Tôhoku Math. Journ.
38 (1986), 269-279.

INFINITESIMAL DEFORMATIONS OF TSUCHIHASHI'S CUSP SINGULARITIES

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(Received April 25, 1985)

0. Introduction. Let (X, x_0) be a normal isolated singularity of dimension n. Assume that X is a Stein neighborhood of the singular point x_0 and that x_0 is the only singularity of X. The set of first order infinitesimal deformations of X is the finite dimensional vector space T_X^1, which is isomorphic to $\mathrm{Ext}^1_{O_X}(\Omega^1_X, O_X)$. In [FK] Freitag and Kiehl proved that Hilbert modular cusp singularities of dimension greater than two are rigid in the sense of Schlessinger [Sc], that is, $T_X^1 = 0$. Behnke and Nakamura computed T_X^1 for two dimensional cusp singularities ([B 1], [B 2] and [N]).

Here we are interested in deformations of normal isolated singularities constructed by Tsuchihashi in [T]. Theorem 3 shows that these singularities of dimension three are not rigid in general.

Thanks are due to Professors T. Oda, M. Namba and M.-N. Ishida for many helpful conversations.

1. Tsuchihashi's cusp singularities. Let N be a free $\boldsymbol{Z}$-module of rank $n > 1$ and $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. Consider a pair (C, Γ) consisting of a nondegenerate open convex cone C in $N_{\boldsymbol{R}}$ and a subgroup Γ in $GL(N) := \mathrm{Aut}_{\boldsymbol{Z}}(N)$ satisfying the following conditions:

(i) C is Γ-invariant.

(ii) The action of Γ on $D := C/\boldsymbol{R}_{>0}$ is properly discontinuous and fixed point free.

(iii) The quotient space D/Γ is compact.

In [T] Tsuchihashi has associated to such a pair (C, Γ) an isolated singularity, which we may call Tsuchihashi's cusp singularity. This is a singular point of the normal analytic space $X := ((N_{\boldsymbol{R}} + \sqrt{-1}C)/N \cdot \Gamma) \cup \{x_0\}$.

Let N^* be the dual $\boldsymbol{Z}$-module of N with the natural pairing $\langle\ ,\ \rangle$: $N^* \times N \to \boldsymbol{Z}$, and let dy and dy' the Lebesgue measures on $N_{\boldsymbol{R}}$ and $N^*_{\boldsymbol{R}}$ respectively. Let $C^* := \{y' \in N^*_{\boldsymbol{R}};\ \langle y', y\rangle > 0 \text{ for all } y \text{ in } \bar{C}\setminus\{0\}\}$ be the dual cone of C. The characteristic function of the cone C is

$$\phi_C(y) := \int_{C^*} \exp(-\langle y', y\rangle)dy'$$

defined by Vinberg [V], which satisfies the properties that $\phi(gy) = |\det(g)|^{-1}\phi(y)$ for a linear automorphism g of $N_{\boldsymbol{R}}$ preserving C, that $\phi(y)$ diverges to infinity when y approaches the boundary of C and that the Hessian of ϕ is positive definite on C. For a point $z = x + \sqrt{-1}y$ in $N_{\boldsymbol{R}} + \sqrt{-1}C$, let $\Phi(z) := \phi(y)$. The real-valued function Φ on $N_{\boldsymbol{R}} + \sqrt{-1}C$, is $N\cdot\Gamma$-invariant and strictly plurisubharmonic. Let $V(c) := \{z \in N_{\boldsymbol{R}} + \sqrt{-1}C;\ \Phi(z) < c\}$ and $\bar{V}(c) := \{y \in C;\ \phi(y) < c\}$ for each positive real number c. Then $V(c) = N_{\boldsymbol{R}} + \sqrt{-1}\bar{V}(c)$. Since the function Φ is $N\cdot\Gamma$-invariant, it may be regarded as a function on X with $\Phi(x_0) := 0$. Let $W(c) := \{z \in X;\ \Phi(z) < c\}$ for each positive number c. Then $W(c) = (V(c)/N\cdot\Gamma) \cup \{x_0\}$.

