^{m+1}) f(\boldsymbol{x} - t\boldsymbol{y}) d\boldsymbol{y}$$

and

$$\tilde{u}_{\Omega,m}(\boldsymbol{x}, t) = c_m''\int_{R^n} \Omega(\boldsymbol{y}')(|\boldsymbol{y}|^{2(m+1)} + 1)^{-(n+1)/2} f(\boldsymbol{x} - t\boldsymbol{y}) d\boldsymbol{y} .$$

Between them, we have the following relation:

THEOREM 8. *If $m > -1/(n + 1)$ and $\alpha = (m - 1)n/2$,*

$$(\tilde{\delta}_{\Omega}^{m,\alpha} f)(\boldsymbol{x}) \approx \tilde{g}_{\Omega,m+1}(f)(\boldsymbol{x})$$

for $f \in \mathscr{S}(\boldsymbol{R}^n)$ *and the relation is uniform in* Ω.

The proof is similar to that of Theorem 7.

If we take $\Omega_j(\boldsymbol{y}) = y_j/|\boldsymbol{y}|$ $(j = 1, 2, \cdots, n)$ as Ω, then we have the relation

$$g_x(f)(\boldsymbol{x}) = \left\{\int_0^\infty |\nabla_x u(\boldsymbol{x}, t)|^2 t dt\right\}^{1/2} \approx \sum_{j=1}^n (\tilde{\delta}_{\Omega_j}^{0,\alpha} f)(\boldsymbol{x})$$

$(\alpha = -n/2 + 1)$, where u is the Poisson integral of f. The left hand side in the above relation is another part of the classical Littlewood-Paley g-function. See Stein [16, p. 83].

7. Applications. Let $H^p(\boldsymbol{R}^n)$, $0 < p < \infty$, be the Hardy spaces in the sense of Fefferman-Stein [4]. If $1 < p < \infty$, then $H^p(\boldsymbol{R}^n)$ coincides with $L^p(\boldsymbol{R}^n)$ and its norms are comparable. So for any p, $0 < p < \infty$, we assume that $\|f\|_p$ denotes the $H^p(\boldsymbol{R}^n)$-norm of f. Moreover, we denote by $\|g\|_{L^p(\boldsymbol{R}^n)}$ the $L^p(\boldsymbol{R}^n)$-norm of $g \in L^p(\boldsymbol{R}^n)$, $0 < p < \infty$.

It is known that the class $\mathscr{S}_0(\boldsymbol{R}^n)$ defined in §5 is dense in $H^p(\boldsymbol{R}^n)$, $0 < p \leqq 1$, and $L^p(\boldsymbol{R}^n) = H^p(\boldsymbol{R}^n)$, $1 < p < \infty$. See Calderón-Torchinsky [1, II, pp. 104-105]. This is useful for extension of f.

The square function arising from the Cesàro summation is generally greater than that arising from the Abel summation, except for a constant factor (Flett [5, p. 116]). Thus concerning the inequality $\|S(f)\|_{L^p(\boldsymbol{R}^n)} \leqq A_p\|f\|_p$ for any square function $S(f)$, if $S(f)$ is generated from a Cesàro type summation, then it is better than the inequality whose $S(f)$ is generated from an Abel type summation.

The following two H^p-boundedness theorems about square functions are fundamental for our argument.

THEOREM A. *For* $0 < p < \infty$,

$$\|f\|_p \leqq A_p\|g_1(f)\|_{L^p(\boldsymbol{R}^n)} \quad \text{and} \quad \|f\|_p \leqq A'_p\|g_x(f)\|_{L^p(\boldsymbol{R}^n)} .$$

This was given by Fefferman-Stein [4, p. 185] and Calderón-Torchinsky [1, I, p. 55].

THEOREM B. *For* $\beta > n(1/p - 1/2) + 1/2$ $(0 < p \leqq 2)$ *and* $\beta > (n-1)(1/2 - 1/p) + 1/2$ $(2 \leqq p < \infty)$,

$$\|h^\beta f\|_{L^p(\boldsymbol{R}^n)} \leqq B_{p,\beta}\|f\|_p .$$

For $1 < p \leqq 2$, Theorem B was given by Sunouchi [22]. We cannot find the case $0 < p \leqq 1$ in the literature, but it can be proved by the atomic decomposition of $H^p(\boldsymbol{R}^n)$; see Latter [10]. Furthermore, when $0 < p < 1$ and $\beta = n(1/p - 1/2) + 1/2$, h^β is weak type (H^p, L^p). For the case $2 \leqq p < \infty$, we can prove Theorem B as follows.

