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ON A METHOD TO CONSTRUCT ANALYTIC ACTIONS OF NON-COMPACT LIE GROUPS ON A SPHERE

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0. Introduction. Let M be a square matrix of degree n with real coefficients, that is, $M \in M_n(\boldsymbol{R})$. We say that M satisfies the *outward transversality condition* if

$$\frac{d}{dt}\|\exp(tM)x\| > 0 \quad \text{for each} \quad x \in \boldsymbol{R}_0^n = \boldsymbol{R}^n - \{0\} \quad \text{and} \quad t \in \boldsymbol{R} .$$

In this case, there exists a unique real valued analytic function τ on $\boldsymbol{R}_0^n$ such that $\|\exp(\tau(x)M)x\| = 1$, and hence we can define an analytic mapping π^M of $\boldsymbol{R}_0^n$ onto the unit $(n-1)$-sphere S^{n-1} by $\pi^M(x) = \exp(\tau(x)M)x$.

Let G be a Lie group, $\rho: G \to GL(n, \boldsymbol{R})$ a matricial representation, and M a square matrix of degree n with real coefficients satisfying the outward transversality condition. We can define an analytic mapping ξ: $G \times S^{n-1} \to S^{n-1}$ by $\xi(g, x) = \pi^M(\rho(g)x)$, and we see that ξ is an analytic G-action on S^{n-1} if $\rho(g)M = M\rho(g)$ for any $g \in G$. We call ξ a *twisted linear action* of G on S^{n-1} associated to the representation ρ. In particular, if M is the identity matrix, we call ξ a *linear action* of G on S^{n-1} associated to the representation ρ.

Let G be a compact Lie group and $\rho: G \to GL(n, \boldsymbol{R})$ a matricial representation. Then we shall show that any twisted linear action of G on S^{n-1} associated to ρ is equivariantly analytically diffeomorphic to the linear action of G on S^{n-1} associated to ρ. On the other hand, if G is a non-compact Lie group, sometimes we can construct uncountably many topologically distinct twisted linear actions of G associated to only one matricial representation (cf. [4, §6]). We shall study such an example in the final section.

1. Outward transversality condition.

1.1. Let $u = (u_i)$ and $v = (v_i)$ be vectors in $\boldsymbol{R}^n$. As usual, we denote their inner product by $u \cdot v = \sum_i u_i v_i$ and the length of u by $\|u\| = \sqrt{u \cdot u}$.

LEMMA 1.1. *Let $M \in M_n(\boldsymbol{R})$ and assume that M satisfies the outward transversality condition. Then,* (i)

$$\lim_{t\to+\infty} \|\exp(tM)x\| = +\infty \quad \textit{and} \quad \lim_{t\to-\infty} \|\exp(tM)x\| = 0$$

for each $x \in \boldsymbol{R}_0^n$, *and* (ii) *there exists a unique real valued analytic function* τ *on* $\boldsymbol{R}_0^n$ *such that* $\|\exp(\tau(x)M)x\| = 1$ *for each* $x \in \boldsymbol{R}_0^n$.

PROOF. Put $f(t; x) = \|\exp(tM)x\|$. Because M satisfies the outward transversality condition, there exists $\varepsilon > 0$ satisfying $f'(0; x) \geqq \varepsilon$ for $x \in S^{n-1}$. Then

$$\begin{aligned} f'(t; x) &= f'(0; \exp(tM)x) \\ &= \|\exp(tM)x\| \cdot f'(0; \|\exp(tM)x\|^{-1} \exp(tM)x) \geqq \varepsilon \cdot f(t; x) \end{aligned}$$

for each $x \in \boldsymbol{R}_0^n$ and $t \in \boldsymbol{R}$. Hence we obtain

$$\frac{d}{dt} \log f(t; x) \geqq \varepsilon \quad \text{for} \quad x \in \boldsymbol{R}_0^n,\ t \in \boldsymbol{R} .$$

