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INFINITESIMAL DEFORMATIONS OF GENERALIZED CUSP SINGULARITIES

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0. Introduction. In [Hz], Hirzebruch studied Hilbert modular surfaces which are the compactifications of $\boldsymbol{H}^2/SL_2(\mathcal{O})$ determined by addition of a finite number of points called "cusps", where $\boldsymbol{H} := \{z \in \boldsymbol{C}; \operatorname{Im} z > 0\}$ is the upper half plane and $\mathcal{O}$ is the ring of integers in a real quadratic field. He also constructed the minimal models of these surfaces by using the method of toroidal embeddings [TE]. This method is local, that is, this is performed only near each cusp. Tsuchihashi constructed in [T1] normal isolated singularities, sometimes called "Tsuchihashi cusps", analogous to Hilbert modular cusp singularities by using toroidal embeddings. A Tsuchihashi cusp singularity (V, p) is of the form $V \setminus \{p\} \cong \mathscr{D}/G$, where $\mathscr{D}$ is a tube domain and G is a subgroup of $\operatorname{Aut}(\mathscr{D})$.

Recall that a tube domain is called a Siegel domain of the *first* kind. We construct in Section 1 a normal isolated singularity (V, p) such that $V \setminus \{p\}$ is isomorphic to a quotient of a Siegel domain of the *second* kind. We would like to call this singularity also a "cusp". It is natural to extend the class of cusp singularities in this way, because the boundary components of the Satake compactification of a quotient of a bounded symmetric domain are also called cusps in a generalized sense.

EXAMPLE. Let F be a totally real algebraic number field of degree ν, F' a totally imaginary quadratic extention of F, B a central division algebra of degree d over F' with an involution of the second kind and $h \in M_\mu(B)$ a Hermitian matrix with Witt index one, i.e., h is conjugate to

$$\left[\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & * \end{array}\right].$$

Set $G_{\boldsymbol{Q}} := R_{F/\boldsymbol{Q}}(SU(h, B/F'/F))$ with Weil's restriction functor $R_{F/\boldsymbol{Q}}$. Then we get

$$G_{\boldsymbol{R}} = \prod_{i=1}^{\nu} SU(p_i, q_i)\,, \quad p_i + q_i = \mu\,, \quad p_i \geqq q_i \geqq d\,.$$

Let K be a maximal compact subgroup of $G_{\boldsymbol{R}}$. When $q_i = d$, we get the

Satake compactification of $K\backslash G_R/G_Z$ by adding a finite number of points, which are called "cusps".

When $\nu = 1$, the homogeneous space $K\backslash G_R$ is isomorphic to the bounded symmetric domain $I_{p,q} := \{Z \in M_{p,q}(C); 1_q - {}^t\bar{Z}Z > 0\}$ $(p \geqq q \geqq 1)$. The domain $I_{p,q}$ can be represented as a Siegel domain of the second kind:

$$\mathscr{D} := \{(Z, u) \in \mathscr{H}_q(C) \otimes_R C \times M_{p-q,q}(C); \operatorname{Im} Z - {}^t\bar{u}u \in \mathscr{P}_q(C)\},$$

where $\mathscr{H}_q(C) := \{Z \in M_q(C); {}^t\bar{Z} = Z\}$ and $\mathscr{P}_q(C) := \{Z \in \mathscr{H}_q(C);\ Z > 0\}$. Here $Z > 0$ means that Z is positive definite.

REMARKS. 1. When $q = 1$, the domain $I_{p,1}$ is the p-ball $B^p := \{(z_i) \in C^p; \sum_{i=1}^p |z_i|^2 < 1\}$ and $\mathscr{D} = \{(z, u_1, \cdots, u_{p-1}) \in C^p; \operatorname{Im} z - \sum_{i=1}^{p-1} |u_i|^2 > 0\}$.

2. When $p = q$, the domain $\mathscr{D} = \mathscr{H}_q(C) + \sqrt{-1}\mathscr{P}_q(C)$ is of tube type.

From this model we derive data necessary for our construction in Section 1.

We show in Section 2 that there exist isomorphisms $T_V^1 \xrightarrow{\sim} H^1(V\setminus\{p\}, \Theta_V)$ and $H^1(U, \Theta_U(-\log X)) \xrightarrow{\sim} H^1(V\setminus\{p\}, \Theta_V)$ for some resolution (U, X) of a "cusp" singularity (V, p) of dimension greater than two. When (V, p) is a Tsuchihashi cusp singularity, we showed in [O] the former isomorphism by using the method analogous to that in [Ft] and [FK]. In the case of a Hilbert modular cusp singularity (V, p) of dimension two, $\dim_C T_V^1$ was calculated by Behnke [B1], [B2] and Nakamura [NK]. The latter isomorphism shows that our generalized cusp singularities are equisingular (cf. [W]).

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1. Construction of cusp singularities.