As proved in Theorem 5, for $\beta = \alpha + n/2, 0 < \alpha < 1$,

$$(h^\beta f)(x) \approx D^\alpha(I_\alpha f)(x) \leqq A_\alpha \mathscr{D}_\alpha(I_\alpha f)(x) .$$

However, for $p \geqq 2$, Stein [15, p. 103, Lemma 1] showed that, for $\alpha > 0$,

$$\| \mathscr{D}_\alpha(I_\alpha f) \|_{L^p(R^n)} \leqq A_{p,\alpha} \| f \|_p .$$

Hence, for $n/2 < \beta < n/2 + 1$ and $p \geqq 2$,

$$\| h^\beta f \|_{L^p(R^n)} \leqq A_{p,\beta} \| f \|_p .$$

So we can get the conclusion by interpolation between $p_1 = 2, \beta > 1/2$ and $p_2 = p, \beta > n/2$.

This result is better than that of Igari-Kuratsubo [9].

Combining these two theorems with our results in the preceding sections, we have following Corollaries 1, 2, 3 and 4.

COROLLARY 1. *For* $\alpha > n/p - n + 3/2$ $(0 < p \leqq 2)$ *and* $\alpha > -(n-1)/p + 1$ $(2 \leqq p < \infty)$,

$$A_{p,\alpha} \| f \|_p \leqq \| \nu^\alpha f \|_{L^p(R^n)} \leqq B_{p,\alpha} \| f \|_p .$$

COROLLARY 2. *For* $\alpha > n/p - n + 1/2$ $(0 < p \leqq 2)$ *and* $\alpha > -(n-1)/p$ $(2 \leqq p < \infty)$,

$$A_{p,\alpha,k} \| \tilde{f}_{Y_k} \|_p \leqq \| \mu^\alpha_{Y_k} f \|_{L^p(R^n)} \leqq B_{p,\alpha,k} \| \tilde{f}_{Y_k} \|_p .$$

Since $\| \tilde{f}_{Y_k} \|_p \leqq C_{p,Y_k} \| f \|_p$, we have

$$\| \mu^\alpha_{Y_k} f \|_{L^p(R^n)} \leqq C_{p,\alpha,Y_k} \| f \|_p \tag{7.1}$$

for the above range. By Theorem 3, we can replace $\mu^\alpha_{Y_k} f$ by $\mu^{*\alpha}_{Y_k} f = \mu^{*\alpha,1}_{Y_k} f$ in (7.1) for $\alpha > -1/2$. In particular for $\alpha \geqq 1/2$, we get

$$\| \mu^{*\alpha}_{Y_k} f \|_{L^p(R^n)} \leqq C_{p,\alpha,Y_k} \| f \|_p \quad (1 < p < \infty) .$$

So the case $\alpha = 1$ is true. This case was studied by Stein [13] and Hörmander [8]. Their operators are more general than ours, but the methods of proofs are different.

In order to get converse inequalities for $\mu^\alpha_{Y_k} f$, we need $\| f \|_p \leqq C \| \tilde{f}_{Y_k} \|_p$. From this point of view, if Y_k is the j-th component of the Riesz transform, i.e., $Y_k(x') = x_j/|x|$, then

$$A_{p,\alpha} \| f \|_p \leqq \sum_{j=1}^n \| \mu^\alpha_j f \|_{L^p(R^n)} \leqq B_{p,\alpha} \| f \|_p$$

for the same range as in Corollary 2, where $\mu^\alpha_j f$ means $\mu^\alpha_\Omega f$ for $\Omega(x') = x_j/|x|$. This was also given by Stein and Hörmander.

COROLLARY 3. *For* $1>\alpha>n/p-n+1/2$ $(2n/(2n+1)<p<2n/(2n-1))$ and $1>\alpha>0$ $(2n/(2n-1)\leqq p<\infty)$,

$$A_{p,\alpha}\|f\|_p \leqq \|D^\alpha(I_\alpha f)\|_{L^p(R^n)} \leqq B_{p,\alpha}\|f\|_p$$

and

$$A_{p,\alpha,k}\|\tilde{f}_{Y_k}\|_p \leqq \|D^\alpha_{Y_k}(I_\alpha f)\|_{L^p(R^n)} \leqq B_{p,\alpha,k}\|\tilde{f}_{Y_k}\|_p ,$$

where Y_k *is a surface spherical harmonic of degree* $k\geqq 1$.

COROLLARY 4. *When* $m\geqq 0$ *and* $\alpha=(m-1)n/2$, *the relation*

$$A_{p,m}\|f\|_p \leqq \|\delta^{m,\alpha}f\|_{L^p(R^n)} \leqq B_{p,m}\|f\|_p$$

holds for $0<p<\infty$.