The elements of the local ring O_{X,x_0} can be represented by holomorphic functions on $V(c)/N\cdot\Gamma$ for sufficiently large $c > 0$ which are continuous at x_0. For a function f in $H^0(W(c), O_X)$, the pull-back $\tilde{f}$ to $V(c)$ of f is an $N\cdot\Gamma$-invariant holomorphic function $\tilde{f}(z)$ bounded for y in $\bar{V}(c)$ with $|y|$ sufficiently large, where $z = x + \sqrt{-1}y$ is in $N_{\boldsymbol{R}} + \sqrt{-1}\bar{V}(c)$ and $|\cdot|$ is a fixed Euclidean norm on $N_{\boldsymbol{R}}$. Since $\tilde{f}(z)$ is N-invariant, it has the Fourier series expansion

$$\tilde{f}(z) = \sum_{\mu \in N^*} a_\mu e(\langle \mu, z\rangle)\ ,$$

where $e(\cdot) := \exp(2\pi i(\cdot))$. The coefficients a_μ vanish for $\mu \notin N^* \cap \bar{C}^*$. Indeed, for $\mu \notin N^* \cap \bar{C}^*$, there exists a y in $\bar{V}(c)$ such that $\langle \mu, y\rangle$ is negative. Since $\tilde{f}(x + \sqrt{-1}y)$ is bounded for $|y|$ sufficiently large, $a_\mu \exp(-2\pi\langle \mu, ty\rangle)$ is bounded for all $t \geqq 1$. Hence we have $a_\mu = 0$ for $\mu \notin N^* \cap \bar{C}^*$. An element $\gamma \in \Gamma$ acts on $\tilde{f}(z)$ as

$$\gamma\tilde{f}(z) = \sum_{\mu \in N^* \cap \bar{C}^*} a_\mu e(\langle \mu, \gamma z\rangle) = \sum_{\mu \in N^* \cap \bar{C}^*} a_\mu e(\langle \gamma\mu, z\rangle) = \sum_{\mu \in N^* \cap \bar{C}^*} a_{\gamma^{-1}\mu} e(\langle \mu, z\rangle)\ .$$

Set $F_\mu(z) := \sum_{\gamma \in \Gamma} e(\langle \gamma\mu, z\rangle)$ for $\mu \in N^* \cap C^*$. Then

$$\begin{aligned} |F_\mu(z)| &\leqq \sum_{\gamma \in \Gamma} |e(\langle \gamma\mu, z\rangle)| = \sum_{\gamma \in \Gamma} |e(\langle \gamma\mu, x + \sqrt{-1}y\rangle)| = \sum_{\gamma \in \Gamma} \exp(-2\pi\langle \gamma\mu, y\rangle) \\ &\leqq K\int_{C^*} \exp(-2\pi\langle y', y\rangle)dy' = K\phi_C(y) \end{aligned}$$

for some positive constant K. Hence $F_\mu(z)$ is a holomorphic function on $N_{\boldsymbol{R}} + \sqrt{-1}C$. Since $\tilde{f}(z)$ is also Γ-invariant, we can express $\tilde{f}(z)$ as

$$\tilde{f}(z) = \sum_{\mu \in (N^* \cap C^*)/\Gamma} b_\mu F_\mu(z) + b_0\ .$$

Hence an element f in $H^0(W(c), O_X)$ is represented by a holomorphic function $\tilde{f}$ on $V(c)$ which has the Fourier series expansion of the form

$$\tilde{f}(z) = \sum_{\mu \in (N^* \cap C^*)/\Gamma} b_\mu F_\mu(z) + b_0\ .$$

LEMMA 1. *The image of* $H^0(X, O_X)$ *in* $H^0(W(c), O_X)$ *is dense with respect to the topology of uniform convergence on compact subsets in* $W(c)$ *for any positive number* c.

PROOF. Since $F_\mu(z)$ for $\mu \in N^* \cap C^*$ is a $N \cdot \Gamma$-invariant holomorphic function on $N_R + \sqrt{-1}C$ and bounded for y in C with $|y|$ sufficiently large, it represents an element of $H^0(X, O_X)$. The function $\tilde{f}$ on $V(c)$, representing an element f in $H^0(W(c), O_X)$, has the Fourier series expansion

$$\tilde{f}(z) = \sum_{\mu \in (N^* \cap C^*)/\Gamma} b_\mu F_\mu(z) + b_0 ,$$

where the series on the right hand side converges absolutely and uniformly on any compact subsets in $W(c)$. Hence $\tilde{f}(z)$ can be approximated by finite sums of $F_\mu(z)$. q.e.d.

2. Main results.

THEOREM 1. *Let* $U := X \setminus \{x_0\}$. *When* $n \geqq 3$, *we have canonical isomorphisms*

$$T_X^1 \simeq H^1(U, \Theta_X) \simeq H^1(\Gamma, N_C) ,$$

where the last term is the first cohomology group of the group Γ *acting naturally on* $N_C := N \otimes_Z C$, *while* Θ_X *is the holomorphic tangent sheaf of* X.

We shall prove Theorem 1 in Section 3.

