Tôhoku Math. Journ.
38 (1986), 551-559.

LINES OF PRINCIPAL CURVATURE FOR MAPPINGS WITH WHITNEY UMBRELLA SINGULARITIES

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(Received November 6, 1985)

Abstract. The pattern of lines of principal curvature near Whitney umbrella singularities of mappings of surfaces M into R^3 are established. Sufficient conditions for the stability of the whole configuration of lines of principal curvature, under small perturbations of the mappings are given. These sufficient conditions are satisfied by a dense set in the space of C^r-mappings of M into R^3, $r \geqq 4$, endowed with the C^2-topology.

1. Introduction. Let M be a compact, oriented smooth two-manifold. This work will be concerned with the study of the different stable patterns through which M can be bended when smoothly mapped into $\boldsymbol{R}^3$. The bending pattern of a mapping $\alpha: M \to \boldsymbol{R}^3$ will be represented here by the singular points, $\mathscr{C}_\alpha$, at which the bending is infinite; the umbilical points $\mathscr{U}_\alpha$ at which the bending is finite but equal in all directions and; by the family of lines of principal curvature $\mathscr{F}_\alpha$, f_α, defined on $M - (\mathscr{U}_\alpha \cup \mathscr{C}_\alpha)$, which represent the directions along which the bending is extremal: maximal along $\mathscr{F}_\alpha$ and minimal along f_α.

These four objects will be assembled together into the principal configuration of the mapping denoted by $\mathscr{P}_\alpha = (\mathscr{C}_\alpha, \mathscr{U}_\alpha, \mathscr{F}_\alpha, f_\alpha)$.

The study of principal configurations on surfaces goes back to the classical works of Dupin, Darboux and Cayley. The reader is refered to the Introduction of Gutierrez—Sotomayor [G-S. 1] for a discussion and bibliographical references concerning the general background for the study of principal configurations of immersions, i.e., for mappings α with $\mathscr{C}_\alpha$ empty.

This paper will be mainly concerned with the study of the global features of principal configurations $\mathscr{P}_\alpha$ that remain undisturbed under small perturbations of the mapping α. The case of immersions was studied in [G-S. 1], [G-S. 2] under the name of Principal Structural Stability.

In this work (Theorem 1) the principal configurations near Whitney umbrella singularities [Wh] of mappings of surfaces M into $\boldsymbol{R}^3$ will be established. The local conclusions obtained here combined with the conditions given in [G-S. 1] for the case of non-singular mappings (immersions), provide sufficient conditions for the stability of the whole configuration of lines of principal curvature, under small perturbations of the mappings (Theorem 2, a).

The genericity of Whitney umbrellas [Wh] together with the C^2-density results of [G-S. 2] for immersions imply that the sufficient conditions given in this work are satisfied by a dense set in the space of C^r-mappings of M into $\boldsymbol{R}^3$, $r \geqq 4$, endowed with the C^2-topology (Theorem 2, b).

2. Statement of the main results. To formulate precisely the main results of the present work, it is necessary to review carefully the elements involved in the principal configuration $\mathscr{P}_\alpha = (\mathscr{C}_\alpha, \mathscr{U}_\alpha, \mathscr{F}_\alpha, f_\alpha)$.

Call by $\mathscr{M}^r = \mathscr{M}^r(M, \boldsymbol{R}^3)$ the space of C^r mappings of M into $\boldsymbol{R}^3$. When endowed with the C^s topology, $s \leqq r$, this space will be denoted by $\mathscr{M}^{r,s}$.

Denote by $\mathscr{C}_\alpha$ the set of *singular points* p of α; that is, where $D\alpha_p$ has rank $\leqq 1$. Call $\mathscr{U}_\alpha$ the set of *umbilical points* p of α, i.e., where the first fundamental form of α, $I_\alpha(p) = \langle D\alpha_p, D\alpha_p\rangle$, is proportional to the second fundamental form $II_\alpha(p) = -\langle DN_\alpha(p), D\alpha(p)\rangle$. Here $\langle\ ,\ \rangle$ is the Euclidean metric on $\boldsymbol{R}^3$ and $N_\alpha: M - \mathscr{C}_\alpha \to S^2$ denotes the normal map of α defined by:

$$N_\alpha(p) = \frac{\alpha_u \wedge \alpha_v}{\|\alpha_u \wedge \alpha_v\|}\ ,$$

where $(u, v): (M, p) \to (\boldsymbol{R}^2, 0)$ is a positive chart of M around p, $\wedge$ denotes the exterior product of vectors in $\boldsymbol{R}^3$, determined by a once-for-all-fixed orientation of $\boldsymbol{R}^3$, $\alpha_u = \partial\alpha/\partial u$, $\alpha_v = \partial\alpha/\partial v$ and $\|\ \| = \langle\ ,\ \rangle^{1/2}$ is the Euclidean norm in $\boldsymbol{R}^3$.

