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CHARACTERIZATION OF QUASI-DISKS AND TEICHMÜLLER SPACES

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1. Introduction and main results. A simply connected domain in the Riemann sphere $\hat{C}$ is called a *quasi-disk* if it is the image of the unit disk by a quasiconformal automorphism of the sphere. Since Ahlfors' investigation [2] in 1963, several characteristic properties of quasi-disks have been studied by many authors. As a result, quasi-disks are related to various topics in analysis. A bird's eye view of these studies are given in Gehring [9]. Among them, the topics with which we are concerned in this article are the BMO *extension property* and the *Schwarzian derivative property*.

Let W be a domain in $\boldsymbol{C}$. Then $f \in L^1_{\mathrm{loc}}(W)$ belongs to BMO(W) if

$$\|f\|_{*,W} = \sup_{B\subset W} \frac{1}{|B|}\int_B |f - f_B|dxdy < +\infty ,$$

where B is a disk in W with $\bar{B}\subset W$, $|B| = \int_B dxdy$ and $f_B = |B|^{-1}\int_B fdxdy$.

Let $\mathscr{F}$ be a subclass of BMO(W). We say that W has the BMO *extension property for* $\mathscr{F}$ if there exists a constant $C_1 > 0$ such that for every $f \in \mathscr{F}$ there is an $F \in \mathrm{BMO}(\boldsymbol{C})$ with $F|W = f$ and

$$\|F\|_{*,C} \leqq C_1\|f\|_{*,W} . \tag{1.1}$$

Jones [11] has shown that a simply connected domain Δ ($\neq \boldsymbol{C}$) in $\boldsymbol{C}$ is a quasi-disk if and only if Δ has the BMO extension property for BMO(Δ) (see also Gehring [9]).

In the first part, we shall strengthen the "if" part of Jones' result.

THEOREM 1. *Let Δ ($\neq \boldsymbol{C}$) be a simply connected domain in $\boldsymbol{C}$. If Δ has the* BMO *extension property for* ABD(Δ), *then Δ is a quasi-disk, where* ABD(Δ) *is the space of all bounded holomorphic functions in Δ with finite Dirichlet integrals.*

In the second part, we shall investigate *Teichmüller spaces* of Fuchsian groups and the *Schwarzian derivative property*, independently

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of Theorem 1. Let Γ be a finitely generated Fuchsian group of the first kind acting on the upper half plane U and let $T(\Gamma)$ be the Teichmüller space of Γ. It is well known (cf. Bers [4]) that $\dim T(\Gamma) < +\infty$ and $T(\Gamma)$ can be idenified with a bounded domain in the Banach space $B_2(L, \Gamma)$ of all holomorphic functions ϕ on the lower half plane L which satisfy

$$\phi(\gamma(z))\gamma'(z)^2 = \phi(z) \quad \text{for all} \quad \gamma \in \Gamma \qquad \text{and}$$
$$\|\phi\| = \sup_{z \in L}(\operatorname{Im} z)^2|\phi(z)| < +\infty .$$

For every ϕ in $B_2(L, \Gamma)$ there exists a meromorphic function W_ϕ defined on L such that the Schwarzian derivative $\{W_\phi, z\}$ of W_ϕ on L is equal to $\phi(z)$ and W_ϕ satisfies the condition

$$W_\phi(z) = (z+i)^{-1} + O(|z+i|) \quad \text{as} \quad z \to -i .$$

We denote by $S(\Gamma)$ the set of all ϕ in $B_2(L, \Gamma)$ such that W_ϕ is univalent on L. It is known that $S(\Gamma)$ is closed and contains $T(\Gamma) \cup \partial T(\Gamma)$. For every ϕ in $B_2(L, \Gamma)$, W_ϕ yields a homomorphism χ_ϕ of Γ with $W_\phi \circ \gamma = \chi_\phi(\gamma) \circ W_\phi$ $(\gamma \in \Gamma)$, and if ϕ is in $S(\Gamma)$, then $\Gamma^\phi = \chi_\phi(\Gamma) = W_\phi \Gamma W_\phi^{-1}$ is a Kleinian group. Furthermore, if ϕ is in $T(\Gamma)$, then Γ^ϕ is a quasi-Fuchsian group, i.e., a Kleinian group with two simply connected invariant components.

We shall show a relation between $S(\Gamma)$ and $T(\Gamma)$.

THEOREM 2. *Int $S(\Gamma)$, the interior of $S(\Gamma)$ on $B_2(L, \Gamma)$, is connected and is equal to $T(\Gamma)$.*

In the proof of Theorem 2, the "λ-lemma" (cf. Mañé, Sad and Sullivan [13]) will play an important role.

COROLLARY. *Let Δ be a simply connected invariant component of a finitely generated non-elementary Kleinian group G. Then Δ is a quasi-disk if and only if there exists a constant $C_2 > 0$ such that every meromorphic function f on Δ satisfying*

$$|\{f, z\}_\Delta| \leqq C_2 \rho_\Delta(z)^2 \tag{1.2}$$

and $\{f, g(z)\}_\Delta g'(z)^2 = \{f, z\}_\Delta$ for all $g \in G$, is univalent, where $\{f, z\}_\Delta$ is the Schwarzian derivative of f in Δ and $\rho_\Delta(z)|dz|$ is the Poincaré metric on Δ.

When $G = \{\text{id.}\}$, Gehring [8] obtained a similar property of quasi-disks called the Schwarzian derivative property,

Furthermore, we shall obtain a geometric property of $T(\Gamma)$ which is an extension of a result in [19].