Stein-Wainger's "Problem 6 (*a*)" in [20, p. 1289] is concerned with $g(f)(x)$ and $(\nu^\alpha f)(x)$ for $\alpha=0$. However, $g_1(f)(x)\approx(\delta^{0,-n/2}f)(x)$ is concerned with the Abel means and $(\nu^\alpha f)(x)$ with the Cesàro means. These facts and Corollaries 1 and 4 may be an answer to the problem.

Let $\mathscr{M}^\alpha f$ be the maximal function for $(M_t^\alpha f)(x)$ of (1.1), i.e.,

$$(\mathscr{M}^\alpha f)(x)=\sup\{|(M_t^\alpha f)(x)|;\ 0<t<\infty\} .$$

COROLLARY 5. *For* $\alpha>n/p-n+1$ $(0<p\leqq 2)$ *and* $\alpha>(-n+2)/p$ $(2\leqq p<\infty)$,

$$\|\mathscr{M}^\alpha f\|_{L^p(R^n)} \leqq C_{p,\alpha}\|f\|_p .$$

PROOF. For $0<p\leqq 2$, we can deduce the conclusion by a routine argument from Corollary 1. The other case is immediate from interpolation between the case $p=2$ and $p=\infty$, which is obvious. q.e.d.

Stein-Wainger [20, p. 1283, Th. 14] and Stein-Taibleson-Weiss [19, Th. II] gave this result. In particular, for $n/(n-1)<p<\infty$, $n\geqq 3$,

$$\|\mathscr{M}f\|_{L^p(R^n)} \leqq C_p\|f\|_{L^p(R^n)} ,$$

where $(\mathscr{M}f)(x)=(\mathscr{M}^0 f)(x)$. This had already been proved by Stein [17].

Let ϕ be a $C_0^\infty(R^n)$-function with $\hat{\phi}(0)=1$ and set $\phi_t(x)=t^{-n}\phi(t^{-1}x)$. Then Stein-Wainger [20, p. 1271] gave the following definition:

$$(7.2)\qquad (g_\alpha f)(x)=\left\{\int_0^\infty|(M_t^\alpha f)(x)-(f*\phi_t)(x)|^2dt/t\right\}^{1/2}$$

and proved for $\alpha>(1-n)/2$,

$$\|g_\alpha f\|_2 \leqq C_\alpha\|f\|_2 .$$

To avoid confusion of this notation $(g_\alpha f)(x)$ in (7.2) with (6.4), we denote (7.2) by $(N^\alpha f)(x)$ instead of $(g_\alpha f)(x)$.

COROLLARY 6. *For* $\alpha>n/p-n+1/2$ $(0<p\leqq 2)$ *and* $\alpha>-(n-1)/p$

$(2 \leqq p < \infty)$,

$$\| N^{\alpha} f \|_{L^p(R^n)} \leqq C_{p,\alpha} \| f \|_p .$$

PROOF. Take N so that $\alpha + N > 2$. Then

$$(N^{\alpha} f)(x) \leqq \sum_{\nu=1}^{N} \left\{ \int_0^{\infty} |(M_t^{\alpha+\nu} f)(x) - (M_t^{\alpha+\nu-1} f)(x)|^2 dt/t \right\}^{1/2} + \left\{ \int_0^{\infty} |(K_t * f)(x)|^2 dt/t \right\}^{1/2} , \tag{7.3}$$

where $K_t(x) = t^{-n} K(t^{-1}x)$ and $K(x) = c_{\alpha+N}(1 - |x|^2)_+^{\alpha+N-1} - \phi(x)$. If we apply Corollary 1 for the first term in (7.3) and apply a multiplier theorem in Stein [16, p. 46, Th. 5] for the last term, then we have the conclusion in the case $1 < p < \infty$. For $0 < p \leqq 1$, it is obtained, if N is taken sufficiently larger and the atomic decomposition of f is applied to the last term on the right hand side of (7.3). q.e.d.

Analogously, if we use Theorem B instead of Corollary 1, then we get the following.

COROLLARY 7. *For* $\beta > n(1/p - 1/2) - 1/2$ $(0 < p \leqq 2)$ *and* $\beta > (n-1)(1/2 - 1/p) - 1/2$ $(2 \leqq p < \infty)$,

$$\| G^{\beta} f \|_{L^p(R^n)} \leqq C_{p,\beta} \| f \|_p ,$$

where

$$(G^{\beta} f)(x) = \left\{ \int_0^{\infty} |(S_R^{\beta} f)(x) - (f * \phi_{1/R})(x)|^2 dR/R \right\}^{1/2}$$

and $(S_R^{\beta} f)(x)$ *is given by* (1.3).

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