Finally, $\mathscr{F}_\alpha$ (resp. f_α) denotes the foliation on $M - (\mathscr{U}_\alpha \cup \mathscr{C}_\alpha)$ by the family of curves of maximal (resp. minimal) principal curvature of α. This means that at each point $p \in M - (\mathscr{U}_\alpha \cup \mathscr{C}_\alpha)$ any vector v which spans the line $\mathscr{L}_\alpha(p)$ (resp. $\ell_\alpha(p)$) tangent to $\mathscr{F}_\alpha(p)$ (resp. $f_\alpha(p)$) provides the maximum $K_\alpha(p)$ (resp. minimum $k_\alpha(p)$) among all possible directions $u \in T_pM$ of the *normal curvature* at p:

$$II_\alpha(p)(u, u)/I_\alpha(p)(u, u)\ .$$

The function K_α (resp. k_α) on $M - \mathscr{C}_\alpha$ is called the *maximal* (resp. *minimal*) *principal* curvature of α. It is of class C^{r-2} on $M - (\mathscr{U}_\alpha \cup \mathscr{C}_\alpha)$.

A mapping $\alpha \in \mathscr{M}^r$ is said to be *C^s-principally structurally stable* if there is a neighborhood $\mathscr{V}_\alpha$ of α in $\mathscr{M}^r$ such that for every $\beta \in \mathscr{V}_\alpha$ there is a homeomorphism $h_\beta: M \to M$ such that for every $\beta \in \mathscr{V}_\alpha$ there is a homeomorphism $h_\beta: M \to M$ such that $h_\beta(\mathscr{C}_\alpha) = \mathscr{C}_\beta$, $h_\beta(\mathscr{U}_\alpha) = \mathscr{U}_\beta$, and h_β maps lines of $\mathscr{F}_\alpha$ (resp. f_α) onto those of $\mathscr{F}_\beta$ (resp. f_β).

The concept of C^s-*principal structural stability of* $\alpha \in \mathscr{M}^r$ *at a point* $p \in M$ can be formulated as follows: For every neighborhood $V(p)$ of p in M there must be a neighborhood $\mathscr{V}_\alpha$ of α in $\mathscr{M}^{r,s}$ such that for $\beta \in \mathscr{V}_\alpha$ there must be a point $q = q(\beta)$ in $V(p)$ and a homeomorphism $h: W(p) \rightarrow W(q)$ between neighborhoods of p and q, which maps p to q and maps $\mathscr{F}_\alpha | W(p)$ and $\mathscr{f}_\alpha | W(p)$ respectively onto $\mathscr{F}_\beta | W(q)$ and $\mathscr{f}_\beta | W(q)$. If $p \in \mathscr{C}_\alpha$, $\mathscr{U}_\alpha$ or $M - (\mathscr{C}_\alpha \cup \mathscr{U}_\alpha)$, it is also required that $q \in \mathscr{C}_\beta$, $\mathscr{U}_\beta$ or $M - (\mathscr{C}_\beta \cup \mathscr{U}_\beta)$.

THEOREM 1. *A mapping* $\alpha \in \mathscr{M}^r$, $r \geqq 2$, *is* C^2-*principally stable at a point* $p \in \mathscr{C}_\alpha$ *provided condition* W *defined below holds*:

Condition W. *There is a chart* $(u, v): (M, p) \rightarrow (\boldsymbol{R}^2, 0)$ *such that*

(a) $\alpha_v(0) = 0$ *and* $\{\alpha_u(0), \alpha_{uv}(0), \alpha_{vv}(0)\}$ *is a basis for* $\boldsymbol{R}^3$,