THEOREM 3. *Let Γ, $T(\Gamma)$ and $B_2(L, \Gamma)$ be as above, and let H be a* *hyperplane in* $B_2(L, \Gamma)$. *Then* $H - H \cap \overline{T(\Gamma)}$ *is connected and* $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) = H \cap \partial T(\Gamma)$, *where* $\hat{\partial}$ *is the boundary operator considered in* H. *In particular,* $\operatorname{Ext} T(\Gamma)$, *the exterior of* $T(\Gamma)$ *in* $B_2(L, \Gamma)$, *is connected and* $\partial(\operatorname{Ext} T(\Gamma)) = \partial T(\Gamma)$.

In the last part, we shall touch upon some results related to the above topics. In fact, we shall extend Theorem 1 to a finitely connected Jordan domain (Theorem 4) and we shall study some properties of Teichmüller spaces (Theorems 5 and 6). Especially, Theorem 5, which shows the complexity of boundaries of Teichmüller spaces in Bers' embedding, is a (strongly) negative answer to a conjecture of Bers [5].

2. Proof of Theorem 1.

LEMMA 1. *Let* Δ $(\neq \boldsymbol{C})$ *be a simply connected domain in* $\boldsymbol{C}$. *Then there exists a constant* $C_3 > 0$ *such that for every harmonic function* u *in* Δ *with the finite Dirichlet integral* $D_\Delta(u)$,

$$\|u\|_{*,\Delta} \leqq C_3 D_\Delta(u)^{1/2} \tag{2.1}$$

holds.

PROOF. From Reimann's theorem (cf. [18]) asserting the quasiconformal invariance of BMO, we may assume that Δ in the unit disk. For a fixed $r > 0$ we consider a disk B in Δ such that the center is $z_0 \in \Delta$ and the hyperbolic diameter is not greater than r. Then we have for all z in B

$$|u(z) - u(z_0)| \leqq \mathrm{d}_H^\Delta(z, z_0)\mathrm{D}_\Delta(u)^{1/2} ,$$

where $\mathrm{d}_H^\Delta(z, z_0) = \sup\{|v(z) - v(z_0)|;\ v \text{ is harmonic in } \Delta \text{ and } \mathrm{D}_\Delta(v) \leqq 1\}$. It is known that $\mathrm{d}_H^\Delta(z, z_0) \leqq \pi^{-1/2}\int_{z_0}^{z} \rho_\Delta(z)|dz| \leqq r\pi^{-1/2}$ (cf. Minda [15]). Hence

$$\begin{aligned}\frac{1}{|B|}\int_B |u(z) - u_B|\,dxdy &= \frac{1}{|B|}\int_B |u(z) - u(z_0)|\,dxdy \\ &\leqq \frac{1}{|B|}\int_B r(\mathrm{D}_\Delta(u)/\pi)^{1/2}dxdy = r(\mathrm{D}_\Delta(u)/\pi)^{1/2} .\end{aligned}$$

Therefore, from [18, I-B, Hilfssatz 2] and its proof, we have the desired assertion (2.1). q.e.d.

LEMMA 2. *Let* Δ $(\neq \boldsymbol{C})$ *be a simply connected domain in* $\boldsymbol{C}$ *having the* BMO *extension property for* $\mathrm{ABD}(\Delta)$. *For* z_1, z_2 *in* Δ, *set*

$$\mathrm{j}_\Delta(z_1, z_2) = \log\left(\frac{|z_1 - z_2|}{\mathrm{d}(z_1, \partial\Delta)} + 1\right)\left(\frac{|z_1 - z_2|}{\mathrm{d}(z_2, \partial\Delta)} + 1\right) ,$$

where $\mathrm{d}(\cdot,\cdot)$ *is the Euclidean distance. Then*

$$\mathrm{h}_\Delta(z_1, z_2) \leqq (\pi/2)(C_1C_3e^2)^2(\mathrm{j}_\Delta(z_1, z_2)+2)^2+\log 2\,, \tag{2.2}$$

where $\mathrm{h}_\Delta(\cdot,\cdot)$ *is the Poincaré distance in* Δ, *and* C_1 *and* C_3 *are the constant as in* (1.1) *and* (2.1), *respectively.*

PROOF. For z_1, z_2 in Δ there exists a harmonic function u such that $\mathrm{D}_\Delta(u)=1$, $u(z_1)=0$ and $u(z_2)=\mathrm{d}_H^\Delta(z_1, z_2)$. Since Δ is conformally equivalent to the unit disk, it is well known (cf. Minda [15]) that

$$u(z_2)^2=\mathrm{d}_H^\Delta(z_1, z_2)^2=(2/\pi)\log\cosh \mathrm{h}_\Delta(z_1, z_2) \leqq (2/\pi)(\mathrm{h}_\Delta(z_1, z_2)-\log 2)\,. \tag{2.3}$$