THEOREM 2. *When the cone* C *is decomposable, that is, the product* $C_1 \times \cdots \times C_r$ *of more than one nontrivial convex cones* $C_1, \cdots, C_r$ *then* $H^1(\Gamma, N_C) = 0$. *Consequently, we have* $T_X^1 = 0$ *when* $n \geqq 3$ *and when the cone* C *is decomposable.*

PROOF. We denote by $\mathrm{Aut}(C)$ the group of linear transformations of N_R preserving C. We may assume that C_i are indecomposable. Then $\prod_{i=1}^r \mathrm{Aut}(C_i)$ is a normal subgroup of $\mathrm{Aut}(C)$ of finite index. Let $\Gamma_0 := \Gamma \cap (\prod_{i=1}^r \mathrm{Aut}(C_i))$. Then Γ_0 is a normal subgroup of Γ of finite index. From the Hochschild-Serre exact sequence

$$\begin{aligned} 0 \to H^1(\Gamma/\Gamma_0, (N_C)^{\Gamma_0}) &\to H^1(\Gamma, N_C) \\ \to H^1(\Gamma_0, N_C)^{(\Gamma/\Gamma_0)} &\to H^2(\Gamma/\Gamma_0, (N_C)^{\Gamma_0}) , \end{aligned}$$

we have an isomorphism

$$H^1(\Gamma, N_C) \simeq H^1(\Gamma_0, N_C)^{(\Gamma/\Gamma_0)} ,$$

the right hand side being the invariants with respect to the natural action of Γ/Γ_0, because higher cohomology groups of a finite group with coefficients in a vector space are trivial. We prove that $H^1(\Gamma_0, N_C) = 0$. Let H be the subgroup of $\prod_{i=1}^r \mathrm{Aut}(C_i)$ consisting of those automorphisms of C which induce homotheties of C_i, thus $(\boldsymbol{R}_{>0})^r \simeq H \subset \prod_{i=1}^r \mathrm{Aut}(C_i)$. Since $H \cap \Gamma_0$ is a normal subgroup of Γ_0, we have an exact sequence

$$1 \to H \cap \Gamma_0 \to \Gamma_0 \to \Gamma_0/(H \cap \Gamma_0) \to 1 .$$

Set $S := H \cap \Gamma_0$ and $Q := \Gamma_0/(H \cap \Gamma_0)$. Let $D_i := C_i/\boldsymbol{R}_{>0}$ $(i = 1, \cdots, r)$. Taking the quotient with respect to the action of $H/(\boldsymbol{R}_{>0}) \simeq (\boldsymbol{R}_{>0})^{r-1}$, we have a natural surjective morphism $D \to (D_1 \times \cdots \times D_r)$. Then we have the fibration

$$D/\Gamma_0 \to (D_1 \times \cdots \times D_r)/Q$$

with the fiber $(\boldsymbol{R}_{>0})^{r-1}/S$, which is compact since D/Γ_0 is compact. Hence S is a free abelian group of rank $r-1$. From the exact sequence

$$1 \to S \to \Gamma_0 \to Q \to 1 ,$$

we have the Hochschild-Serre exact sequence

$$0 \to H^1(Q, (N_C)^S) \to H^1(\Gamma_0, N_C) \to H^1(S, N_C)^Q \to H^2(Q, (N_C)^S) .$$

Since $(N_C)^S = 0$, it suffices to show that $H^1(S, N_C) = 0$. Since $C = C_1 \times \cdots \times C_r$ is the decomposition of C into the product of indecomposable cones, we have $N_C = V_1 \oplus \cdots \oplus V_r$. An element h in H acts on $v = (v_1, \cdots, v_r)$ in $N_C = V_1 \oplus \cdots \oplus V_r$ as $hv = (\varepsilon_1(h)v_1, \cdots, \varepsilon_r(h)v_r)$ with $\varepsilon_j(h) > 0$, that is, h acts on each V_j as a scalar multiplication. Hence we have $H^1(S, N_C) = \bigoplus_{j=1}^r H^1(S, V_j)$. We claim that $H^1(S, V_j) = 0$ for $j = 1, \cdots, r$. Indeed, a 1-cocycle $f \in H^1(S, V_j)$ is a function from S to V_j which satisfies $f(s_1 s_2) = \varepsilon_j(s_1)f(s_2) + f(s_1)$ for all s_1, s_2 in S. Since S is an abelian group, $f(s_1 s_2) = f(s_2 s_1)$, hence we have

$$(\varepsilon_j(s_1) - 1)f(s_2) = (\varepsilon_j(s_2) - 1)f(s_1) .$$

Since there exists an s_0 in S with $\varepsilon_j(s_0) \neq 1$, for any s in S we have

$$f(s) = (\varepsilon_j(s) - 1)(f(s_0)/(\varepsilon_j(s_0) - 1)) .$$

This shows that f is a coboundary, and hence $H^1(S, V_j) = 0$. q.e.d.