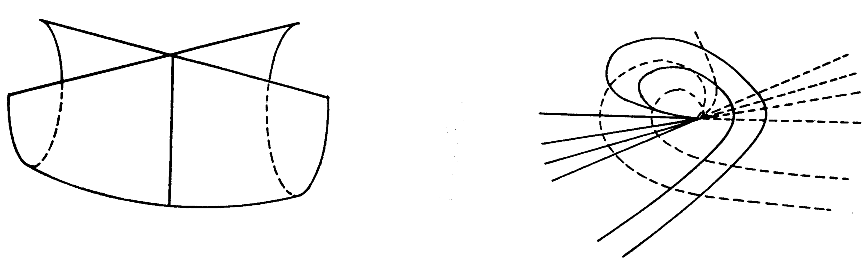
(b) $\langle \alpha_u(0), \alpha_{vv}(0) \rangle^2 + \langle \alpha_u(0), \alpha_{uv}(0) \rangle^2 \neq 0$.

A singular point which verifies (a) of condition W is called a *Whitney umbrella singular point*. Canonical forms for these points under diffeomorphisms at the source and target were provided in [Wh], for mappings of class C^r, $r \geqq 12$.

Condition W, (a) is independent of the choice of charts (u, v) (cf. [Wh], [L-Th]). Actually it is equivalent to the transversality of $J^1\alpha$, the first jet extension of α, to $S_1(2, 3)$, the submanifold of codimension one of rank one jets in $J^1(2, 3) \cong \boldsymbol{R}^6$.

Condition W, (b) is also intrinsic as follows from direct computation. Moreover, after a linear change of variables in the (u, v)-plane it can always be assumed that $\langle \alpha_u, \alpha_{vv} \rangle |_{(0,0)}$ is different from zero.

In Section 3, it will be proved that the principal configuration around a Whitney umbrella singular point of type W is as in Figure 1.



point of type W principal configuration

FIGURE 1.

To formulate the global principal stability theorem for mappings, it will be convenient to previously reformulate some results from [G-S. 1], [G-S. 2].

PROPOSITION 1 ([G-S. 1]). *A mapping $\alpha \in \mathscr{M}^r$, $r \geqq 4$, is C^3-principally structurally stable at a point $p \in \mathscr{U}_\alpha$ provided the following condition is satisfied:*

Condition D: *There is a chart $(u, v): (M, p) \to (\boldsymbol{R}^2, 0)$ and an isometry Γ of $\boldsymbol{R}^3$ with $\Gamma(\alpha(p)) = 0$ such that:*

$$(\Gamma \circ \alpha)(u, v) = (u, v, (k/2)(u^2 + v^2) + (a/6)u^3 + (b/2)uv^2 + (c/6)v^3 + O((u^2 + v^2)^2) ,$$

where

(τ) $b(b - a) \neq 0$

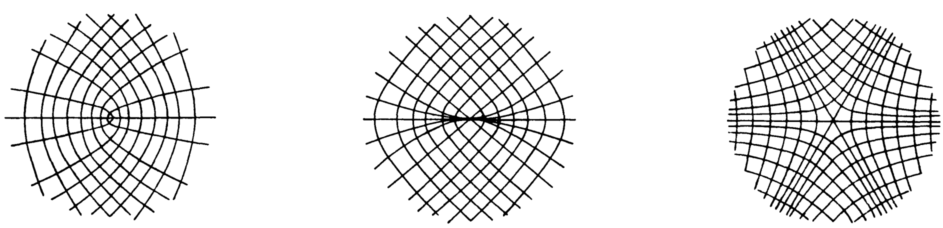
(δ) *either one of the following inequalities hold:*

D_1: $a/b > (c/2b)^2 + 2$

D_2: $(c/2b)^2 + 2 > a/b > 1$, $a \neq 2b$

D_3: $1 > a/b$.

The local principal configurations are illustrated in Figure 2. Condition D amounts to a condition given by Darboux.



D_1: $b(b-a)\neq 0$, $a/b > (c/2b)^2+2$

D_2: $b(b-a)\neq 0$, $1<a/b<(c/2b)^2+2$

D_3: $b(b-a)\neq 0$, $1>a/b$

FIGURE 2.

The index i refers to the number of *umbilical separatrices* [G-S. 1] of D_i, $i = 1, 2, 3$, in each of the foliations $\mathscr{F}_\alpha$, $\mathscr{f}_\alpha$.