Furthermore, u is $\mathrm{Re} f$ for some $f\in \mathrm{ABD}(\Delta)$, because u is harmonic on a neighbourhood of $\bar{\Delta}$ when Δ is the unit disk. Hence u has an extension $U\in \mathrm{BMO}(\boldsymbol{C})$ satisfying (1.1). Let B_j be the disk of radius $\mathrm{d}(z_j, \partial\Delta)$ centered at z_j $(j=1, 2)$. From Lemma 1 and the argument in Gehring [9, Chap. III, 10.2], we have

$$\begin{aligned}|U_{B_1}-U_{B_2}| &\leqq (e^2\mathrm{j}_\Delta(z_1, z_2)+2e^2)\|U\|_{*,C} \leqq C_1e^2(\mathrm{j}_\Delta(z_1, z_2)+2)\|u\|_{*,\Delta}\\ &\leqq C_1C_3e^2(\mathrm{j}_\Delta(z_1, z_2)+2)\mathrm{D}_\Delta(u)^{1/2}=C_1C_3e^2(\mathrm{j}_\Delta(z_1, z_2)+2)\,.\end{aligned}$$

On the other hand, $U_{B_1}=u(z_1)=0$ and $U_{B_2}=u(z_2)$, because B_1 and B_2 are contained in Δ. Therefore,

$$0\leqq u(z_2)\leqq C_1C_3e^2(\mathrm{j}_\Delta(z_1, z_2)+2)\,. \tag{2.4}$$

By (2.3) and (2.4) we have the desired inequality (2.2).

PROOF OF THEOREM 1. We shall show that Δ has the hyperbolic segment property, that is, there exist constants A and B such that for every z_1, z_2 in Δ $(z_1\neq z_2)$ and for all $z\in\alpha$

$$l(\alpha)\leqq A|z_1-z_2| \quad \text{and} \quad \min_{j=1,2} l(\alpha_j)\leqq B\mathrm{d}(z, \partial\Delta)\,, \tag{2.5}$$

where α is the hyperbolic segment from z_1 to z_2, $l(\alpha)$ is the Euclidean length of α and α_j $(j=1, 2)$ are components of $\alpha-\{z\}$. If this is done, Theorem 1 is proved, because a simply connected domain with the hyperbolic segment property is a quasi-disk ([9, Chap. III]).

Set $r=\min(\sup_{z\in\alpha}\mathrm{d}(z, \partial\Delta), 2|z_1-z_2|)$. First, we suppose that $r\leqq \max_{j=1,2}\mathrm{d}(z_j, \partial\Delta)$. Let m_j $(j=1, 2)$ be the largest integers for which $2^{m_j}\mathrm{d}(z_j, \partial\Delta)\leqq r$ and let w_j $(j=1, 2)$ be the nearest point on α from z_j satisfying $\mathrm{d}(w_j, \partial\Delta)=2^{m_j}\mathrm{d}(z_j, \partial\Delta)$. Obviously, we may assume that $\mathrm{d}(w_1, \partial\Delta)\leqq \mathrm{d}(w_2, \partial\Delta)$. Then there exist constants B_1, B_2 which do not depend on α and the following inequalities hold.

$$(2.6)\qquad \begin{aligned} &l(\alpha(z_j, w_j)) \leqq B_1 \mathrm{d}(w_j, \partial\Delta) ,\\ &l(\alpha(z_j, z)) \leqq B_1 \mathrm{d}(z, \partial\Delta) \quad \text{for all} \quad z \in \alpha(z_j, w_j) , \end{aligned}$$

and

$$(2.7)\qquad \begin{aligned} &l(\alpha(w_1, w_2)) \leqq B_2 \mathrm{d}(w_1, \partial\Delta) ,\\ &\mathrm{d}(w_2, \partial\Delta) \leqq B_2 \mathrm{d}(z, \partial\Delta) \quad \text{for all} \quad z \in \alpha(w_1, w_2) , \end{aligned}$$

where $\alpha(z, z')$ $(z, z' \in \alpha)$ stands for the open subarc of α from z to z'.

Our proofs of the inequalities (2.6) and (2.7) are slight modifications of those for the inequalities (4) and (9) given in [9, Chap. III, 11.3]. But for completeness, we shall give them.

In showing the inequality (2.6), we may assume that $j = 1$ and $m_1 \geqq 1$. Now, we take points $z_1 = \zeta_1, \zeta_2, \cdots, \zeta_{m_1}, \zeta_{m_1+1} = w_1$ on $\alpha(z_1, w_1)$ so that ζ_k is the nearest point from z_1 on $\alpha(z_1, w_1)$ satisfying $\mathrm{d}(\zeta_k, \partial\Delta) = 2^{k-1}\mathrm{d}(z_1, \partial\Delta)$. Then fix k and set $t = l(\alpha(\zeta_k, \zeta_{k+1}))(\mathrm{d}(\zeta_k, \partial\Delta))^{-1}$. We have

$$(2.8)\qquad \begin{aligned} t &\leqq (\mathrm{d}(\zeta_k, \partial\Delta))^{-1} \int_{\alpha(\zeta_k,\zeta_{k+1})} |dz| \leqq 2 \int_{\alpha(\zeta_k,\zeta_{k+1})} (\mathrm{d}(z, \partial\Delta))^{-1} |dz| \\ &\leqq 4\mathrm{h}_\Delta(\zeta_k, \zeta_{k+1}) , \end{aligned}$$

because $(2\mathrm{d}(z, \partial\Delta))^{-1} \leqq \rho_\Delta(z)$ and $\mathrm{d}(z, \partial\Delta) \leqq \mathrm{d}(\zeta_{k+1}, \partial\Delta) = 2\mathrm{d}(\zeta_k, \partial\Delta)$ for $z \in \alpha(\zeta_k, \zeta_{k+1})$. Hence

$$(2.9)\qquad \begin{aligned} \mathrm{j}_\Delta(\zeta_k, \zeta_{k+1}) &= \log\left(\frac{|\zeta_k - \zeta_{k+1}|}{\mathrm{d}(\zeta_k, \partial\Delta)} + 1\right)\left(\frac{|\zeta_k - \zeta_{k+1}|}{\mathrm{d}(\zeta_{k+1}, \partial\Delta)} + 1\right) \\ &\leqq 2 \log\left(\frac{|\zeta_k - \zeta_{k+1}|}{\mathrm{d}(\zeta_k, \partial\Delta)} + 1\right) \leqq 2 \log(t + 1) . \end{aligned}$$

