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NEGATIVITY OF THE CURVATURE OPERATOR OF A BOUNDED DOMAIN

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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Introduction. Let D be a bounded domain in C^n equipped with the Bergman metric g. Let $R_{a\bar{b}c\bar{d}}$ be the components of the Riemannian curvature tensor of g. The curvature operator Q of g at a point $p \in D$ is, by definition, the endomorphism

$$\xi_{ab} \mapsto R_a{}^{cd}{}_b(p)\xi_{cd}$$

of the 2-symmetric tensor product of the holomorphic tangent space at p. The eigenvalues of Q are holomorphically invariant and are all real because Q is self-adjoint with respect to the Hermitian inner product induced from g. In particular, if D is homogeneous, then the eigenvalues of Q do not depend on the point of D under consideration. The following is well-known ([4], [5]): If D is irreducible symmetric and the operator Q is negative definite, then D is holomorphically equivalent to a ball. Concerning this, we consider the following two problems:

(A) Let D be a, not necessarily irreducible, homogeneous domain in C^n. Suppose Q is negative definite. Then is D holomorphically equivalent to a ball?

(B) Does there exist a bounded domain which is not holomorphically equivalent to a ball and for which Q is negative definite?

Our aim of the present note is to show that problem (A) has an affirmative answer by means of the theory of normal j-algebras by Pyatetskii-Shapiro [8] (Theorem 1), and to show that a Thullen domain, which is holomorphically inequivalent to a ball, has negative definite curvature operator (Proposition 4). Problem (B) is also affirmative in view of the deformation theory by Greene and Krantz [6], [7].

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1. Homogeneous bounded domains. In this section we shall show the following.

THEOREM 1. *Let D be a homogeneous bounded domain. Assume that the curvature operator of the Bergman metric on D is negative semi-definite. Then D is holomorphically equivalent to a product of balls. In particular, if the operator is negative definite, then D is holomorphically equivalent to a ball.*

Let D be a homogeneous bounded domain and p be a point of D. Then the real tangent space T_p^R at p possesses the structure $(\mathfrak{g}, j, \omega)$ of a normal j-algebra such that $\mathfrak{g}$ is a Lie algebra, which coincides with T_p^R as a real vector space, j is the complex structure of T_p^R, and ω is a form on $\mathfrak{g}$ with the property $g(x, y) = \omega[jx, y]$ for $x, y \in T_p^R = \mathfrak{g}$, where g is the Bergman metric on D (cf. [8], [2]).

Let PI be the set of all primitive idempotents in $\mathfrak{g}$, i.e.,

$$\mathrm{PI} = \{r \in \mathfrak{g} - \{0\}; \{x \in \mathfrak{g}; [jr, x] = x\} = \boldsymbol{R}r\} .$$

It is well-known ([8], [9]) that PI is nonempty and linearly independent. The cardinality R of PI is called the rank of D. It is also known ([8]) that there exists a numbering $r_1, \cdots, r_R$ of the set PI such that if

$$\mathfrak{n}_{ab} = \{x \in \mathfrak{g}; [jr_c, x] = ((\delta_{ca} + \delta_{cb})/2)x \quad (c = 1, \cdots, R)\}$$

for $a, b \in \{1, \cdots, R\}$ with $a \leqq b$, and

$$\mathfrak{n}_a = \{x \in \mathfrak{g}; [jr_c, x] = (\delta_{ca}/2)x \quad (c = 1, \cdots, R)\}$$

for $a \in \{1, \cdots, R\}$, then $j\mathfrak{n}_{ab} = \{x \in \mathfrak{g}; [jr_c, x] = ((\delta_{ac} - \delta_{bc})/2)x \quad (c = 1, \cdots, R)\}$ for $a < b$ and the following orthogonal decomposition, with respect to $g(\, , \,)$, holds:

$$\mathfrak{g} = \sum_{a \leqq b} \mathfrak{n}_{ab} + \sum_{a \leqq b} j\mathfrak{n}_{ab} + \sum_{a} \mathfrak{n}_a .$$

In particular, $\mathfrak{n}_{aa} = \boldsymbol{R}r_a$ for all $a \in \{1, \cdots, R\}$.