Integrating both sides of the inequality, we obtain

$$\|\exp(tM)x\| \geqq \|x\| \exp(\varepsilon t) \quad \text{for } t > 0 ,$$

$$\|\exp(tM)x\| \leqq \|x\| \exp(\varepsilon t) \quad \text{for } t < 0 .$$

The condition (i) follows from these inequalities. The function $f(t; x)$ is strictly monotone by the assumption on M. Thus the condition (i) assures the unique existence of $\tau: \boldsymbol{R}_0^n \to \boldsymbol{R}$ satisfying $\|\exp(\tau(x)M)x\| = 1$ for each $x \in \boldsymbol{R}_0^n$. On the other hand, we see that τ is analytic, applying the implicit function theorem to the analytic function $(x, t) \to \|\exp(tM)x\|$, because M satisfies the outward transversality condition. q.e.d.

REMARK. Conversely, we can prove that the conditions (i), (ii) are sufficient for M to satisfy the outward transversality condition.

By this lemma, we can define an analytic mapping $\pi^M: \boldsymbol{R}_0^n \to S^{n-1}$ by $\pi^M(x) = \exp(\tau(x)M)x$, if M satisfies the outward transversality condition.

1.2. Let G be a Lie group and $\rho: G \to GL(n, \boldsymbol{R})$ a matricial representation. Denote by $\mathrm{End}_G(\rho)$ the set of all matrices $X \in M_n(\boldsymbol{R})$ satisfying $X\rho(g) = \rho(g)X$ for $g \in G$. The set $GL(n, \boldsymbol{R}) \cap \mathrm{End}_G(\rho)$ is denoted by $\mathrm{Aut}_G(\rho)$. If $M \in \mathrm{End}_G(\rho)$ and M satisfies the outward transversality condition, we call (ρ, M) a TC-pair of degree n. In this case, we can define an analytic mapping $\xi: G \times S^{n-1} \to S^{n-1}$ by $\xi(g, x) = \pi^M(\rho(g)x)$ and we see easily that ξ is an action of G on S^{n-1}. We call ξ a *twisted linear action* of G on S^{n-1} determined by the TC-pair (ρ, M). In particular, if M is the identity matrix I_n, we call ξ a *linear action* of G on S^{n-1} associated to ρ.

Let (ρ, M) and (σ, N) be TC-pairs of degree n. We say that (ρ, M)

is equivalent to (σ, N) if there exist $A \in GL(n, \boldsymbol{R})$ and a positive real number c such that $cN = AMA^{-1}$ and $\sigma(g)A = A\rho(g)$ for any $g \in G$.

LEMMA 1.2. *If (ρ, M) and (σ, N) are equivalent as* TC-*pairs, then the twisted linear action of G on a sphere determined by (ρ, M) is equivariantly analytically diffeomorphic to the one determined by (σ, N).*

PROOF. It is easy to see that the twisted linear action of G determined by (σ, cN) coincides with the one determined by (σ, N) for any positive real number c. So we assume that there exists $A \in GL(n, \boldsymbol{R})$ such that

$$(*) \qquad N = AMA^{-1} \quad \text{and} \quad \sigma(g)A = A\rho(g) \quad \text{for any} \quad g \in G\,.$$

Define analytic mappings h_A, k_A of S^{n-1} into itself by $h_A(x) = \pi^N(Ax)$ and $k_A(y) = \pi^M(A^{-1}y)$. Then we see that the composites $h_A k_A$ and $k_A h_A$ are the identity mappings on S^{n-1} by the condition $N = AMA^{-1}$, and hence $h_A\colon S^{n-1} \to S^{n-1}$ is an analytic diffeomorphism. In addition, we see that

$$h_A(\pi^M(\rho(g)x)) = \pi^N(\sigma(g)h_A(x)) \quad \text{for} \quad g \in G,\ x \in S^{n-1}$$

by the condition $(*)$. q.e.d.