By (2.2), (2.8) and (2.9) we have

$$t/4 \leqq \mathrm{h}_\Delta(\zeta_k, \zeta_{k+1}) \leqq (\pi/2)(C_1C_3e^2)^2(\mathrm{j}_\Delta(\zeta_k, \zeta_{k+1}) + 2)^2 + \log 2$$

and

$$(2.10)\qquad t \leqq 8\pi(C_1C_3e^2)^2(\log(t + 1)e)^2 + 4 \log t .$$

Obviously, the range of t satisfying (2.10) is bounded and depends only on C_1 and C_3. Therefore, there exist constants C_4 and C_5 depending only on C_1 and C_3 such that $t \leqq C_4$ and $\mathrm{h}_\Delta(\zeta_k, \zeta_{k+1}) \leqq C_5$. Thus we have

$$(2.11)\qquad \begin{aligned} &l(\alpha(\zeta_k, \zeta_{k+1})) \leqq C_4 \mathrm{d}(\zeta_k, \partial\Delta) ,\\ &\mathrm{d}(\zeta_{k+1}, \partial\Delta) \leqq \mathrm{d}(z, \partial\Delta)\exp(2C_5) \quad \text{for} \quad z \in \alpha(\zeta_k, \zeta_{k+1}) . \end{aligned}$$

By using the Gehring-Palka inequality (cf. [9, Chap. III, p. 84 and p. 88]) we have

$$0 < \log \mathrm{d}(\zeta_{k+1}, \partial\Delta)(\mathrm{d}(z, \partial\Delta))^{-1} \leqq 2\mathrm{h}_\Delta(z, \zeta_{k+1}) .$$

Hence $l(\alpha(z_1, w_1)) = \sum_{k=1}^{m_1} l(\alpha(\zeta_k, \zeta_{k+1})) \leqq C_4 \sum_{k=1}^{m_1} \mathrm{d}(\zeta_k, \partial\Delta) = C_4(2^{m_1}-1)\mathrm{d}(z_1, \partial\Delta) \leqq C_4\mathrm{d}(w_1, \partial\Delta)$. Let $z \in \alpha(z_1, w_1)$. Then $z \in \alpha(\zeta_k, \zeta_{k+1})$ for some k and $l(\alpha(z_1, z)) \leqq \sum_{i=1}^{k} l(\alpha(\zeta_i, \zeta_{i+1})) \leqq C_4 \sum_{i=1}^{k} \mathrm{d}(\zeta_i, \partial\Delta) \leqq C_4\mathrm{d}(z, \partial\Delta)\exp(2C_5)$. This completes the proof of (2.6).

In proving (2.7), we may assume that $w_1 \neq w_2$. If $r = \sup_{z\in\alpha} \mathrm{d}(z, \partial\Delta)$, we set $t = l(\alpha(w_1, w_2))(\mathrm{d}(w_1, \partial\Delta))^{-1}$. Then we have

$$t = (\mathrm{d}(w_1, \partial\Delta))^{-1} \int_{\alpha(w_1,w_2)} |dz| \leqq 2 \int_{\alpha(w_1,w_2)} (\mathrm{d}(z, \partial\Delta))^{-1}|dz| \leqq 4\mathrm{h}_\Delta(w_1, w_2) ,$$

because $\mathrm{d}(z, \partial\Delta) \leqq r < 2\mathrm{d}(w_1, \partial\Delta)$. Hence

$$t < 4\mathrm{h}_\Delta(w_1, w_2) \leqq 8\pi(C_1C_3e^2)^2(\log(t + 1)e)^2 + 4\log 2$$

and by the same argument as in the proof of (2.11) we obtain (2.7) in this case. If $r = 2|z_1 - z_2|$, then by (2.6)

$$|w_1 - w_2| \leqq l(\alpha(z_1, w_1)) + l(\alpha(z_1, w_2)) + |z_1 - z_2| \leqq (3B_1 + 1)\mathrm{d}(w_1, \partial\Delta) ,$$

because $\mathrm{d}(w_2, \partial\Delta) \leqq r < 2\mathrm{d}(w_1, \partial\Delta)$. Therefore $\mathrm{j}_\Delta(w_1, w_2) \leqq 2\log(3B_1 + 2)$, and $\mathrm{h}_\Delta(w_1, w_2) \leqq 2\pi(C_1C_3e^2)^2(\log(3B_1 + 2)e)^2 + \log 2$ by (2.2).