1.1. Siegel domains of the second kind. For integers $r \geqq 1$, $m \geqq 0$, let us denote $n := r + m$. Fix a free Z-module N of rank r. Let $C \subset N_R := N \otimes_Z R$ be an open convex cone with $\bar{C} \cap (-\bar{C}) = \{0\}$ and let $H: C^m \times C^m \to N_C := N \otimes_Z C$ be a Hermitian form satisfying the following conditions:

(i) $H(\lambda_1 u_1 + \lambda_2 u_2, v) = \lambda_1 H(u_1, v) + \lambda_2 H(u_2, v)$ for $\lambda_i \in C$, u_i, $v \in C^m$ $(i = 1, 2)$.

(ii) $H(u, v) = H(v, u)^-$, where $^-$ denotes the complex conjugation.

(iii) $H(u, u) \in \bar{C}$, where $\bar{C}$ is the closure of C in N_R.

(iv) $H(u, u) = 0$ implies $u = 0$.

Then we set

$$\mathscr{D} = \mathscr{D}(H, C) := \{(z, u) \in N_C \times \boldsymbol{C}^m;\ \operatorname{Im} z - H(u, u) \in C\}$$

and call it a *Siegel domain of the second kind* associated with H and C. Note that the group $N(\mathscr{D}) := \{(a, c) \in N_R \times \boldsymbol{C}^m\}$ acts on $\mathscr{D}$ by

$$(a, c)\cdot(z, u) = (z + a + 2\sqrt{-1}H(u, c) + \sqrt{-1}H(c, c),\ c + u)\ .$$

1.2. Lattice data. Let $L \subset \boldsymbol{C}^m$ be a free $\boldsymbol{Z}$-module of rank $2m$ with the compact quotient $\boldsymbol{C}^m/L$ and $\Gamma \subset \operatorname{Aut}(N)$ a subgroup preserving C and satisfying the following conditions:

(a) The induced action of Γ on $D := C/\boldsymbol{R}_{>0}$ is properly discontinuous and fixed point free.

(b) The quotient D/Γ is compact.

(c) There exists a homomorphism of groups sending $g \in \Gamma$ to $\tilde{g} \in GL(m, \boldsymbol{C})$ so that $gH(u, u) = H(\tilde{g}u, \tilde{g}u)$ and $\tilde{g}L = L$ for all $g \in \Gamma$, $u \in \boldsymbol{C}^m$.

(d) $H(l, l') - H(l', l) \in \sqrt{-1}N$ for all $l, l' \in L$.

1.3. Construction. In the following, we use the notation as in [MO]. Let $T_N := N \otimes_{\boldsymbol{Z}} \boldsymbol{C}^*$ be an algebraic torus of dimension r. Regarding N, L as subgroups of $N(\mathscr{D})$, construct the following diagram:

$$\begin{array}{ccc} \mathscr{D}/N & \subset & T_N \times \boldsymbol{C}^m \\ \downarrow & & \downarrow \\ \mathscr{D}/N\cdot L & \subset & (T_N \times \boldsymbol{C}^m)/L\ . \end{array}$$

Here $(T_N \times \boldsymbol{C}^m)/L$ is a T_N-bundle over the Abelian variety $A := \boldsymbol{C}^m/L$ with $\tilde{p}\colon (T_N \times \boldsymbol{C})/L \to A$ as the projection, and its transition function is

$$\exp(2\pi(2H(u, l) + H(l, l))) \in T_N \quad \text{for} \quad u \in \boldsymbol{C}^m\ , \quad l \in L\ ,$$

where $\exp\colon N_C \to T_N = N_C/N$. Now take a Γ-admissible rational partial polyhedral decomposition (r.p.p. decomposition, for short) $\tilde{\Delta}$ of $C \cup \{0\}$ with $\tilde{\Delta}$ modulo Γ finite. Then construct a diagram

$$\begin{array}{ccc} T_N \times \boldsymbol{C}^m & \subset & T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m \\ \downarrow & & \downarrow \\ (T_N \times \boldsymbol{C}^m)/L & \subset & (T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m)/L\ . \end{array}$$

We also use the same notation $\tilde{p}\colon (T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m)/L \to A$.