To prove Theorem 1, suppose that D is not holomorphically equivalent to any product of balls. There then exists a pair (a, b) with $a < b$ such that the dimension N of the subspace $\mathfrak{n}_{ab}$ is positive. Let $\{m_1, \cdots, m_N\}$ be an orthogonal basis of $\mathfrak{n}_{ab}$. If we denote by χ the mapping from T_p^R to the holomorphic tangent space T_p at p given by $x \mapsto (x - \sqrt{-1}jx)/2$, then $f = (\chi(r_a)^2, \chi(m_1)^2, \cdots, \chi(m_N)^2)$ is an orthogonal system in the 2-symmetric tensor product of the space T_p. Let Q be the curvature operator of g at p. In this situation we know the following.

LEMMA 2 ([2; Proposition 6.5]). *The matrix representing Q modulo $\mathrm{span}_C f$ with respect to the basis f has the form*

$$(1.1) \qquad -\begin{bmatrix} 0 & 2^{-1/2}\omega_a^{-1}e_N \\ 2^{-1/2}\omega_a^{-1}\,{}^te_N & \omega_a^{-1}I_N - \omega_b^{-1}E_N \end{bmatrix},$$

where $\omega_c = \omega(r_c) > 1$ $(c = a, b)$, $e_N = (1, \cdots, 1)$ *(N-times)*, $E_N = (\xi_{st})$ *with* $\xi_{st} = 1$ $(s, t \in \{1, \cdots, N\})$, I_N *is the identity matrix of order N, and* ${}^te_N$ *means the transpose of* e_N.

It follows from Lemma 2 that the matrix representing Q possesses a principal minor which is not negative semi-definite, because the (1, 2)-principal minor of the negative of the matrix (1.1) has the determinant

$$\det\begin{bmatrix} 0 & 2^{-1/2}\omega_a^{-1} \\ 2^{-1/2}\omega_a^{-1} & \omega_a^{-1} - \omega_b^{-1} \end{bmatrix} = -(2\omega_a^2)^{-1} < 0 .$$

Therefore, Q itself is not negative semi-definite. Thus, the main assertion of Theorem 1 is proved.

To prove the second assertion of Theorem 1, assume Q is negative definite. Then D is holomorphically equivalent to a product of balls. It is well-known (cf., e.g., [2; Proposition 1.4]) that if D is not irreducible, i.e., if D is holomorphically equivalent to a product of two lower dimensional domains, then zero is an eigenvalue of Q. From this, it follows that D is holomorphically equivalent to a ball. The proof is completed.

REMARK 3. Let G be a maximal triangular analytic Lie subgroup in the group of all biholomorphic transformations of D. Theorem 1 holds if the Bergman metric is replaced by a G-invariant Kähler metric, since Lemma 2 holds true also for this metric.

2. Thullen domains. Let D_p be a Thullen domain in $\boldsymbol{C}^2$ with parameter $p > 0$:

$$D_p = \{z \in \boldsymbol{C}^2;\ |z_1| < 1,\ |z_2|^2 < (1 - |z_1|^2)^p\} .$$

It is well-known (cf., e.g., [1], [3]) that for every $z \in D_p$ one can find a biholomorphic transformation Φ of D_p such that $\Phi(z) \in \{(0, \omega);\ 0 \leqq \omega < 1\}$. As in [1], [3] we make use of the auxiliary variables

$$r = (1 - p)/(1 + p) \qquad (p > 0) ,$$
$$t = (1 - \omega^2)/(1 - r\omega^2) \quad (0 \leqq \omega < 1)$$

and the functions

$$\alpha = 3 + rt^2 ,$$
$$\beta = 3 - rt^2 ,$$
$$A = 6 + 4rt^2 + (1 + r)rt^3 ,$$

$$B = 2(9 + 3rt^2 - 3(1 + r)rt^3 + 2r^2t^4)/\alpha ,$$
$$C = 3(6 - 6rt^2 + (1 + r)rt^3)/\beta .$$

Then the Bergman metric tensor $g_{a\bar{b}}$ at $(0, \omega) \in D_p$ with $0 \leqq \omega < 1$ is given by

$$(2.1) \qquad \begin{aligned} g_{1\bar{1}} &= \alpha/(1 + r)t , \\ g_{2\bar{2}} &= \beta(1 - rt)^2/(1 - r)^2t^2 , \\ g_{1\bar{2}} &= 0 , \end{aligned}$$

and the curvature tensor $R_{a\bar{b}c\bar{d}}$ is given by

$$(2.2) \qquad \begin{aligned} R_{1\bar{1}1\bar{1}} &= 4A/(1 + r)^2t^2 - 2(g_{1\bar{1}})^2 , \\ R_{1\bar{1}2\bar{2}} &= 2(1 - rt)^2B/(1 + r)(1 - r)^2t^3 - g_{1\bar{1}}g_{2\bar{2}} , \\ R_{2\bar{2}2\bar{2}} &= 4(1 - rt)^4C/(1 - r)^4t^4 - 2(g_{2\bar{2}})^2 , \\ R_{1\bar{1}1\bar{2}} &= R_{1\bar{2}1\bar{2}} = R_{1\bar{2}2\bar{2}} = 0 . \end{aligned}$$