For each $z \in \alpha(w_1, w_2)$ we have $\mathrm{h}_\Delta(w_1, w_2) \geqq \mathrm{h}_\Delta(z, w_j) \geqq 2^{-1}|\log \mathrm{d}(z, \partial\Delta)(\mathrm{d}(w_j, \partial\Delta))^{-1}|$ $(j = 1, 2)$ by using the Gehring-Palka inequality again. Hence

$$\mathrm{d}(w_2, \partial\Delta)\exp(-2C_6) \leqq \mathrm{d}(z, \partial\Delta) \leqq \mathrm{d}(w_1, \partial\Delta)\exp(2C_6) ,$$

where $C_6 = 2\pi(C_1C_3e^2)^2(\log(3B_1 + 2)e)^2 + \log 2$. Thus we have the second inequality of (2.7). From this

$$\begin{aligned} l(\alpha(w_1, w_2)) &\leqq \int_{\alpha(w_1,w_2)} \mathrm{d}(w_1, \partial\Delta)(\mathrm{d}(z, \partial\Delta))^{-1}|dz|\exp(2C_6) \\ &\leqq 2\mathrm{d}(w_1, \partial\Delta)\mathrm{h}_\Delta(w_1, w_2)\exp(2C_6) \leqq 2C_6\mathrm{d}(w_1, \partial\Delta)\exp(2C_6) . \end{aligned}$$

This completes the proof of (2.7).

By the definitions of r and w_j $(j = 1, 2)$ and by assumption $\max_{j=1,2} \{\mathrm{d}(z_j, \partial\Delta), \mathrm{d}(w_j, \partial\Delta)\} \leqq r$. Hence we have

$$\begin{aligned} l(\alpha) &\leqq l(\alpha(z_1, w_1)) + l(\alpha(z_2, w_2)) + l(\alpha(w_1, w_2)) \leqq (2B_1 + B_2)\mathrm{d}(w_2, \partial\Delta) \\ &\leqq (2B_1 + B_2)r \leqq 2(2B_1 + B_2)|z_1 - z_2| , \end{aligned}$$

by (2.6) and (2.7). This establishes the first inequality of (2.5). As for the second inequality, if $z \in \alpha$, then either $z \in \alpha(z_j, w_j)$ and

$$\min_{j=1,2} l(\alpha(z_j, z)) \leqq l(\alpha(z_j, z)) \leqq B_1\mathrm{d}(z, \partial\Delta)$$

by (2.6), or $z \in \alpha(w_1, w_2)$ and

$$\min_{j=1,2} l(\alpha(z_j, z)) \leqq l(\alpha)/2 \leqq (2B_1 + B_2)\mathrm{d}(w_2, \partial\Delta)/2 \leqq B_2(2B_1 + B_2)\mathrm{d}(z, \partial\Delta)/2$$

by (2.7). Hence we have also obtained the second inequality of (2.5).

Next, we suppose that $r < \mathrm{d}(z_1, \partial\Delta)$. Then $r = 2|z_1 - z_2|$. For any z on the Euclidean line segment β from z_1 to z_2 we have $\mathrm{d}(z, \partial\Delta) \geqq \mathrm{d}(z_1, \partial\Delta)/2 \geqq |z_1 - z_2|$, and hence

$$\mathrm{h}_\Delta(z_1, z_2) \leqq \int_\beta 2(\mathrm{d}(z, \partial\Delta))^{-1}|dz| \leqq 4|z_1 - z_2|/\mathrm{d}(z_1, \partial\Delta) \leqq 2 .$$

By the Gehring-Palka inequality, we have

$$l(\alpha) \leqq e^4\mathrm{d}(z_1, \partial\Delta)\int_\alpha (\mathrm{d}(z, \partial\Delta))^{-1}|dz| \leqq 2e^4 d(z_1, \partial\Delta)\mathrm{h}_\Delta(z_1, z_2) \leqq 8e^4|z_1 - z_2| .$$

For $z \in \alpha$, $l(\alpha(z_1, z)) \leqq l(\alpha) \leqq 4e^4\mathrm{d}(z_1, \partial\Delta) \leqq 4e^8\mathrm{d}(z, \partial\Delta)$. This establishes (2.5) in the case where $r < \mathrm{d}(z_1, \partial\Delta)$. Similarly we obtain (2.5) in the case where $r < \mathrm{d}(z_2, \partial\Delta)$. Hence we completely proved (2.5).

3. Proofs of Theorem 2 and Corollary.

PROOF OF THEOREM 2. Žuravlev [21] showed that $T(\Gamma)$ is equal to the component of $\mathrm{Int}\, S(\Gamma)$ containing the origin. Hence it suffices to show that $\mathrm{Int}\, S(\Gamma)$ has no other component than $T(\Gamma)$. Let S be such a component of $\mathrm{Int}\, S(\Gamma)$. Then for each $\phi \in S$, $\Gamma^\phi = \chi_\phi(\Gamma) = W_\phi \Gamma (W_\phi)^{-1}$ is a Kleinian group with a simply connected invariant component $W_\phi(L)$. Indeed, let Ω_ϕ be a component of containing $W_\phi(L)$. Suppose that there exists a point p in $\Omega_\phi - W_\phi(L)$. Then for any $\varepsilon > 0$, $N_\varepsilon(p) = \{z \in \boldsymbol{C}; |z - p| < \varepsilon\}$ is not containd in $W_\phi(L) \cup \{p\}$ because $W_\phi(L)$ is simply connected. This implies that $N_\varepsilon(p)$ contains infinitely many points of $\Omega_\phi - W_\phi(L)$ for any $\varepsilon > 0$ and the Riemann surface Ω_ϕ/Γ^ϕ contains infinitely many points which are not contained in $W_\phi(L)/\Gamma^\phi$ conformally equivalent to L/Γ. However, L/Γ is a Riemann surface of conformally finite type and, by Ahlfors' finiteness theorem, so is Ω_ϕ/Γ^ϕ. This is absurd because $L/\Gamma \cong W_\phi(L)/\Gamma^\phi$. Thus $\Omega_\phi = W_\phi(L)$. Clearly, $W_\phi(L)$ is invariant under Γ^ϕ. Hence $W_\phi(L)$ is a simply connected invariant component of Γ^ϕ.