A map $\alpha \in \mathscr{M}^r$ is C^s-principally stable at a *principal cycle* c if for every neighborhood $V(c)$ of c in M there is a neighborhood $\mathscr{V}(\alpha)$ of α on $\mathscr{M}^r$ such that for $\beta \in \mathscr{V}(\alpha)$ there is a principal cycle $d = d(\beta)$, of β, contained in $V(c)$ and homeomorphism $h: W(c) \to W(d)$ between neighborhoods of c and d, which maps c to d and maps $\mathscr{F}_\alpha | W(c)$ and $\mathscr{f}_\alpha | W(c)$ respectively onto $\mathscr{F}_\beta | W(d)$ and $\mathscr{f}_\beta | W(d)$.

PROPOSITION ([G-S. 1]). *A mapping $\alpha \in \mathscr{M}^r$, $r \geqq 4$, is C^3-principally structurally stable at a principal cycle c provided one of the following conditions,* H_1 *or* H_2, *which are equivalent, is satisfied:*

Condition H:

$$(\mathrm{H}_1) \qquad \int_c \frac{dk_\alpha}{K_\alpha - k_\alpha} = \int_c \frac{dK_\alpha}{K_\alpha - k_\alpha} \neq 0 .$$

(H_2) *The cycle c is a hyperbolic cycle of the foliation to which it belongs. That is, the Poincaré first return map h associated to a transversal to c at a point q is such that $h'(q) \neq 1$.*

DEFINITION. Let $\mathscr{S}^r(M)$ denote the set of $\alpha \in \mathscr{M}^r$, $r \geqq 4$, such that

(a) α satisfies condition D at every point in $\mathscr{U}_\alpha$ and condition W at every point in $\mathscr{C}_\alpha$.

(b) α satisfies condition H at each of its of principal cycles.

(c) The limit set of every principal line of α is the union of singular points, umbilical points and principal cycles.

(d) There is no umbilical separatrix of α which is separatrix of two different umbilical points or twice a separatrix of the same umbilical point.

THEOREM 2. *Let $r \geqq 4$ and M be a compact oriented two-manifold.*

(a) *The set $\mathscr{S}^r = \mathscr{S}^r(M)$ defined above is open in $\mathscr{M}^{r,3}$ and every $\alpha \in \mathscr{S}^r$ is C^3-principally structurally stable.*

(b) *Furthermore, $\mathscr{S}^r$ is dense in $\mathscr{M}^{r,2}$.*

3. Principal configuration near Whitney Umbrellas. In what follows $(\partial/\partial u)\alpha$, $(\partial^2/\partial u\partial v)\alpha, \cdots$ will be denoted by $\alpha_u, \alpha_{uv}, \cdots$ respectively. Let $\alpha_u(0)$, $\alpha_{uv}(0)$, $\alpha_{vv}(0)$ be a positive basis for $\boldsymbol{R}^3$, $\alpha_v(0) = 0$; $\partial/\partial u$, $\partial/\partial v$ be a positive basis for M. Let

$$\begin{aligned} E &= \langle \alpha_u, \alpha_u \rangle & e &= \langle \alpha_u \wedge \alpha_v, \alpha_{uu} \rangle \\ F &= \langle \alpha_u, \alpha_v \rangle & f &= \langle \alpha_u \wedge \alpha_v, \alpha_{uv} \rangle \\ G &= \langle \alpha_v, \alpha_v \rangle & g &= \langle \alpha_u \wedge \alpha_v, \alpha_{vv} \rangle . \end{aligned}$$

The differential equation of lines of principal curvature is given by:

$$(1) \qquad (dv)^2(Fg - Gf) + (du)(dv)(Eg - Ge) + (du)^2(Ef - eF) = 0 .$$

Let

$$\begin{aligned} a &= 1/2\langle \alpha_u, \alpha_{uv} \rangle \langle \alpha_u \wedge \alpha_{uv}, \alpha_{vv} \rangle |_{(0,0)} \\ b &= \langle \alpha_u, \alpha_{vv} \rangle \langle \alpha_u \wedge \alpha_{uv}, \alpha_{vv} \rangle |_{(0,0)} \\ c &= \langle \alpha_u, \alpha_u \rangle \langle \alpha_u \wedge \alpha_{uv}, \alpha_{vv} \rangle |_{(0,0)} . \end{aligned}$$