In order to take the quotient with respect to the action of Γ, we need to shrink $(T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m)/L$. A real analytic mapping sending $(t, u) \in (T_N \times \boldsymbol{C}^m)$ to $\operatorname{ord}(t) - H(u, u) \in N_R$ extends to a mapping $\Phi\colon T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m \to \operatorname{Mc}(N, \tilde{\Delta})$, which is L-invariant and hence induces a mapping from $(T_N\operatorname{emb}(\tilde{\Delta}) \times \boldsymbol{C}^m)/L$ to $\operatorname{Mc}(N, \tilde{\Delta})$. We also denote it by the same letter Φ. We see that Φ is Γ-equivariant. Set

$$\tilde{U} := \Phi^{-1}(\text{the interior of the closure of } C \text{ in } \mathrm{Mc}(N, \tilde{\Delta}))$$

and

$$\tilde{Y} := \tilde{U} \setminus \Phi^{-1}(C) ,$$

Γ acts on $\tilde{U}$ and $\tilde{Y}$ properly discontinuously and without fixed points. Therefore we can take the quotients:

$$U := \tilde{U}/\Gamma \quad \text{and} \quad X := \tilde{Y}/\Gamma .$$

In order to contract X to a normal isolated singular point, we use the kernel function of $\mathscr{D}$ (cf. [Sal] and [Ro]). For $(z, u) \in \mathscr{D}$, set

$$\Psi(z, u) := \int_{C^*} \exp(-\langle \mathrm{Im}\, z - H(u, u), t\rangle) \frac{\det M(t)}{\phi_{C^*}(t)} dt ,$$

where $C^* := \{y \in N_R^*; \langle x, y\rangle > 0 \text{ for all } x \in \bar{C} \setminus \{0\}\}$ is the dual cone of C, the function ϕ_{C^*} is the characteristic function of C^* defined by Vinberg [V] and $M(t) \in M_m(C)$ is defined for a fixed inner product (,) in $\boldsymbol{C}^m$ by

$$\langle H(u, v), t\rangle = (M(t)u, v) \quad \text{for all} \quad u, v \in \boldsymbol{C}^m , \quad t \in N_R^* .$$

$M(t)$ is Hermitian symmetric. Moreover, it is positive definite for $t \in C^*$. The function Ψ is N- and L-invariant, and has positive values on $\mathscr{D}$, and its Hessian is positive definite. For $g \in \Gamma$, we have

$$\Psi(gz, \tilde{g}u) = |\det g|^{-2} |\det \tilde{g}|^{-2} \Psi(z, u) .$$

Therefore Ψ induces a function on $U \setminus X$, which we also denote by the same letter Ψ. Set $\Psi \equiv 0$ on X. Then the function Ψ is plurisubharmonic on U and strictly plurisubharmonic on $U \setminus X$. Thus we can contract X to a point p (see [GR]):

$$\pi: (U, X) \to (V, p) .$$

2. **Results.** In this paper we consider a singularity (V, p) constructed in Section 1 which satisfies an additional condition $(C, \Gamma) \in \mathscr{S}_0$ in the sense of Tsuchihashi [T1], that is, there exists a duality between Γ-admissible decompositions $\tilde{\square}$ and $\tilde{\square}^*$ induced by the convex hulls of $C \cap N$ and $C^* \cap N^*$, respectively.

THEOREM 2.1. *For the normal isolated singularity* (V, p) *constructed in Section* 1, *we have isomorphisms*

$$R^i\pi_*\mathscr{O}_U \xrightarrow{\sim} H^i(X, \mathscr{O}_X) \qquad \textit{and} \qquad R^i\pi_*\mathscr{O}_U(-X) = 0 \quad \textit{for} \quad i \geqq 1 .$$

REMARK. Theorem 2.1 implies that (V, p) is an isolated Du Bois singularity (cf. [St]).

2.2. Infinitesimal deformations of (V, p). By a deformation of (V, p)

we mean a pair of a flat morphism of complex analytic spaces $f: (\mathscr{V}, v_0) \to (T, t_0)$ and an isomorphism $(V, p) \xrightarrow{\sim} (f^{-1}(t_0), v_0)$. A first order infinitesimal deformation of (V, p) is a deformation $f: (\mathscr{V}, v_0) \to (T, 0)$ of (V, p) with $T = \operatorname{Specan} \boldsymbol{C}[\varepsilon]/(\varepsilon^2)$, We are interested in $T_V^1 := \operatorname{Ext}^1_{\mathscr{O}_V}(\Omega_V^1, \mathscr{O}_V)$, which parametrizes the set of first order infinitesimal deformations of (V, p). For this purpose the following theorem due to Schlessinger [Sc] is useful:

COMPARISON THEOREM (Schlessinger). *Let $(V, p) \to (\boldsymbol{C}^d, 0)$ be a closed embedding. Then we have an exact sequence*

$$0 \to T_V^1 \to H^1(V \setminus \{p\}, \Theta_V) \to H^1(V \setminus \{p\}, (\Theta_{\boldsymbol{C}^d})|_V) ,$$

where Θ_V is the holomorphic tangent sheaf on V and $(\Theta_{\boldsymbol{C}^d})|_V$ is the restriction to V of the holomorphic tangent sheaf on $\boldsymbol{C}^d$.