Therefore Γ^ϕ has one or two simply connected invariant components by a theorem of Accola (cf. [4], [14]). Namely, Γ^ϕ is a quasi-Fuchsian group or a b-group.

If Γ^ϕ is a quasi-Fuchsian group, then the limit set $\Lambda(\Gamma^\phi)$ of Γ^ϕ is a quasi-circle (Maskit [14]). Therefore, W_ϕ has a quasiconformal extension to $\hat{\boldsymbol{C}}$ by a theorem in Ahlfors [3] and ϕ belongs to $T \cap B_2(L, \Gamma)$, where T is the universal Teichmüller space. On the other hand, Kra [12] showed that $T(\Gamma) = T \cap B_2(L, \Gamma)$ if Γ is a finitely generated Fuchsian group of the first kind. Thus, ϕ is in $T(\Gamma)$. But this is a contradiction. Hence Γ^ϕ is a b-group.

Since a function $(\operatorname{trace} \chi_\phi(\gamma))^2$ for a fixed $\gamma \in \Gamma$ is analytic on $B_2(L, \Gamma)$ and Γ consists of countable number of elements, there exists a ϕ in S such that $(\operatorname{trace} \chi_\phi(\gamma))^2 \neq 4$ for every non-parabolic element γ in Γ, namely, a b-group Γ^ϕ is not a cusp. Therefore, Γ^ϕ is a totally degenerate group with $\Omega(\Gamma^\phi) = W_\phi(L)$ by Maskit [14, Theorem 4], where $\Omega(\Gamma^\phi)$ is the region of discontinuity of Γ^ϕ. From now on, we shall consider such ϕ and Γ^ϕ.

Here, we note the following fact called the "λ-lemma".

PROPOSITION (Mañé, Sad and Sullivan [13]). *Let A be a subset of $\boldsymbol{C}$ and $\{i_\lambda\}$ be a family of injections of A into $\hat{\boldsymbol{C}}$, where λ is in the unit disk D. Furthermore, let $i_\lambda(z)$ be analytic with respect to $\lambda \in D$ for each $z \in A$ and $i_0(z) \equiv z$. Then i_λ for each $\lambda \in D$ is automatically a quasiconformal mapping on $\bar{A}$, that is, i_λ is a homeomorphism of $\bar{A}$ into $\hat{\boldsymbol{C}}$ with*

$$\sup_{z \in \bar{A}} \varlimsup_{r \to 0} \frac{\inf\{\delta(i_\lambda(z), i_\lambda(z')) \colon \delta(z, z') = r,\ z' \in \bar{A}\}}{\sup\{\delta(i_\lambda(z), i_\lambda(z')) \colon \delta(z, z') = r,\ z' \in \bar{A}\}} < +\infty ,$$

where $\delta(\cdot, \cdot)$ is the spherical distance in $\hat{\boldsymbol{C}}$.

We proceed to prove Theorem 2. Since ϕ is in S, there exists a constant $r > 0$ such that $\{\psi \in B_2(L, \Gamma) \colon \|\psi - \phi\| < r\}$ is contained in $\operatorname{Int} S(\Gamma)$. For each $\lambda \in D$ we set $\phi_\lambda = \phi + \lambda(\psi_0 - \phi)$ and $i_\lambda = W_{\phi_\lambda} \circ (W_\phi)^{-1}$ on $W_\phi(L)$, where ψ_0 is in $B_2(L, \Gamma)$ with $0 < \|\psi_0 - \phi\| < r$. Then i_λ is conformal on $W_\phi(L) = \Omega(\Gamma^\phi)$ and satisfies the condition of the above proposition for $A = \Omega(\Gamma^\phi)$. Hence i_λ for each $\lambda \in D$ can be extended to $\overline{\Omega(\Gamma^\phi)} = \hat{\boldsymbol{C}}$ quasiconformally. On the other hand, i_λ is a Γ^ϕ-compatible quasiconformal mapping and Γ^ϕ is a finitely generated Kleinian group. Thus, the Beltrami differential of i_λ vanishes almost everywhere on $\Lambda(\Gamma^\phi)$ from Sullivan's theorem in [20]. This implies that i_λ is conformal on $\hat{\boldsymbol{C}}$ for each $\lambda \in D$ and $\{i_\lambda, z\} = 0$ on $\boldsymbol{C}$. But this is absurd because $\{i_\lambda, z\} = \lambda(\psi_0 - \phi)(W_\phi^{-1}(z)) \cdot ((W_\phi^{-1})'(z))^2 \neq 0$ for $\lambda \neq 0$. Therefore, we complete the proof of Theorem 2.