LEMMA 1.3. *Let $M = (m_{ij})$ be a square matrix of degree n with real coefficients. Then M satisfies the outward transversality condition if and only if the quadratic form*

$$x \cdot Mx = \sum_{i,j} m_{ij} x_i x_j$$

is positive definite.

PROOF. The result follows immediately from the equality:

$$\begin{aligned} 2(\exp(tM)x) \cdot (M \exp(tM)x) &= \frac{d}{dt}\|\exp(tM)x\|^2 \\ &= 2\,\|\exp(tM)x\|\frac{d}{dt}\|\exp(tM)x\|\,. \end{aligned}$$

q.e.d.

2. Positive definite quadratic forms.

2.1. Let $\boldsymbol{F}$ denote the field of real numbers $\boldsymbol{R}$, complex numbers $\boldsymbol{C}$, or quaternions $\boldsymbol{Q}$. As usual, let $M_n(\boldsymbol{F})$ denote the set of all matrices of degree n with coefficients in $\boldsymbol{F}$, and let $GL(n, \boldsymbol{F})$ denote the general linear group consisting of regular matrices in $M_n(\boldsymbol{F})$. Let $u = (u_i)$ and $v = (v_i)$ be vectors in $\boldsymbol{F}^n$, the n-dimensional cartesian space over the field $\boldsymbol{F}$. As usual, we define their inner product by $u \cdot v = \sum_i \bar{u}_i v_i$, and the length of u to be the number $\|u\| = \sqrt{u \cdot u}$.

We define $\iota_1\colon M_n(\boldsymbol{C}) \to M_{2n}(\boldsymbol{R})$ and $\iota_2\colon M_n(\boldsymbol{Q}) \to M_{2n}(\boldsymbol{C})$ by

$$\iota_1(A+iB)=\begin{pmatrix}A & -B\\ B & A\end{pmatrix} \quad\text{and}\quad \iota_2(C+jD)=\begin{pmatrix}C & -\bar{D}\\ D & \bar{C}\end{pmatrix}$$

where $A, B \in M_n(\boldsymbol{R})$ and $C, D \in M_n(\boldsymbol{C})$. Then we see that ι_1 and ι_2 are injective ring homomorphisms. We define $\iota\colon M_n(\boldsymbol{F}) \to M_{kn}(\boldsymbol{R})$ by $(k, \iota) = (1, \text{id.})$, $(2, \iota_1)$ and $(4, \iota_1\iota_2)$ for $\boldsymbol{F} = \boldsymbol{R}$, $\boldsymbol{C}$ and $\boldsymbol{Q}$, respectively.

If $u = x + iy \in \boldsymbol{C}^n$ and $v = z + jw \in \boldsymbol{Q}^n$, we assign to u and v the vectors $u' = \begin{pmatrix}x\\ y\end{pmatrix} \in \boldsymbol{R}^{2n}$ and $v' = \begin{pmatrix}z\\ w\end{pmatrix} \in \boldsymbol{C}^{2n}$, respectively. Moreover, we assign to $v \in \boldsymbol{Q}^n$ the vector $v'' = (v')' \in \boldsymbol{R}^{4n}$. We have the following.

$$\text{(2.1)} \qquad \begin{aligned} &\operatorname{Re}(u \cdot Xu) = u' \cdot \iota(X)u' \quad \text{for} \quad X \in M_n(\boldsymbol{C}),\ u \in \boldsymbol{C}^n\,,\\ &\operatorname{Re}(v \cdot Xv) = v'' \cdot \iota(X)v'' \quad \text{for} \quad X \in M_n(\boldsymbol{Q}),\ v \in \boldsymbol{Q}^n\,, \end{aligned}$$

where Re( ) denotes the real part.