We choose a nonsingular r.p.p. decomposition $\tilde{\Delta}$ of $C \cup \{0\}$. Then we get a desingularization $\pi: (U, X) \to (V, p)$. Let $X = \bigcup_i X_i$ be the decomposition of X into irreducible components and $N_{X_i/U}$ the normal sheaf of X_i in U. Then we define the logarithmic tangent sheaf of (U, X) by

$$\Theta_U(-\log X) := \operatorname{Ker}(\Theta_U \to \bigoplus_i N_{X_i/U}) .$$

THEOREM 2.2. *When $n \geqq 3$, we have isomorphisms*

$$H^1(U, \Theta_U(-\log X)) \xrightarrow{\sim} T_V^1 \xrightarrow{\sim} H^1(V \setminus \{p\}, \Theta_V) .$$

THEOREM 2.3. *When $m = 0$, that is, (V, p) is a Tsuchihashi cusp singularity, we have*

$$H^i(U, \Theta_U(-\log X)) \cong H^i(\Gamma, N_{\boldsymbol{C}}) \quad \text{for} \quad i \geqq 1 ,$$

where the right hand side is the i-th group cohomology of the natural action of $\Gamma \subset \operatorname{Aut}(N)$ on $N_{\boldsymbol{C}}$.

REMARK. From the exact sequence

$$0 \to H^1(\Theta_U(-\log X)) \to H^1(\Theta_U) \to \bigoplus_i H^1(N_{X_i/U}) ,$$

we see that $H^1(\Theta_U(-\log X))$ parametrizes the set of first order infinitesimal deformations of U for which none of X_i vanish (cf. [W]). If there exists a versal family of such deformations, Theorem 2.2 implies that (V, p) is equisingular.

3. Proof of Theorems.

3.1. First we prove the following two propositions. Let F be a finite dimensional complex vector space with a Γ-action. Set $\mathscr{F} := q_*^{\Gamma}(F \otimes_{\boldsymbol{C}} \mathscr{O}_{\tilde{U}})$, where $q: \tilde{U} \to U = \tilde{U}/\Gamma$ is the natural projection.

PROPOSITION 3.1. *$H^i(X, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_X(-kX)) = 0$ for $i > 0$ and $k > 0$.*

PROPOSITION 3.2. *The local cohomology groups $H^i_X(U, \mathscr{F})$ vanish for $i < n$.*

3.2. PROOF OF PROPOSITION 3.1. Let $\tilde{\square}$ be the r.p.p. decomposition of $C \cup \{0\}$ induced by a natural Γ-invariant polyhedral decomposition of the boundary of the convex hull of $C \cap N$, that is, every member of $\tilde{\square}$ is written in the form

$$\boldsymbol{R}_{\geqq 0}\alpha = \{rx \in N_{\boldsymbol{R}};\ x \in \alpha,\ r \geqq 0\}$$

with a polyhedron α appearing in the boundary of the convex hull of $C \cap N$ (see [TE] and [T1]). Then we can get a nonsingular r.p.p. decomposition $\tilde{\Delta}$ of $C \cup \{0\}$ by subdividing $\tilde{\square}$. Let $(\tilde{U}', \tilde{Y}')$ be those constructed as in Section 1 corresponding to $\tilde{\square}$ and $U' := \tilde{U}'/\Gamma$, $X' := \tilde{Y}'/\Gamma$. Then we have the morphism $\tau\colon (U, X) \to (U', X')$ induced by the subdivision $\tilde{\Delta}$ of $\tilde{\square}$, and have

$$R^i\tau_*\mathscr{O}_U = \begin{cases} \mathscr{O}_{U'} & \text{if } i = 0\,, \\ 0 & \text{if } i \geqq 1\,, \end{cases}$$

$$R^i\tau_*\mathscr{O}_U(-X) = \begin{cases} \mathscr{O}_{U'}(-X') & \text{if } i = 0\,, \\ 0 & \text{if } i \geqq 1\,. \end{cases}$$

Let $q'\colon \tilde{U}' \to U' = \tilde{U}'/\Gamma$ and $\mathscr{F}' := q'^{\Gamma}_*(F \otimes_C \mathscr{O}_{\tilde{U}'})$. Then $\tau_*\mathscr{F} = \mathscr{F}'$ and $H^i(X, \mathscr{F} \otimes \mathscr{O}_U(-kX)) = H^i(X', \mathscr{F}' \otimes \mathscr{O}_{U'}(-kX'))$ for $i \geqq 0$. Hence we may assume that $\tilde{\Delta}$ is the r.p.p. decomposition of $C \cup \{0\}$ induced by the convex hull of $C \cap N$ and that $\pi\colon (U, X) \to (V, p)$ is a partial resolution of singularities corresponding to $\tilde{\Delta}$.