PROOF OF COROLLARY. We may assume that $\infty \in \Delta$. Let h be a conformal mapping of L onto Δ satisfying $h(z) = (z + i)^{-1} + O(|z + i|)$ as $z \to -i$. Then $\Gamma = h^{-1}Gh$ is a finitely generated Fuchsian group of the first kind and $\{h, z\}$ is in $B_2(L, \Gamma)$ by Nehari's theorem in [16]. So, if all f satisfying (1.2) are schlicht on Δ, then $\{f \circ h, z\} = \{f, h(z)\}(h'(z))^2 + \{h, z\}$ is in $S(\Gamma)$, and $\{h, z\}$ is in $\operatorname{Int} S(\Gamma)$ because $\{f, h(z)\}(h'(z))^2$ is in $B_2(L, \Gamma)$ and $\sup_{w \in \Delta} \rho_\Delta(w)^{-2} |\{f, w\}| = \|\{f, h(z)\}(h'(z))^2\|$. Hence $\{h, z\}$ is in $T(\Gamma)$ from Theorem 2, that is, $h(L) = \Delta$ is a quasi-disk.

Conversely, if Δ is a quasi-disk, then Δ has the Schwarzian derivative property (cf. [8], [9]). Hence all f satisfying (1.2) are schlicht on Δ.

4. Proof of Theorem 3. Suppose that $H - H \cap \overline{T(\Gamma)}$ is not connected. Then there exists a bounded component of $H - H \cap \overline{T(\Gamma)}$ in H, say V, because $H \cap \overline{T(\Gamma)}$ is bounded in H. Obviously, $\hat{\partial} V \subset S(\Gamma)$ and therefore we can show that V is contained in $S(\Gamma)$ by the same argument as in the proof of [19, Theorem 2]. For convenience, we shall sketch the proof.

For each $\phi \in B_2(L, \Gamma)$ we set $w_\phi(z) = 2i W_\phi(i(1-z)(1+z)^{-1})$ on $\{|z| > 1\}$. Then w_ϕ is schlicht on $\{|z| > r\}$ for some $r \geqq 1$. So, we can define the Grunsky coefficients $b_{ij}(\phi)$ $(i, j = 1, 2, \cdots)$, namely,

$$\log \frac{w_\phi(z) - w_\phi(\zeta)}{z - \zeta} = -\sum_{i,j=1}^{\infty} b_{ij}(\phi) z^{-i} \zeta^{-j}$$

holds on $|z|, |\zeta| > r$. It is known (cf. [17]) that w_ϕ is schlicht on $|z| > 1$ if and only if

$$\left| \sum_{i,j=1}^{\infty} b_{ij}(\phi) \lambda_i \lambda_j \right| \leqq \sum_{n=1}^{\infty} |\lambda_n|^2 / n \tag{4.1}$$

holds for an arbitrary sequence $\{\lambda_n\}$ of complex numbers.

Let ϕ be in $\hat{\partial} V$. Then w_ϕ is schlicht on $|z| > 1$. Hence we have

$$\left| \sum_{i,j=1}^{N} b_{ij}(\phi) \lambda_i \lambda_j \right| \leqq \sum_{n=1}^{N} |\lambda_n|^2 / n \tag{4.2}$$

for an arbitrary natural number N. Since $b_{ij}(\phi)$ is analytic with respect to $\phi \in B_2(L, \Gamma)$, we can verify that (4.2) holds for all ϕ in V by the maximum principle, and (4.1) holds for every ϕ in V. So, V is contained in $S(\Gamma)$.

For a non-parabolic element $\gamma \in \Gamma$, $(\operatorname{trace} \chi_\phi(\gamma))^2 - 4$ is analytic in $B_2(L, \Gamma)$ and does not vanish identically on H, because $H \cap T(\Gamma) \neq \varnothing$. Therefore, $\{\phi \in V; (\operatorname{trace} \chi_\phi(\gamma))^2 - 4 = 0\}$ is a nowhere dense subset of V, and by the same argument as in the proof of Theorem 2 we can take such a ϕ in V that $(\operatorname{trace} \chi_\phi(\gamma))^2 \neq 4$ for every non-parabolic element $\gamma \in \Gamma$. Since ϕ is in $S(\Gamma) - T(\Gamma)$, Γ^ϕ is a totally degenerate Kleinian group. By using Proposition (the λ-lemma) and Sullivan' theorem [20] again as in the proof of Theorem 2 for a small disk in V centered at ϕ, we have a contradiction. Since we have already shown that $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) \supset H \cap \partial T(\Gamma)$ in [19, Theorem 2], we have $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) = H \cap \partial T(\Gamma)$ by a general relation $\hat{\partial}(H - H \cap \overline{T(\Gamma)}) \subset H \cap \partial T(\Gamma)$. Thus, we complete the proof of Theorem 3.

5. Remarks.

(1) Let W be a bounded domain in $\boldsymbol{C}$ whose boundary consists of a

finite number of mutually disjoint closed Jordan curves, say $\alpha_1, \alpha_2, \cdots, \alpha_N$, and let W_j $(j = 1, 2, \cdots, N)$ be a domain in $\hat{C}$ with $\partial W_j = \alpha_j$ and $W_j \supset W$. Then we have the following:

THEOREM 4. *If W has the* BMO *extension property for* $\cup_{j=1}^N \mathrm{ABD}(W_j)|W$, *then $\alpha_1, \alpha_2, \cdots, \alpha_N$ are all quasi-circles.*

PROOF. From the hypothesis, there exists a constants $C_1 > 0$ such that for every $g \in \cup_{j=1}^N \mathrm{ABD}(W_j)$ there exists a $G \in \mathrm{BMO}(C)$ with $G|W = g|W$ and

$$\|G\|_{*,C} \leqq C_1 \|g|W\|_{*,W} . \tag{5.1}$$

We may take g as an arbitrary function in $\mathrm{ABD}(W_j)$ for a fixed j $(1 \leqq j \leqq N)$. Let β_j be a circle in $C - W_j$ and let Δ_j be the component of $\hat{C} - \beta_j$ containing W_j. We define a function $\tilde{G}$ in Δ_j by

$$\tilde{G}(z) = \begin{cases} G(z) , & z \in \Delta_j - W_j , \\ g(z) , & z \in W_j . \end{cases} \tag{5.2}$$