LEMMA 2.2. *Let $X \in M_n(\boldsymbol{F})$ and assume that all the eigenvalues of $\iota(X)$ have positive real parts. Then there exists $P \in GL(n, \boldsymbol{F})$ such that $\operatorname{Re}(u \cdot PXP^{-1}u) > 0$ for $u \in \boldsymbol{F}^n - \{0\}$.*

PROOF. Notice that if $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $X \in M_n(\boldsymbol{C})$ then $\lambda_1, \cdots, \lambda_n, \bar{\lambda}_1, \cdots, \bar{\lambda}_n$ are the eigenvalues of $\iota_1(X)$, and hence the result for $\boldsymbol{F} = \boldsymbol{R}$ and $\boldsymbol{C}$ is proved essentially as in the case of Lyapunov functions in [2, §22.3–§22.5]. Here we shall prove the result for $\boldsymbol{F} = \boldsymbol{Q}$ by the same method for completeness. Let $X \in M_n(\boldsymbol{Q})$ and assume that all the eigenvalues of $\iota_2(X)$ have positive real parts. Let λ be an eigenvalue of $\iota_2(X)$. Then there exists a unit vector $v \in \boldsymbol{Q}^n$ such that $\iota_2(X)v' = v'\lambda$; hence we have $Xv = v\lambda$. There exists $P_0 \in Sp(n)$ such that $P_0^{-1}e_1 = v$ (cf. [3, ch. I, §VII]). Then $P_0XP_0^{-1}e_1 = e_1\lambda$; in other words,

$$P_0XP_0^{-1} = \begin{pmatrix}\lambda & \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}.$$

By induction on n, we have an element $P_1 \in Sp(n)$ such that

$$P_1XP_1^{-1} = \begin{pmatrix}\lambda_1 & & x_{ij}\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix},$$

where $\lambda_1, \cdots, \lambda_n$ are complex numbers and x_{ij}'s are quaternions. Then we see that $\lambda_1, \cdots, \lambda_n, \bar{\lambda}_1, \cdots, \bar{\lambda}_n$ are the eigenvalues of $\iota_2(X)$. We define positive real numbers a, b by

$$a = \min_i \operatorname{Re}(\lambda_i)\,, \qquad b = n(n-1)(a + \max_{i<j}|x_{ij}|)a^{-1}\,.$$

Let P_2 be the diagonal matrix with diagonal entries $b, b^2, \cdots, b^n$, and put $P = P_2P_1$. Then we have

$$PXP^{-1} = \begin{pmatrix} \lambda_1 & & a_{ij} \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

where $a_{ij} = b^{i-j}x_{ij}$. We shall show that P is a desired matrix. We have

$$u \cdot PXP^{-1}u = \sum_i \bar{u}_i\lambda_i u_i + \sum_{i<j} \bar{u}_i a_{ij} u_j$$

for $u = (u_i) \in \boldsymbol{Q}^n$, and hence

$$\mathrm{Re}(u \cdot PXP^{-1}u) = \sum_i \mathrm{Re}(\lambda_i)|u_i|^2 + \sum_{i<j} \mathrm{Re}(\bar{u}_i a_{ij} u_j) .$$

Therefore

$$|\mathrm{Re}(u \cdot PXP^{-1}u) - \sum_i \mathrm{Re}(\lambda_i)\,|u_i|^2| \leqq \sum_{i<j} |a_{ij}|\,||u||^2 \leqq \frac{a}{2}||u||^2 ,$$

because $|a_{ij}| \leqq a/n(n-1)$ for $i < j$. Consequently, we obtain

$$\mathrm{Re}(u \cdot PXP^{-1}u) \geqq \sum_i \mathrm{Re}(\lambda_i)\,|u_i|^2 - \frac{a}{2}||u||^2 \geqq \frac{a}{2}||u||^2 > 0$$

for $u \in \boldsymbol{Q}^n - \{0\}$. q.e.d.

2.2. Next we shall show the following.

THEOREM 2.3. *The following three conditions are equivalent for* $X \in M_n(\boldsymbol{R})$.