First assume that the dual graph of X is orientable and fine in the sense of Tsuchihashi [T1], that is, $\{\gamma \in \Gamma;\ \gamma\alpha \cap \beta \neq \varnothing\} = \{1\}$ for $\alpha, \beta \in \tilde{\Delta}$ with $\alpha \cap \beta \neq \varnothing$. Let $\tilde{\Delta}(j) := \{\sigma \in \tilde{\Delta};\ \dim \sigma = j\}$ and $\Delta(j) := \tilde{\Delta}(j)/\Gamma$. For each cone $\alpha \in \tilde{\Delta}(j)$ let $\bar{X}_\alpha$ be the toric subvariety $\mathrm{orb}(\alpha)^-$ in $T_N\,\mathrm{emb}(\tilde{\Delta})$ corresponding to α and $X_\alpha := q(\Phi^{-1}(\mathrm{ord}(\bar{X}_\alpha)))$. Then we have an exact sequence

$$0 \to \mathscr{O}_X \to \bigoplus_{\alpha \in \Delta(1)} \mathscr{O}_{X_\alpha} \to \bigoplus_{\beta \in \Delta(2)} \mathscr{O}_{X_\beta} \to \cdots \to \bigoplus_{\omega \in \Delta(r)} \mathscr{O}_{X_\omega} \to 0\,.$$

The sequence we obtain from this by tensoring $\mathscr{O}_U(-kX) \otimes_{\mathscr{O}_U} \mathscr{F}$ for a nonnegative integer k is also exact. Hence we get a spectral sequence

$$\text{(I)} \qquad E_1^{p,q}(\mathscr{F}(-kX)) := \bigoplus_{\alpha \in \Delta(p+1)} H^q(X_\alpha, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_{X_\alpha}(-kX)) \Rightarrow H^{p+q}(X, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_X(-kX))\,.$$

Set $\tilde{X}_\alpha := q^{-1}(X_\alpha)$, which is the disjoint union of $Y_{\gamma\alpha} := \Phi^{-1}(\mathrm{ord}(\bar{X}_{\gamma\alpha})$ for $\gamma \in \Gamma$. Since $q\colon \tilde{X}_\alpha \to X_\alpha$ is unramified and $\mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_{X_\alpha}(-kX) \cong q^{\Gamma}_*(F \otimes_C \mathscr{O}_{\tilde{X}_\alpha}(-k\tilde{Y}))$, we have a spectral sequence

(II) $$E_2^{p,q}(\Gamma, \mathscr{F}(-kX)) := H^p(\Gamma, H^q(\tilde{X}_\alpha, F \otimes_C \mathscr{O}_{\tilde{X}_\alpha}(-k\tilde{Y}))) \Rightarrow H^{p+q}(X_\alpha, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_{X_\alpha}(-kX)) .$$

We have an isomorphism $H^q(\tilde{X}_\alpha, F \otimes_C \mathscr{O}_{\tilde{X}_\alpha}(-k\tilde{Y})) \cong \bigoplus_{\gamma \in \Gamma} (F \otimes_C H^q(Y_{\gamma\alpha}, \mathscr{O}_{Y_{\gamma\alpha}}(-k\tilde{Y})))$ as vector spaces and have a Leray spectral sequence

(III) $$E_2^{p,q}(A, \mathscr{O}_X(-kX)) := H^p(A, R^q\tilde{p}_*\mathscr{O}_{Y_\alpha}(-k\tilde{Y})) \Rightarrow H^{p+q}(Y_\alpha, \mathscr{O}_{Y_\alpha}(-k\tilde{Y})) .$$

For each point $a \in A$,

$$R^q\tilde{p}_*\mathscr{O}_{Y_\alpha}(-k\tilde{Y}) \otimes_{\mathscr{O}_A} C(a) \cong H^q(\tilde{p}^{-1}(a), \mathscr{O}_{Y_\alpha}(-k\tilde{Y})|_{\tilde{p}^{-1}(a)}) \cong H^q(\bar{X}_\alpha, \mathscr{O}_{\bar{X}_\alpha}(-k\bar{X}))$$

vanishes for $q > 0$ and $k \geqq 0$, because $\tilde{\Delta}$ is convex (see, for instance, [TE]). Hence we have $H^p(Y_\alpha, \mathscr{O}_{Y_\alpha}(-k\tilde{Y})) \cong H^p(A, \tilde{p}_*\mathscr{O}_{Y_\alpha}(-k\tilde{Y}))$. Since Y_α and $Y_{\gamma\alpha}$ are isomorphic for every $\gamma \in \Gamma$, we have an isomorphism as $C[\Gamma]$-modules

$$\begin{aligned} H^q(\tilde{X}_\alpha, F \otimes_C \mathscr{O}_{\tilde{X}_\alpha}(-k\tilde{Y})) &= H^q(\bigcup_{\gamma \in \Gamma} Y_{\gamma\alpha}, F \otimes_C \mathscr{O}_{Y_{\gamma\alpha}}(-k\tilde{Y})) \\ &\cong \mathrm{Hom}_C(C[\Gamma], F) \otimes_C H^q(A, \tilde{p}_*\mathscr{O}_{Y_\alpha}(-k\tilde{Y})) . \end{aligned}$$