Let $f = (\partial_1\cdot\partial_1/\sqrt{2},\ \partial_2\cdot\partial_2/\sqrt{2},\ \partial_1\cdot\partial_2)$ with $\partial_i = (\partial/\partial z_i)_{(0,\omega)}$. It follows from (2.1) and (2.2) that the matrix representing the curvature operator at $(0, \omega)$ with respect to the basis f is written as

$$\begin{bmatrix} R^{11}_{11} & R^{11}_{22} & \sqrt{2}R^{11}_{12} \\ R^{22}_{11} & R^{22}_{22} & \sqrt{2}R^{22}_{12} \\ \sqrt{2}R^{12}_{11} & \sqrt{2}R^{12}_{22} & 2R^{12}_{12} \end{bmatrix} = -4\begin{bmatrix} A/\alpha^2 - 1/2 & 0 & 0 \\ 0 & C/\beta^2 - 1/2 & 0 \\ 0 & 0 & B/\alpha\beta - 1/2 \end{bmatrix},$$

where $R^{cd}_{ab} = R_a{}^{cd}{}_b$. Using Fourier's theorem concerning the zeros of a polynomial (cf. [3; Appendices]), we see that for any $r \in (-1, 1)$ the functions A/α^2, C/β^2, and $B/\alpha\beta$ are all greater than 1/2 for every $t \in (0, 1]$. Thus we have proved the following.

PROPOSITION 4. *Let D_p be a Thullen domain with $p \neq 1$. Then the curvature operator of the Bergman metric on D_p is negative definite at every point of D_p, and D_p is not holomorphically equivalent to a ball.*

The latter assertion of Proposition 4 is well-known (cf., e.g., [1; Proposition 2.8]).

POSTSCRIPT. We would like to take this opportunity to point out necessary corrections to our previous paper [2].

Page 201, $\downarrow$14: "$g(\Phi_* \circ \rho(x),\ \Phi_* \circ \rho(p))$" should read "$g(\Phi_* \circ \rho(x),\ \Phi_* \circ \rho(y))$".

Page 205, $\downarrow$9: "$X(D)$" should read "$\mathscr{X}(D)$".

Page 206, $\uparrow$15: "$(jr_a/2\omega_a + jr_b/\omega_b)\langle x, y\rangle_\omega$" should read "$(jr_a/2\omega_a + jr_b/2\omega_b)\langle x, y\rangle_\omega$".

Page 209, $\uparrow$16: "$\omega\nabla_{v_4}u_1$" should read "${}^\omega\nabla_{v_4}u_1$".

Page 209, ↑16: "${}^{\omega}\nabla_{[u_3,u_4]}u_2)$" should read "${}^{\omega}\nabla_{[u_3,u_4]}u_2$".
Page 210, ↓2: "$(\delta_{bs}+\delta_{bt})y'/^{2}$" should read "$(\delta_{bs}+\delta_{bt})y'/2$".
Page 211, ↑4: "Lemma 5.4" should read "Lemma 5.3".
Page 212, ↓15: "$(\mathrm{ii})_2$ $\alpha_1=\alpha_2\neq\alpha_3=\alpha_4$" should read "$(\mathrm{ii})_1$ $\alpha_1=\alpha_2\neq\alpha_3=\alpha_4$".
Page 213, ↓13: "$\langle\ ,^{-}\rangle$" should read "the Hermitian inner product inherited from $\langle\ ,^{-}\rangle$ (see (1.4))".
Page 215, ↑3: "$\langle E(u, v), y\rangle$" should read "$\langle F(u, v), y\rangle$".
Page 216, ↓16: "$-1/k_R$" should read "$-1/\kappa_R$".
Page 217, ↓12: "$u, u'\in\mathfrak{u}_{a*}$" should read "$u, u'\in\mathfrak{n}_{a*}$".
Page 219, ↑10: "min HSC $\geqq$ " should read "min HSC $\leqq$ ".
Page 221, ↓9: "$B_B^2(X_1)$" should read "$B_D^2(X_1)$".

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