THEOREM 3. *When $n = 3$, we have*

$$3(1 - \chi(D/\Gamma)) \geqq \dim_C T_X^1 \geqq -3\chi(D/\Gamma) ,$$

where $\chi(D/\Gamma)$ is the Euler number of the compact real manifold D/Γ.

PROOF. First we prove that $(N_C)^\Gamma = 0$. It is enough to prove that $(N_R)^\Gamma = 0$ because $(N_C)^\Gamma = (N_R)^\Gamma \otimes_R \boldsymbol{C}$. Assume that there exists a nonzero element v in N_R invariant under the action of all elements in Γ. By the assumption that Γ acts on $D = C/\boldsymbol{R}_{>0}$ fixed point freely, we see that v is not contained in $C \cup (-C)$. Set $H_v := \{v' \in N_R^*;\ \langle v', v\rangle = 0\}$. Then $H_v \cap (\bar{C}^* \setminus \{0\}) \neq \emptyset$. Set $S' := \{v' \in C^*;\ \phi_{C^*}(v') = 1\}$. Then Γ acts on S' properly discontinuously and fixed point freely as well as on $D' := C^*/\boldsymbol{R}_{>0}$, and we have $S'/\Gamma \simeq D'/\Gamma$. For any real number r set $H_v(r) := \{v' \in N_R^*;\ \langle v', v\rangle = r\}$, and we have that $H_v(r) \cap S'$ is Γ-invariant. Set $R' := \{r \in \boldsymbol{R};\ H_v(r) \cap S' \neq \emptyset\}$. Then $R' = \boldsymbol{R}$ or $\boldsymbol{R}_{>0}$. Hence we have a disjoint union $\cup_{r \in R'} (H_v(r) \cap S')$ and a fibration $S'/\Gamma \to R'$. This contradicts to the assumption that S'/Γ is compact, because the image R' of the compact set S'/Γ by the continuous map must be compact, but R' is really not compact. Thus we see that $(N_R)^\Gamma = 0$, hence $(N_C)^\Gamma = 0$.

Assume that Γ is the quotient group of a free group F generated by s ($\geqq 2$) elements. Then we have a natural injection $H^1(\Gamma, N_C) \to H^1(F, N_C)$. We take the $\boldsymbol{C}[F]$-free resolution of the trivial $\boldsymbol{C}[F]$-module $\boldsymbol{C}$ as

$$0 \to IF \to \boldsymbol{C}[F] \to \boldsymbol{C} \to 0\ ,$$

where IF is a complex vector space generated by $(f - 1)$ for $f \in F \setminus \{1\}$ and is known to be a free $\boldsymbol{C}[F]$-module of rank s (see, for example, [HS]). From this we have the exact sequence

$$0 \to \mathrm{Hom}_{C[F]}(\boldsymbol{C}, N_C) \to \mathrm{Hom}_{C[F]}(\boldsymbol{C}[F], N_C) \to \mathrm{Hom}_{C[F]}(IF, N_C) \to H^1(F, N_C) \to 0\ ,$$

where $\mathrm{Hom}_{C[F]}(\boldsymbol{C}, N_C) \simeq (N_C)^\Gamma = 0$ and $\mathrm{Hom}_{C[F]}(IF, N_C) \simeq (N_C)^s$. Hence

$$\dim_C H^1(F, N_C) = (s - 1)\cdot\dim_C(N_C)\ .$$

Since Γ is isomorphic to the fundamental group of the $(n-1)$-dimensional compact real manifold D/Γ, we can choose s so that $s = 2 - \chi(D/\Gamma)$ when $n = 3$. Thus we proved that

$$3(1 - \chi(D/\Gamma)) \geqq \dim_C H^1(\Gamma, N_C) = \dim_C T_X^1\ .$$

On the other hand, since Γ is isomorphic to the fundamental group of D/Γ, the chain complex of the universal covering space D of D/Γ gives a finite $\boldsymbol{C}[\Gamma]$-free resolution of the trivial $\boldsymbol{C}[\Gamma]$-module $\boldsymbol{C}$ (cf. [Se]); more precisely, let Σ be a finite triangulation of D/Γ. The triangulation Σ determines a Γ-invariant triangulation $\tilde{\Sigma}$ of D. Let $C.(\tilde{\Sigma})$ be the chain complex with coefficients in $\boldsymbol{C}$ obtained from $\tilde{\Sigma}$. Since D is contractible, we have an exact sequence of complex vector spaces