Thus (see, for instance, [HS])

$$H^p(\Gamma, H^q(\tilde{X}_\alpha, F \otimes_C \mathscr{O}_{\tilde{X}_\alpha}(-k\tilde{Y})) = 0 \quad \text{for} \quad p > 0 .$$

On the other hand, for a positive integer k, the sheaf $\tilde{p}_*\mathscr{O}_{Y_\alpha}(-k\tilde{Y})$ corresponds to the holomorphic vector bundle which is the direct sum of the line bundles $\mathscr{L}(m)$ associated to positive definite Hermitian forms $4\langle m, H(\ , \ )\rangle$ for $m \in N^* \cap k\alpha^*$. Here α^* is the cone in $\mathfrak{n}^*$ dual to α. Hence $H^q(A, \tilde{p}_*\mathscr{O}_{\tilde{Y}_\alpha}(-k\tilde{Y})) = 0$ for $q > 0$ and $k > 0$. Thus for a positive integer k we have

$$E_1^{p,q}(\mathscr{F}(-kX)) = 0 \quad \text{if} \quad q > 0 ,$$

and

$$\begin{aligned} E_1^{p,0}(\mathscr{F}(-kX)) &= \bigoplus_{\alpha \in \Delta(p+1)} H^0(\Gamma, \mathrm{Hom}_C(C[\Gamma], F \otimes_C H^0(A, \tilde{p}_*\mathscr{O}_{Y_\alpha}(-kY)))) \\ &= \bigoplus_{\alpha \in \Delta(p+1)} H^0(\Gamma, \mathrm{Hom}_C(C[\Gamma], F \otimes_C (\bigoplus_{m \in N^* \cap k\alpha^*} H^0(A, \mathscr{L}(m))))) . \end{aligned}$$

For each $m \in N^* \cap k\alpha^*$ there exists a unique β in Δ of the smallest dimension among cones β satisfying $m \in N^* \cap k\beta^*$. Thus we have an exact sequence as in [T1]

$$\begin{aligned} 0 \to F \otimes_C H^0(A, \mathscr{L}(m)) &\to \bigoplus_{\alpha \in \Delta(\beta,1)} F \otimes_C H^0(A, \mathscr{L}(m)) \\ &\to \bigoplus_{\delta \in \Delta(\beta,2)} F \otimes_C H^0(A, \mathscr{L}(m)) \to \cdots , \end{aligned}$$

where $\Delta(\beta, j) := \{\alpha \in \Delta; \alpha < \beta \text{ and } \dim \alpha = j\}$. The complex

$$K^p := \bigoplus_{\alpha \in \Delta(p+1)} F \otimes_C (\bigoplus_{m \in N^* \cap k\alpha^*} H^0(A, \mathscr{L}(m)))$$

is the direct sum of the complexes

$$\mathscr{K}^j(m) := \bigoplus_{\alpha \in \Delta(\beta, j)} F \otimes_C H^0(A, \mathscr{L}(m)) .$$

Thus the E_2-term of the spectral sequence (I), which on the one hand satisfies $E_2^{p,0}(\mathscr{F}(-kX)) \cong H^p(X, \mathscr{F} \otimes \mathscr{O}_X(-kX))$, is the direct sum of the p-th cohomology groups $\mathscr{H}^p(H^0(\Gamma, \mathrm{Hom}_C(C[\Gamma], \mathscr{K}^{\cdot}(m)))) \cong \mathscr{H}^p(\mathscr{K}^{\cdot}(m))$, which vanish for $p > 0$, because β is contractible.

In the general case, we take a normal subgroup Γ' of finite index in Γ so that for the pair (U', X') constructed as in Section 1 for Γ' the dual graph of X' is orientable and fine. Then we have

$$H^i(X, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_X(-kX)) \cong H^i(X', \mathscr{F} \otimes_{\mathscr{O}_{U'}} \mathscr{O}_{X'}(-kX'))^{\Gamma/\Gamma'} = 0$$

for $i > 0$ and $k > 0$. Thus we finish the proof of Proposition 3.1.

For $k = 0$, we also have

$$\begin{aligned} E_1^{p,q}(\mathscr{F}) = & \bigoplus_{\alpha \in \Delta(p+1)} H^0(\Gamma, \mathrm{Hom}_C(C[\Gamma], \mathscr{F} \otimes H^q(A, \mathscr{O}_A))) \\ & \Rightarrow H^{p+q}(X, \mathscr{F} \otimes \mathscr{O}_X) . \end{aligned}$$

COROLLARY. *When* $\dim A = 0$, *i.e.*, $m = 0$, *we have*

$$H^p(X, \mathscr{F} \otimes_{\mathscr{O}_U} \mathscr{O}_X) \cong H^p(\Gamma, F) \quad \textit{for} \quad p \geqq 0 .$$