(1) *All the eigenvalues of X have positive real parts.*

(2) *There exists $P \in GL(n, \boldsymbol{R})$ such that the quadratic form $u \cdot PXP^{-1}u$ is positive definite.*

(3) $\lim_{t\to+\infty} ||\exp(tX)u|| = +\infty$, $\lim_{t\to-\infty} ||\exp(tX)u|| = 0$ *for* $u \in \boldsymbol{R}^n_0$.

PROOF. The condition (1) implies (2) by Lemma 2.2. If $A \in GL(n, \boldsymbol{R})$ and $x \in \boldsymbol{R}^n$, then we have

$$||A^{-1}||^{-1}||x|| \leqq ||Ax|| \leqq ||A||\,||x|| ,$$

where $||A||^2 = \mathrm{trace}\,{}^tAA$. In particular,

$$||P||^{-1}||\exp(tPXP^{-1})Pu|| \leqq ||\exp(tX)u|| \leqq ||P^{-1}||\,||\exp(tPXP^{-1})Pu||$$

for $X \in M_n(\boldsymbol{R})$, $P \in GL(n, \boldsymbol{R})$ and $u \in \boldsymbol{R}^n$. Therefore Lemma 1.1 and Lemma 1.3 assure that the condition (2) implies (3). Finally, we shall show that the condition (3) implies (1). Let $\lambda = a + ib$ be an eigenvalue of X, and let $z = x + iy$ be a unit eigenvector of X in C^n belonging to λ. Then

$$\|\exp(tX)x\|^2 + \|\exp(tX)y\|^2 = e^{2ta}\|z\|^2 = e^{2ta} .$$

The condition (3) for the matrix X implies $a > 0$. q.e.d.

3. Twisted linear actions for compact Lie groups. Let $\alpha = (a_{ij})$ and $\beta = (b_{kl})$ be matrices of degrees p and q, respectively. We denote by $\alpha \oplus \beta$ the matrix $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ of degree $p + q$, and denote by $\alpha \otimes \beta$ the Kronecker product, that is, the matrix (c_{rs}) of degree pq whose coefficients are given by $c_{rs} = a_{ij}b_{kl}$ for $r = i + p(k-1)$, $s = j + p(l-1)$.

Let $\rho: G \to GL(n, \boldsymbol{R})$ be a matricial representation of a Lie group G. We say that ρ is in standard form, if there exist irreducible representations $\rho_i: G \to GL(n_i, \boldsymbol{F}_i)$ $(i = 1, 2, \cdots, r)$ such that

$$\begin{aligned} &\rho = (\rho_1 \otimes I_{k_1}) \oplus \cdots \oplus (\rho_r \otimes I_{k_r}) , \\ &\mathrm{End}_G(\rho) = (I_{n_1} \otimes M_{k_1}(\boldsymbol{F}_1)) \oplus \cdots \oplus (I_{n_r} \otimes M_{k_r}(\boldsymbol{F}_r)) , \end{aligned} \tag{3.1}$$

where $\boldsymbol{F}_i = \boldsymbol{R}$, $\boldsymbol{C}$ or $\boldsymbol{Q}$. It is well known that any matricial representation of a compact Lie group is equivalent to one in standard from (cf. [1, ch. 3], [3, ch. VI]).

LEMMA 3.2. *Let ρ be a matricial representation in standard form of a Lie group G. Let $X \in \mathrm{End}_G(\rho)$ and assume that all the eigenvalues of X have positive real parts. Then there exists $P \in \mathrm{Aut}_G(\rho)$ such that PXP^{-1} satisfies the outward transversality condition.*

PROOF. The result follows immediately from (3.1), (2.1), Lemma 2.2 and Lemma 1.3. q.e.d.

REMARK. If $\rho: G \to GL(n, \boldsymbol{R})$ is an irreducible representation which has no complex structure, then the linear action is the unique twisted linear action of G on S^{n-1} associated to ρ.