$$0 \to C_{n-1}(\tilde{\Sigma}) \to \cdots \to C_1(\tilde{\Sigma}) \to C_0(\tilde{\Sigma}) \to \boldsymbol{C} \to 0\ ,$$

which is also an exact sequence of $\boldsymbol{C}[\Gamma]$-modules. Then we see that the Euler-Poincaré characteristic $\chi(\Gamma) := \sum_{j=0}^{n-1} (-1)^j\cdot\dim_C H^j(\Gamma, \boldsymbol{C})$ of Γ is equal to the Euler number $\chi(D/\Gamma)$ of D/Γ. Moreover, since $H^j(\Gamma, N_C)$ are the cohomology groups of the complex consisting of the invariants of $\mathrm{Hom}_C(C.(\tilde{\Sigma}), N_C)$ with respect to the action of Γ, we have

$$\sum_{j=0}^{n-1}(-1)^j \dim_C H^j(\Gamma, N_C) = \dim_C(N_C)\cdot\chi(\Gamma) .$$

Since $H^0(\Gamma, N_C) = (N_C)^\Gamma = 0$, we have

$$\dim_C T_X^1 = \dim_C H^1(\Gamma, N_C) \geqq -3\cdot\chi(D/\Gamma) \quad \text{if} \quad n = 3 . \qquad \text{q.e.d.}$$

REMARK. Tsuchihashi proved in [T] that if the compact real manifold D/Γ is a two-dimensional real torus, then the normal isolated singularity X associated to (C, Γ) is a Hilbert modular cusp singularity and that D/Γ cannot be a Klein bottle. Hence we see from Theorem 3 that Tsuchihashi's cusp singularity X of dimension three is not rigid if the cone C is indecomposable, since $\chi(D/\Gamma)$ is then necessarily negative.

REMARK. Recently Tsuchihashi proved that $\dim_C T_X^1 = -3\chi(D/\Gamma)$ when $n = 3$, and he succeeded in constructing a versal family of Tsuchihashi cusp singularities of dimension greater than two by using our main result, namely Theorem 1. His results say that Tsuchihashi cusp singularities of dimension greater than two are not taut and that these singularities have no smoothing.

3. Proof of Theorem 1.

We utilize a method partly analogous to that employed by Freitag in [F].

LEMMA 2. *The complex manifold* $Y := (N_R + \sqrt{-1}C)/N$ *and the complex analytic space* $X = ((N_R + \sqrt{-1}C)/N\cdot\Gamma) \cup \{x_0\}$ *are Stein spaces.*

PROOF. The algebraic torus $N \otimes_Z \boldsymbol{C}^\times = (N \otimes_Z \boldsymbol{C})/N$ is obviously a Stein space. Since Y is a strictly pseudoconvex domain in $N \otimes_Z \boldsymbol{C}^\times$, it is a Stein space as well.

The continuous function $\Phi\colon X \to \boldsymbol{R}_{>0}$ is strictly plurisubharmonic on $Y \setminus \{x_0\}$. Hence the relatively compact subset $W(c) = \{z \in X;\ \Phi(z) < c\}$ is a Stein space for any positive number c. Indeed, since $W(c)$ is a strictly Levi pseudoconvex domain in X, there exist a Stein space $\tilde{W}(c)$ and a proper morphism $\psi\colon W(c) \to \tilde{W}(c)$ satisfying the following conditions

(a) $H^0(W(c), O_{W(c)}) \simeq H^0(\tilde{W}(c), O_{\tilde{W}(c)})$,

(b) there exist finite number of points $z_1', \cdots, z_r'$ in $\tilde{W}(c)$ such that $\psi^{-1}(z_j')$ are compact subvarieties in $W(c)$ of positive dimensions and that $\psi\colon W(c) \setminus \bigcup_{j=1}^r \psi^{-1}(z_j') \to \tilde{W}(c) \setminus \{z_1', \cdots, z_r'\}$ is an isomorphism ([GR, Chap. IX, Theorem C.4]).

From its proof, we see that $\psi\colon W(c) \setminus \{x_0\} \to \psi(W(c) \setminus \{x_0\})$ is an isomorphism, because Φ is strictly plurisubharmonic on $W(c)\setminus\{x_0\}$. Hence $\psi^{-1}(\psi(x_0))=\{x_0\}$, and ψ is an isomorphism.

Let $\{c_l\}$ be a sequence of positive numbers with $0 < c_1 < c_2 < \cdots \to \infty$. Then $W(c_l)\subset W(c_{l+1})$ and X is the union of Stein spaces $\bigcup_{l=1}^{\infty} W(c_l)$. Since $H^0(W(c_{l+1}), O_X)$ is dense in $H^0(W(c_l), O_X)$ by Lemma 1, we see that X is a Stein space ([GR, Chap. VII, Theorem B.10]). q.e.d.