By the comparison theorm in [BS], we have for $i > 0$

$$(R^i\pi_*\mathscr{F})_p^{\wedge} = \mathrm{proj\,lim}_k H^i(U, \mathscr{F}/\mathscr{F}(-kX)) ,$$
$$(R^i\pi_*\mathscr{F}(-X))_p^{\wedge} = \mathrm{proj\,lim}_k H^i(U, \mathscr{F}(-X)/\mathscr{F}(-(k+1)X)) .$$

The exact sequences

$$0 \to \mathscr{O}_X(-kX) \to \mathscr{O}_U/\mathscr{O}_U(-(k+1)X) \to \mathscr{O}_U/\mathscr{O}_U(-kX) \to 0 ,$$
$$0 \to \mathscr{O}_X(-kX) \to \mathscr{O}_U(-X)/\mathscr{O}_U(-(k+1)X) \to \mathscr{O}_U(-X)/\mathscr{O}_U(-kX) \to 0 ,$$

tensored with $\mathscr{F}$ remain exact. Applying Proposition 3.1 and the above comparison theorem, we have

$$R^i\pi_*\mathscr{F} = H^i(X, \mathscr{F} \otimes \mathscr{O}_X) \quad \text{and} \quad R^i\pi_*\mathscr{F}(-X) = 0 \quad \text{for} \quad i > 0 .$$

This proves Theorem 2.1.

Since $\pi(U) = V$ is a Stein space, we also have

$$H^i(U, \mathscr{F}) = H^0(V, R^i\pi_*\mathscr{F}) = H^i(X, \mathscr{F} \otimes \mathscr{O}_X) \quad \text{and}$$
$$H^i(U, \mathscr{F}(-X)) = 0 \quad \text{for} \quad i > 0 .$$

This, combined with the corollary, proves Theorem 2.3 because in that case $\Theta_U(-\log X) \cong q_*^{\Gamma}(N \otimes_Z \mathscr{O}_{\tilde{U}})$ holds.

3.3. PROOF OF PROPOSITION 3.2. We use the following lemma:

LEMMA ([BS]). *Let Z be a topological space and K a compact subset with a countable fundamental system of neighborhoods. Then for a sheaf $\mathscr{G}$ of abelian groups on Z, we have a surjective mapping*

$$(*) \qquad H^q(Z \setminus K, \mathscr{G}) \to \operatorname{proj\,lim}_{W \supset K} H^q(Z \setminus W, \mathscr{G}) \quad for \quad q \geqq 0 .$$

Moreover, $(*)$ is an isomorphism if for every member W of a fundamental system of neighborhoods of K the mapping induced by restriction

$$H^{q-1}(Z, \mathscr{G}) \to H^{q-1}(Z \setminus W, \mathscr{G})$$

is surjective.

In our situation, choose a fundamental system of neighborhoods of X consisting of relatively compact and holomorphically convex neighborhoods U_ν ($\nu = 1, 2, \cdots$) with $U_\nu \supset U_{\nu+1}$. Consider the commutative diagram of long exact sequences

$$\begin{array}{ccccccc} H^i_X(U, \mathscr{F}) & \longrightarrow & H^i(U, \mathscr{F}) & \longrightarrow & H^i(U \setminus X, \mathscr{F}) & \longrightarrow & H^{i+1}_X(U, \mathscr{F}) \\ \downarrow & & \| & & \downarrow & & \downarrow \\ H^i_c(U_\nu, \mathscr{F}) & \longrightarrow & H^i(U, \mathscr{F}) & \longrightarrow & H^i(U \setminus U_\nu, \mathscr{F}) & \longrightarrow & H^{i+1}_c(U_\nu, \mathscr{F}) . \end{array}$$

The cohomology group $H^i_c(U_\nu, \mathscr{F})$ with compact support is the algebraic dual of $H^{n-i}(U_\nu, \mathscr{F}^\vee \otimes \Omega^n_{U_\nu})$.

LEMMA. $H^{n-i}(U_\nu, \mathscr{F}^\vee \otimes \Omega^n_{U_\nu}) = 0$ *for* $i < n$.

PROOF. First assume that $\Gamma \subset SL(N)$. Then we have $\Omega^n_U \cong \mathscr{O}_U(-X)$, and hence

$$\begin{aligned} H^{n-i}(U_\nu, \mathscr{F}^\vee \otimes \Omega^n_{U_\nu}) &= H^{n-i}(U_\nu, \mathscr{F}^\vee \otimes \mathscr{O}_{U_\nu}(-X)) \\ &= H^0(V_\nu, R^{n-i}\pi_*\mathscr{F}^\vee(-X)) = 0 \end{aligned}$$

for $i < n$ by Proposition 3.1, because $V_\nu := \pi(U_\nu)$ is a Stein space. Next, for a general Γ we take a normal subgroup Γ' of finite index in Γ so that $\Gamma' \subset SL(N)$. Let (U', X') be the pair constructed as in Section 1 for Γ'. Then $\Omega^n_{U'} \cong \mathscr{O}_{U'}(-X')$, and hence for $i < n$

$$H^{n-i}(U_\nu, \mathscr{F}^\vee \otimes \Omega^n_{U_\nu}) \cong H^{n-i}(U'_\nu, \mathscr{F}^\vee \otimes \mathscr{O}_{U'_\nu}(-X'))^{\Gamma/\Gamma'} = 0 . \quad \text{q.e.d.}$$