Since $(Fg - Gf)(0, 0) = (Fg - Gf)_u(0, 0) = (Fg - Gf)_v(0, 0) = (Fg - Gf)_{vv}(0, 0) = 0$, $(Eg - Ge)(0, 0) = (Eg - Ge)_v(0, 0) = 0$ and $(Ef - Fe)(0, 0) = (Ef - Fe)_u(0, 0) = 0$, it is found that

$$2(Fg - Gf) = 2au^2 + 2buv + 2R(u, v)$$

$$Eg - Ge = cu + N(u, v)$$
$$Ef - Fe = -cv - M(u, v) ,$$

where $R(u, v) = O((u^2 + v^2)^{3/2})$, $N(u, v) = O(u^2 + v^2)$ and $M(u, v) = O(u^2 + v^2)$.

It follows from Hadamard's Lemma that

$$\begin{aligned}(2)\qquad R(u, v) &= u^3R_1(u, v) + u^2vR_2(u, v) + uv^2R_3(u, v) + v^3R_4(u, v)\\ N(u, v) &= u^2N_1(u, v) + uvN_2(u, v) + v^2N_3(u, v)\\ M(u, v) &= u^2M_1(u, v) + uvM_2(u, v) + v^2M_3(u, v) ,\end{aligned}$$

where R_i, M_i, N_i are C^1 functions defined in a neighborhood of (0, 0).

Consider the vector field

$$X(u, v) = P(\partial/\partial u) + Q(\partial/\partial v) ,$$

where

$$\begin{aligned}P &= 2au^2 + 2buv + 2R(u, v)\\ Q &= -(cu + N(u, v))\\ &\quad - \{(cu + N(u, v))^2 + 2(2au^2 + 2buv + 2R(u, v))(cv + M(u, v))\}^{1/2} .\end{aligned}$$

Except possibly when $P = 0$, X is tangent to one of the principal foliations. To study the principal configuration in an angular sector which contains the positive u semi-axis it is appropriate to perform the following blowing up:

$H(u, w) = (u, uw)$ (defined in the half-plane $u > 0$). The induced vector field $H_*X = U(\partial/\partial u) + W(\partial/\partial w)$ is such that:

$$\begin{aligned}U &= 2au^2 + 2bu^2w + 2R(u, uw)\\ W &= (-w/u)(2au^2 + 2bu^2w + 2R(u, uw))\\ &\quad + (1/u)[-(cu + N(u, uw)) - \{(cu + N(u, uw))^2\\ &\quad + 2(2au^2 + 2bu^2w + 2R(u, uw))(cuw + M(u, uw))\}^{1/2}] .\end{aligned}$$

Further simplification leads to

$$\begin{aligned}W &= (-w)(2au + 2buw + (2/u)R(u, uw)) - (c + (1/u)N(u, uw))\\ &\quad - \{(c + (1/u)N(u, uw))^2 + 2(2a + 2bw + (2/u^2)R(u, uw))\\ &\quad \times (cuw + M(u, uw))\}^{1/2} .\end{aligned}$$

It follows from (2) that the vector field H_*X extends differentiably to a neighborhood of (0, 0) and that, for all w, $H_*X(0, w) = (0, -2c)$. When restricted to the half-plane $u > 0$, the diffeomorphism H takes orbits of H_*X into integral curves of (1). Therefore, locally around (0, 0), in any angular sector having (0, 0) as a vertex and contained in the half-plane $u > 0$, the principal lines of (1), tangent to X, are distributed as in

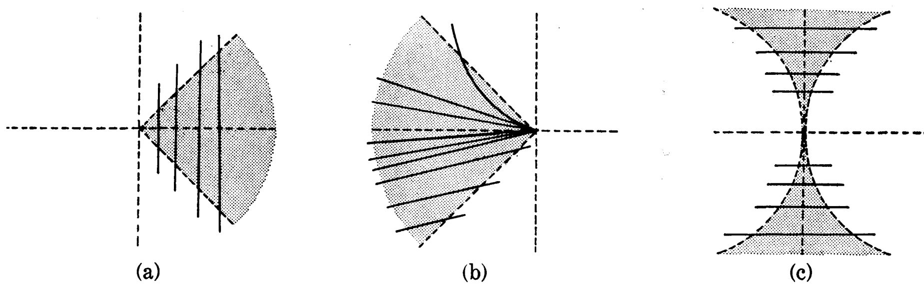


FIGURE 3.