LEMMA 3 (Schlessinger [Sc]). *Let $i\colon X\to \boldsymbol{C}^d$ be a closed embedding. Then there exists an exact sequence*

$$0 \to T_X^1 \to H^1(U, \Theta_X) \xrightarrow{\chi^*} H^1(U, i^*\Theta_{C^d}) .$$

Since Γ acts on the Stein manifold Y properly discontinuously and freely, $H^\nu(Y, O_Y) = H^\nu(Y, \Theta_Y) = 0$ for all $\nu > 0$ and $R^\nu p_* O_Y = R^\nu p_* \Theta_Y = 0$ for all $\nu > 0$ ($p\colon Y \to U = Y/\Gamma$ is a canonical projection). Hence we have:

LEMMA 4 (Grothendieck [G]). *In our situation, we have natural isomorphisms*

$$H^\nu(U, O_U) \simeq H^\nu(\Gamma, H^0(Y, O_Y))$$

and

$$H^\nu(U, \Theta_U) \simeq H^\nu(\Gamma, H^0(Y, \Theta_Y)) ,$$

for all ν.

PROPOSITION 1. *The canonical inclusion $\boldsymbol{C} \to H^0(Y, O_Y)$ as constant holomorphic functions induces the natural homomorphisms*

$$H^\nu(\Gamma, \boldsymbol{C}) \to H^\nu(\Gamma, H^0(Y, O_Y)) ,$$

which are isomorphisms for $1 \leqq \nu \leqq n-2$.

PROOF. The vector spaces $H^\nu(\Gamma, H^0(Y, O_Y))$ are finite dimensional for $1 \leqq \nu \leqq n-2$. Indeed, we have

$$H^\nu(\Gamma, H^0(Y, O_Y)) \simeq H^\nu(U, O_U) \quad \text{and}$$
$$H^\nu(U, O_U) \simeq H^{\nu+1}_{x_0}(X, O_X) \quad \text{for all } \nu .$$

The local cohomology groups $H^j_{x_0}(X, O_X)$ are finite dimensional vector spaces for $j < n$, since $U = X\setminus\{x_0\}$ is smooth ([BS, Chapter II, Corollary 4.5]).

Put $M := \{f\colon N_{\boldsymbol{R}} + \sqrt{-1}C \to \boldsymbol{C}$ holomorphic functions such that $f(\mu+z) = f(z)$ for all $\mu \in N\}$. Then $H^0(Y, O_Y) \simeq M$. A function f in M has the Fourier series expansion

$$f(z) = \sum_{\mu\in N^*} a_\mu e(\langle \mu, z\rangle) ,$$

and the coefficients $a_\mu = a_\mu(f)$ are written as

$$a_\mu(f) := \mathrm{vol}(D(N))^{-1}\int_{D(N)} f(z+u)e(-\langle \mu, z+u\rangle)du\ ,$$

which are independent of z, where du denotes a fixed Lebesgue measure on N_R, $D(N)$ is the fundamental domain of N in N_R and $\mathrm{vol}(D(N))$ is the volume of $D(N)$ with respect to du. Set $M^0 := \{f \in M;\ a_0(f) = 0\}$. The Γ-module M decomposes into $M = \boldsymbol{C} \oplus M^0$. The proposition is equivalent to $H^\nu(\Gamma, M^0) = 0$ for $1 \leqq \nu \leqq n-2$. Set $N_0^* := N^* \setminus \{0\}$. An element of M^0 is a convergent Fourier series

$$f(z) = \sum_{\mu \in N_0^*} a_\mu e(\langle \mu, z\rangle)\ ,$$

on which $\gamma \in \Gamma$ acts as

$$\gamma f(z) = \sum_{\mu \in N_0^*} a_\mu e(\langle \mu, \gamma z\rangle) = \sum_{\mu \in N_0^*} a_\mu e(\langle \gamma\mu, z\rangle) = \sum_{\mu \in N_0^*} a_{\gamma^{-1}\mu} e(\langle \mu, z\rangle)\ .$$

We can decompose M^0 into Γ-invariant submodules. We associate to $\mu \in N_0^*$ the submodule

$$M_\mu^0 := \{f \in M^0;\ a_\nu(f) = 0 \quad \text{for} \quad \nu \notin \Gamma\mu\}\ .$$