THEOREM 3.3. *Let G be a compact Lie group and $\rho: G \to GL(n, \boldsymbol{R})$ a matricial representation. Then any twisted linear action of G on S^{n-1} associated to ρ is equivariantly analytically diffeomorphic to the linear action of G on S^{n-1} associated to ρ.*

PROOF. Let $M \in \mathrm{End}_G(\rho)$ and assume that M satisfies the outward transversality conditition. We shall show that the twisted linear action of G on S^{n-1} determined by the TC-pair (ρ, M) is equivariantly analytically diffeomorphic to the linear action of G on S^{n-1} associated to ρ. Since G is compact, there are $P_1 \in GL(n, \boldsymbol{R})$ and an orthogonal representation

σ in standard form satisfying $\sigma(g)P_1 = P_1\rho(g)$ for any $g \in G$. Then $P_1MP_1^{-1} \in \mathrm{End}_G(\sigma)$ and all the eigenvalues of $P_1MP_1^{-1}$ have positive real parts. Thus there exists $P_2 \in \mathrm{Aut}_G(\sigma)$ such that $P_2P_1MP_1^{-1}P_2^{-1}$ satisfies the outward transversality condition by Lemma 3.2. Let $P = P_2P_1$ and $N = PMP^{-1}$. Define analytic diffeomorphisms h_P, k_P of S^{n-1} onto itself by $h_P(x) = \pi^N(Px)$ and $k_P(x) = \pi^{I_n}(P^{-1}x)$. As in the proof of Lemma 1.2, we see that h_P is an equivariant analytic diffeomorphism from S^{n-1} with the twisted linear action determined by the TC-pair (ρ, M) onto S^{n-1} with the one determined by the TC-pair (σ, N), and k_P is an equivariant analytic diffeomorphism from S^{n-1} with the linear action associated to σ onto S^{n-1} with the one associated to ρ. Since σ is an orthogonal representation, the twisted linear action of G on S^{n-1} determined by the TC-pair (σ, N) coincides with the linear action associated to σ. Therefore the composite k_Ph_P is an equivariant analytic diffeomorphism from S^{n-1} with the twisted linear action determined by the TC-pair (ρ, M) onto S^{n-1} with the linear action associated to ρ. q.e.d.

4. **Typical example.** Here we shall study twisted linear actions of $G = SL(n, \boldsymbol{R})$ on S^{2n-1} associated to $\rho_n \otimes I_2$, where $\rho_n: SL(n, \boldsymbol{R}) \to GL(n, \boldsymbol{R})$ is the natural inclusion. We have $\mathrm{End}_G(\rho_n \otimes I_2) = I_n \otimes M_2(\boldsymbol{R})$. Let $e_1, \cdots, e_n$ be the standard base of $\boldsymbol{R}^n$.

LEMMA 4.1. *Let u, v be vectors in $\boldsymbol{R}^n$. If u, v are linearly independent and $n \geqq 3$, then there exists $P \in SL(n, \boldsymbol{R})$ such that $Pu = (1/\sqrt{2})e_1$ and $Pv = (1/\sqrt{2})e_2$.*

PROOF. Since u, v are linearly independent, there exists $P_1 \in SO(n)$ such that $P_1u = pe_1$ and $P_1v = qe_1 + re_2$ for some real numbers p, q, r satisfying $pr \neq 0$. Next, since $n \geqq 3$, there exists $P_2 \in SL(n, \boldsymbol{R})$ such that $P_2e_1 = (1/p\sqrt{2})e_1$ and $P_2e_2 = (-q/pr\sqrt{2})e_1 + (1/r\sqrt{2})e_2$. We are done by letting $P = P_2P_1$. q.e.d.

By this lemma, we see that the orbit through $(1/\sqrt{2})(e_1 \oplus e_2)$ is open and dense in S^{2n-1} for any twisted linear action of $SL(n, \boldsymbol{R})$ associated to $\rho_n \otimes I_2$, because the orbit consists of all $u \oplus v \in S^{2n-1}$ such that u, v are linearly independent.