Set $d_j = \min\{h_{\Delta_j}(\alpha_j, \alpha_k) : k \neq j\}$, where $h_{\Delta_j}(\cdot, \cdot)$ is the hyperbolic distance in Δ_j. Then $d_j > 0$ and for every disk B in Δ_j whose hyperbolic diameter is not greater than d_j, we have

$$\frac{1}{|B|}\int_B |\tilde{G} - \tilde{G}_B| dxdy = \frac{1}{|B|}\int_B |g - g_B| dxdy \leqq \|g\|_{*,W_j}$$

if $\bar{B}$ is contained in W_j, and

$$\frac{1}{|B|}\int_B |\tilde{G} - \tilde{G}_B| dxdy = \frac{1}{|B|}\int_B |G - G_B| dxdy \leqq \|G\|_{*,C}$$
$$\leqq C_1 \|g|W\|_{*,W} \leqq C_1 \|g\|_{*,W_j}$$

if $B \cap (\Delta_j - W_j) \neq \emptyset$. Therefore, from [18, I-B, Hilfssatz 2] and its proof we conclude that $\tilde{G}$ belongs to $\mathrm{BMO}(\Delta_j)$ and

$$\|\tilde{G}\|_{*,\Delta_j} \leqq C(d_j, C_1)\|g\|_{*,W_j} , \tag{5.3}$$

where $C(d_j, C_1)$ is a constant depending only on d_j and C_1. On the other hand, Δ_j is a (quasi-)disk. Hence there exists a constant C_1' not depending on $\tilde{G}$ such that $\tilde{G}$ has an extension $G_j \in \mathrm{BMO}(C)$ satisfying

$$\|G_j\|_{*,C} \leqq C_1' \|\tilde{G}\|_{*,\Delta_j} \leqq C_1' C(d_j, C_1)\|g\|_{*,W_j} . \tag{5.4}$$

Since $G_j|W_j = \tilde{G}|W_j = g$ from (5.2), the inequality (5.4) implies that W_j has the BMO extension property for $\mathrm{ABD}(W_j)$. Thus α_j must be a quasi-circle from Theorem 1, if $\infty \notin W_j$. If $\infty \in W_j$, then we consider a certain Möbius transformation A such that $A(W_j) \not\ni \infty$. By using the conformal invariance of BMO, we have also the assertion in this case.

NOTE. Since $\mathrm{BMO}(W) \supset \mathrm{ABD}(W) \supset \bigcup_{j=1}^{N} \mathrm{ABD}(W_j) | W$, we see that if W has the BMO extension property for $\mathrm{ABD}(W)$ ($\mathrm{BMO}(W)$), then $\alpha_1, \cdots, \alpha_N$ are all quasi-circles. Conversely, if $\alpha_1, \cdots, \alpha_N$ are all quasi-circles, then W has the BMO extension property for $\mathrm{BMO}(W)$ (Mr. Y. Gotoh, oral communication).

(2) Bers conjectured that for every $\phi \in \partial T(\Gamma)$, there are complex manifold M isomorphic to a product of Teichmüller spaces, with $\phi \in M \subset \partial T(\Gamma)$ and a quasiconformal deformation Γ^{ψ} of Γ^{ϕ} for every ψ in M (cf. [5, p. 296]).

Abikoff ([1, § 5, Corollary]) showed that the conjecture is affirmative when Γ^{ϕ} is a regular b-group. In contrast with this result we have the following theorem for $\phi \in \partial T(\Gamma)$ corresponding to a totally degenerate group, which is a strongly negative answer to the conjecture.

THEOREM 5. *For each ϕ corresponding to a totally degenerate group there exists no complex manifold in $\overline{T(\Gamma)}$ containing ϕ.*

PROOF. If such a complex manifold exists, then there is a holomorphic injection f of the unit disk in $\boldsymbol{C}$ into $\overline{T(\Gamma)}$ with $f(0) = \phi$. Set $i_\lambda(z) = W_{f(\lambda)} \circ W_\phi^{-1}(z)$ on $\Omega(\Gamma^\phi)$ for $\lambda \in D$. By the same argument as in the proof of Theorem 2, we have $\{i_\lambda, z\} = 0$ on $\boldsymbol{C}$ for all $\lambda \in D$ and this yields a contradiction, because $f(\lambda) \neq \phi$ for $\lambda \in D - \{0\}$.

(3) We shall suppose that Γ has no elliptic transformation and $\dim T(\Gamma) = 1$. Then Bers [6] showed that all modular transformation of $T(\Gamma)$ can be extended to $\partial T(\Gamma)$ continuously. Since $\overline{T(\Gamma)}$ is compact and the complement is connected in $B_2(L, \Gamma)$ ($\cong \boldsymbol{C}$) from Theorem 3, we have the following from Mergelyan's theorem (cf. [7]).

THEOREM 6. *Let Γ be as above and consider $T(\Gamma)$ as a bounded domain in $\boldsymbol{C}$. Then every modular transformation can be approximated uniformly on $\overline{T(\Gamma)}$ by polynomials.*

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