If $\mu_1, \cdots, \mu_m$ are elements in N_0^* such that $\Gamma\mu_1, \cdots, \Gamma\mu_m$ are disjoint, then we have

$$M^0 = M_{\mu_1}^0 \oplus \cdots \oplus M_{\mu_m}^0 \oplus L\ ,$$

where any element f of L has the Fourier coefficients $a_{\gamma\mu_j}(f) = 0$ for all γ in Γ and $j = 1, \cdots, m$. When $\mu_r = r\mu$ $(r = 1, \cdots, m)$, the modules $M_\mu^0, M_{2\mu}^0, \cdots, M_{m\mu}^0$ are isomorphic to each other under the mappings

$$M_\mu^0 \ni f(z) \to f(rz) \in M_{r\mu}^0\ .$$

Therefore

$$\dim_C H^\nu(\Gamma, M^0) \geqq m \dim_C H^\nu(\Gamma, M_\mu^0) \qquad (1 \leqq \nu \leqq n-2)$$

for all positive integers m and $\mu \in N_0^*$. Since $H^\nu(\Gamma, M^0)$ are finite dimensional, we have $H^\nu(\Gamma, M_\mu^0) = 0$ for $1 \leqq \nu \leqq n-2$ and $\mu \in N_0^*$.

Note that M^0 is a Fréchet space with the topology of uniform convergence on compact subsets. We can calculate the cohomology groups $H^\nu(\Gamma, M^0)$ by $H^\nu(\Gamma, M^0) = \mathrm{Ext}_A^\nu(\boldsymbol{C}, M^0)$ where $A = \boldsymbol{C}[\Gamma]$ is the group ring of Γ.

Since Γ is isomorphic to the fundamental group of the compact real manifold D/Γ, there exists a finite A-free resolution of the trivial A-module $\boldsymbol{C}$

$$\cdots \to E^2 \to E^1 \to E^0 \to \boldsymbol{C} \to 0\ .$$

For example, the chain complex determined by a Γ-invariant triangulation of D as in the proof of Theorem 3 is a finite free resolution of the trivial A-module $\boldsymbol{C}$ (see [Se, Proposition 9]). Then we have complexes $L^{\cdot}$ and $L^{\cdot}_{\mu}$ from which we can calculate the cohomology groups of M^0 and M^0_{μ}:

$$L^{\cdot}: 0 \to L^0 \to L^1 \to L^2 \to \cdots$$

and

$$L^{\cdot}_{\mu}: 0 \to L^0_{\mu} \to L^1_{\mu} \to L^2_{\mu} \to \cdots ,$$

where L^{ν} and L^{ν}_{μ} are the direct sums of finitely many copies of M^0 and M^0_{μ}, respectively. By Z^{ν} and B^{ν} we denote the subspaces of cocycles and coboundaries in L^{ν}, respectively. By Z^{ν}_{μ} and B^{ν}_{μ} we denote those in L^{ν}_{μ}. Since $Z^{\nu}_{\mu} = B^{\nu}_{\mu}$ for $1 \leqq \nu \leqq n-2$ by the vanishing of the cohomology groups $H^{\nu}(\Gamma, M^0_{\mu})$ and since the direct sum of the submodules M^0_{μ} is dense in M^0, the direct sum of the submodules $B^{\nu}_{\mu} \subset B^{\nu}$, where μ runs through N^*_0 modulo Γ, is dense in Z^{ν}. Hence B^{ν} is a dense submodule of Z^{ν}. Note that L^{ν} is a Fréchet space, since it is the direct sum of finitely many Fréchet spaces. It remains to prove that $B^{\nu} \subset L^{\nu}$ are closed subspaces for $1 \leqq \nu \leqq n-2$.

Since Z^{ν}/B^{ν} is finite dimensional for $1 \leqq \nu \leqq n-2$, there exists a finite dimensional vector subspace V of Z^{ν} such that $Z^{\nu} = V + B^{\nu}$ and that the natural mapping

$$V \oplus L^{\nu-1} \to Z^{\nu}$$

is surjective. By the open mapping theorem (see, for instance, [Y]), this mapping is open, i.e., Z^{ν} can be identified with the quotient space of $V \oplus L^{\nu-1}$. Therefore the image B^{ν} of $L^{\nu-1}$ is closed in Z^{ν}. q.e.d.

COROLLARY. *We have isomorphisms*

$$H^{\nu}(\Gamma, N_C) \simeq H^{\nu}(\Gamma, H^0(Y, \Theta_Y)) \quad \textit{for} \quad 1 \leqq \nu \leqq n-2 .$$

PROOF. The dimensions of the cohomology groups $H^{\nu}(U, \Theta_U)$ for $1 \leqq \nu \leqq n-2$ are finite since the local cohomology groups $H^{\nu+1}_{x_0}(X, \Theta_X)$ for $1 \leqq \nu \leqq n-2$ are finite dimensional and since $H^{\nu}(U, \Theta_U) \simeq H^{\nu+1}_{x_0}(X, \Theta_X)$ for $1 \leqq \nu \leqq n-2$. Thus we can apply the argument in Proposition 1 to this case by noting $H^{\nu}(Y, \Theta_Y) \simeq N \otimes_{\boldsymbol{Z}} H^{\nu}(Y, O_Y)$. q.e.d.