Figure 3. (a).

To study the principal configuration in an angular sector which contains the negative u semi-axis it is appropriate to perform the blowing up $H(u, w) = (u, uw)$ (now defined in the half-plane $u < 0$). The induced vector field $H_*X = U_1(\partial/\partial u) + W_1(\partial/\partial w)$ satisfies:

$$\begin{aligned} U_1 &= 2au^2 + 2bu^2w + 2R(u, uw) \\ W_1 &= (-w)(2au + 2buw + (2/u)R(u, uw)) - (c + (1/u)N(u, uw)) \\ &\quad + \{(c + (1/u)N(u, uw))^2 + 2(2a + 2bw \\ &\quad + (2/u^2)R(u, uw))(cuw + M(u, uw))\}^{1/2} . \end{aligned}$$

Let

$$\begin{aligned} \tau = &-(c + (1/u)N(u, uw)) - \{(c + (1/u)N(u, uw))^2 \\ &+ 2(2a + 2bw + (2/u^2)R(u, uw))(cuw + M(u, uw))\}^{1/2} . \end{aligned}$$

It follows from (2) that the vector field $(1/u^2)\tau(H_*X)$ extends differentiably to a neighborhood of $(0, 0)$. An easy computation shows that the value of $(1/u^2)\tau(U_1)$ at $(0, w)$, for all w, is $2a$ which can be assumed to be different from 0 by Remark in Section 2.

Therefore, locally around $(0, 0)$, in any angular sector having $(0, 0)$ as a vertex and contained in the half-plane $u < 0$, the principal lines of (1) tangent to X are distributed as in Figure 3. (b).

Since $R_4(0, 0) = 2\langle \alpha_{vv}, \alpha_{vv}\rangle\langle \alpha_u \wedge \alpha_v, \alpha_{vv}\rangle|_{(0,0)} \neq 0$, it follows that there exists a $\theta > 0$ such that, around $(0, 0)$, for all (u, v) such that $u^2 < \theta v^4$, $(1/v^3)Q(u, v) \neq 0$. This implies that, in this region, the principal foliation of (1) tangent to X is as illustrated in Figure 3. (c).

In conclusion, by patching together the foliations obtained in these sectors, results that the principal foliation tangent to X is as illustrated in Figure 1.

REMARK. The singular points $\mathscr{C}_\alpha$ of a map α can be considered as ends for the immersion $\alpha | M - \mathscr{C}_\alpha$. It is interesting to notice that the principal configuration shown in Figure 1 for Whitney umbrella singularities also appear in a case of elementary ends for immersions with constant mean curvature [G-S. 3].

4. **Proof of Theorem 1.** Let α satisfy condition W at a point p of M. By the intrinsic transversality characterization of Whitney umbrellas, any map β, C^2-close to α, has a unique Whitney umbrella singular point $p(\beta)$, near p. The C^s-principal structural stability of α at p follows by using the methods of [G-S. 1, Section 5].

5. **Proof of Theorem 2.** (a) The openness of $\mathscr{S}^r$ follows from the openness of condition W and of the other conditions involved in its definition (cf. [G-S. 1]).

The method of canonical regions used in [G-S. 1, Section 5] extends to the present case leading the C^3-principal structural stability of any α of $\mathscr{S}^r$.

(b) The density of condition W in $\mathscr{M}^{r,2}$ (cf. [Wh]) as well as of the other conditions which define $\mathscr{S}^r$ (cf. [G-S. 2, Theorem 3.1]) prove part b.

FINAL REMARK. Surfaces with Whitney umbrellas have been considered by Kuiper [Kp] in the problem of minimizing $\int_M |\mathscr{K}_\alpha dA_\alpha|$ among the class of mappings $\alpha: M \to \boldsymbol{R}^3$, where M is a compact two-manifold, $\mathscr{K}_\alpha$ is the Gaussian Curvature of α (outside $\mathscr{C}_\alpha$) and dA_α is the area element associated to I_α on $M - \mathscr{C}_\alpha$. This infimum is not attained for immersions α but it does so for maps whose only singularities are Whitney Umbrellas.

ACKNOWLEDGEMENT. The authors are grateful for the partial support received from California Institute of Technology (C. Gutierrez) and Michigan State University (J. Sotomayor) during the preparation of this paper.

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