Let $M \in M_2(\boldsymbol{R})$ and assume that M satisfies the outward transversality condition. Then $(\rho_n \otimes I_2, I_n \otimes M)$ is a TC-pair. In fact, $I_n \otimes M$ satisfies the outward transversality condition if and only if M satisfies the condition. Denote by $I^n(M)$ the isotropy group at $(1/\sqrt{2})(e_1 \oplus e_2)$ with respect to the twisted linear action of $SL(n, \boldsymbol{R})$ on S^{2n-1} determined by the TC-pair $(\rho_n \otimes I_2, I_n \otimes M)$. We see easily $X \in I^n(M)$ if and only if

$$X = \left(\begin{array}{c|c} {}^t\exp(\theta M) & * \\ \hline 0 & * \end{array}\right) \tag{4.2}$$

for some $\theta \in \boldsymbol{R}$.

LEMMA 4.3. *With respect to the natural action of $I^n(M)$ on $\boldsymbol{R}^n$ as a subgroup of $SL(n, \boldsymbol{R})$, the subspace spanned by $\{e_1, e_2\}$ is the unique invariant 2-dimensional linear subspace.*

PROOF. Let V be an invariant linear subspace of $\boldsymbol{R}^n$, and assume that V contains a vector which is not a linear combination of e_1, e_2. Then we see that V contains e_1 and e_2, because any matrix of the form

$$\begin{pmatrix} I_2 & * \\ 0 & I_{n-2} \end{pmatrix}$$

is contained in $I^n(M)$. q.e.d.

Let $M, N \in M_2(\boldsymbol{R})$. We say that M is similar to N up to positive scalar multiplication, if there exist $P \in GL(2, \boldsymbol{R})$ and a positive real number c such that $cN = PMP^{-1}$.

LEMMA 4.4. *Let $M, N \in M_2(\boldsymbol{R})$ and assume that M, N satisfy the outward transversality condition. If $n \geqq 3$, then the following two conditions are equivalent.*

(1) *M is similar to N up to positive scalar multiplication.*

(2) *$I^n(M)$ and $I^n(N)$ are conjugate in $SL(n, \boldsymbol{R})$.*

PROOF. By Lemma 1.2, we see that the condition (1) implies (2). Now we shall show that the condition (2) implies (1). Assume that there exists $A \in SL(n, \boldsymbol{R})$ such that $I^n(N) = AI^n(M)A^{-1}$. Then, by Lemma 4.3, we see that the subspace spanned by $\{e_1, e_2\}$ is A-invariant, and hence $A = \begin{pmatrix} B & * \\ 0 & * \end{pmatrix}$ for some $B \in GL(2, \boldsymbol{R})$. By (4.2), we obtain $cN = {}^tB^{-1}M{}^tB$ for a real number c. We see $c > 0$, because M, N satisfy the outward transversality condition. q.e.d.

By this lemma, if $(\rho_n \otimes I_2, I_n \otimes M)$ and $(\rho_n \otimes I_2, I_n \otimes N)$ are not equivalent as TC-pairs, then there is no equivariant homeomorphism from S^{2n-1} with the twisted linear action of $SL(n, \boldsymbol{R})$ determined by the TC-pair $(\rho_n \otimes I_2, I_n \otimes M)$ onto S^{2n-1} with the one determined by the TC-pair $(\rho_n \otimes I_2, I_n \otimes N)$.

We see easily the following. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then M satisfies the outward transversality condition if and only if $a > 0$ and $4ad - (b + c)^2 > 0$, by Lemma 1.3. The following matrices satisfy the outward transversality condition.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \quad (0 < x \leqq 1), \quad \begin{pmatrix} 1 & y \\ -y & 1 \end{pmatrix} \quad (y > 0).$$

Moreover, no two of them are similar up to positive scalar multiplication, and any matrix of degree 2 satisfying the outward transversality condition is similar to one of the above matrices up to positive scalar multiplication.

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