PROPOSITION 2. *The morphism $H^1(U, \Theta_X) \to H^1(U, i^*\Theta_{C^d})$ is the zero map, i.e., $T^1_X \simeq H^1(U, \Theta_X)$.*

PROOF. For any element f in O_{X,x_0}, there exists a positive number c such that f is a holomorphic function on $(N_{\boldsymbol{R}} + \sqrt{-1}\,\bar{V}(c))/N\cdot\Gamma$. The pull-back of f has the Fourier series expansion

$$\sum_{\mu \in N^* \cap \bar{C}^*} a_{\mu} e(\langle \mu, z \rangle) .$$

We can easily see that $a_{\gamma\mu} = a_\mu$ for all $\gamma \in \Gamma$ and that the maximal ideal of O_{X,x_0} is generated by elements of the form $F_\mu(z) = \sum_{\gamma\in\Gamma} e(\langle\gamma\mu, z\rangle)$ with $\mu \in N_0^* \cap C^*$. Let $f_1, \cdots, f_d$ be generators of the maximal ideal of O_{X,x_0} so that $f_j(z)$ are of the form $F_\mu(z)$ for some $\mu \in N_0^* \cap C^*$. Let $x_1, \cdots, x_d$ be the coordinates for $\boldsymbol{C}^d$ such that $x_j = f_j(z)$. The tangent sheaf of $\boldsymbol{C}^d$ is free with $\{\partial/\partial x_1, \cdots, \partial/\partial x_d\}$ as a basis:

$$H^1(U, i^*\Theta_{\boldsymbol{C}^d}) = \bigoplus_{j=1}^{d} H^1(U, O_U)\frac{\partial}{\partial x_j} = \bigoplus_{j=1}^{d} H^1(\Gamma, H^0(Y, O_Y))\frac{\partial}{\partial x_j}\,.$$

Let

$$\chi\colon H^0(Y, \Theta_Y) \to \bigoplus_{j=1}^{d} H^0(Y, O_Y)(\partial/\partial x_j)$$

be the linear mapping with components χ_j ($j = 1, \cdots, d$), where

$$\chi_j\Big(\sum_{k=1}^{n} h_k(\partial/\partial z_k)\Big) = \Big(\sum_{k=1}^{n} h_k(\partial f_j/\partial z_k)\Big)(\partial/\partial x_j)$$

for any section $\sum_{k=1}^{n} h_k(\partial/\partial z_k)$ of Θ_Y.

To calculate the mapping

$$\chi_j^*\colon H^1(\Gamma, H^0(Y, \Theta_Y)) \to H^1(\Gamma, H^0(Y, O_Y))(\partial/\partial x_j)\,,$$

we consider the composite

$$(*)\qquad N\otimes_{\boldsymbol{Z}}\boldsymbol{C} \hookrightarrow H^0(Y, \Theta_Y) \simeq N\otimes_{\boldsymbol{Z}} H^0(Y, O_Y) \xrightarrow{\chi_j} H^0(Y, O_Y) \simeq \boldsymbol{C}\oplus M^0\,.$$

Let $\{u_1, \cdots, u_n\}$ be a $\boldsymbol{Z}$-basis of N. Then a point z in $N_{\boldsymbol{C}}$ is expressed as $z = z_1u_1 + \cdots + z_nu_n$. Taking $z_1, \cdots, z_n$ as the coordinates for $N_{\boldsymbol{C}}$, we have

$$\partial F_\mu(z)/\partial z_k = \sum_{\gamma\in\Gamma} 2\pi\sqrt{-1}\langle\gamma\mu, u_k\rangle e(\langle\gamma\mu, z\rangle)\,.$$

Hence the image of the composite (*) is in M^0. By Corollary to Proposition 1, the mapping χ_j^* factors as

$$H^1(\Gamma, N_{\boldsymbol{C}}) \xrightarrow{\sim} H^1(\Gamma, H^0(Y, \Theta_Y)) \to H^1(\Gamma, M^0) \hookrightarrow H^1(\Gamma, H^0(Y, O_Y))\,.$$

By Proposition 1, we have $H^1(\Gamma, M^0) = 0$ hence $\chi_j^* = 0$ for all j. q.e.d.

This completes the proof of Theorem 1.

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