Applying the lemmas, we see that the mapping $H^i(U \setminus X, \mathscr{F}) \to \operatorname{proj\,lim}_\nu H^i(U \setminus U_\nu, \mathscr{F})$ is isomorphic for $i < n - 1$ and surjective for $i = n - 1$, and hence that $H^i(U, \mathscr{F}) \to H^i(U \setminus X, \mathscr{F})$ is isomorphic for $i < n - 1$ and injective for $i = n - 1$. This implies that $H^i_X(U, \mathscr{F}) = 0$ for $i < n$.

3.4. PROOF OF THEOREM 2.2. The logarithmic tangent sheaf

$\Theta_U(-\log X)$ splits as

$$0 \to q_*^\Gamma(N \otimes_Z \mathcal{O}_{\tilde{U}}) \to \Theta_U(-\log X) \to q_*^\Gamma(H^0(A, \Theta_A) \otimes_C \mathcal{O}_{\tilde{U}}) \to 0 .$$

Applying Propositions 3.1 and 3.2 to this we have

$$H^i(X, \Theta_U(-\log X) \otimes \mathcal{O}_X(-kX)) = 0 \quad \text{for} \quad i > 0 \text{ and } k > 0 ,$$
$$H_X^i(U, \Theta_U(-\log X)) = 0 \quad \text{for} \quad i < n .$$

Thus we have for $n \geqq 3$

$$\begin{array}{ccc} H^1(U, \Theta_U(-\log X)) & \xrightarrow{\sim} & H^1(U \setminus X, \Theta_U) \\ & & \| \\ T_V^1 & \longrightarrow & H^1(V \setminus \{p\}, \Theta_V) . \end{array}$$

In order to prove Theorem 2.2 it is sufficient to show that the isomorphism $H^1(\Theta_U(-\log X)) \xrightarrow{\sim} H^1(V \setminus \{p\}, \Theta_V)$ factors through T_V^1. This follows from the following proposition applied to $S = \mathrm{Specan}\, \boldsymbol{C}[\varepsilon]/(\varepsilon^2)$:

PROPOSITION 3.4. *For a germ (S, s_0) of complex analytic spaces which need not be reduced, let $\omega: (\mathscr{U}, u_0) \to (S, s_0)$ be a deformation of $U \cong \mathscr{U}_0 := \omega^{-1}(s_0)$ for which none of X_i disappear, that is, there exists a subvariety $\mathscr{X}$ of $\mathscr{U}$ such that, after possible shrinking of S, the restriction $\omega' := \omega|_{\mathscr{X}}: (\mathscr{X}, x_0) \to (S, s_0)$ is a deformation of $X := \cup_i X_i$. We assume that $H^1(U, \mathcal{O}_U(-X)) = 0$. Then there exist neighborhoods $\mathscr{U}'$ of $\mathscr{X}_0 := \omega'^{-1}(s_0)$ in $\mathscr{U}$ and S' of s_0 in S so that in the canonical reduction diagram of ω' over S' in the sense of Riemenschneider* [R2]

$$\begin{array}{ccc} \mathscr{U}' & \xrightarrow{\tau} & \mathscr{V} \\ & {\scriptstyle \omega'}\searrow & \downarrow{\scriptstyle \rho} \\ & & S' \end{array}$$

τ is a proper morphism and $\rho: (\mathscr{V}, v_0) \to (S', s_0)$ is a deformation of (V, p).

PROOF. By shrinking $\mathscr{U}$ and S if necessary, we may assume that ω is a 1-convex holomorphic mapping with an exhaustion function φ and a convexity bound c_* and that S is a Stein space [R2]. Then ω can be factored as follows:

$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{\tau} & \mathscr{V} \\ & {\scriptstyle \omega}\searrow & \downarrow{\scriptstyle \rho} \\ & & S . \end{array}$$

In this diagram τ is proper and biholomorphic outside the union $\mathscr{X}$ of all maximal compact analytic subsets $\mathscr{X}_s \subset \mathscr{U}_s$ for $s \in S$, ρ is a Stein morphism and $\rho|\tau(\mathscr{X})$ is finite. Further $\mathscr{U}$ is holomorphically convex and $\mathscr{V}$ is the Remmert quotient of $\mathscr{U}$, i.e., $\mathscr{O}_{\mathscr{V}} = \tau_* \mathscr{O}_{\mathscr{U}}$. Now we have an exact sequence

$$0 \to \mathscr{O}_{\mathscr{U}}(-\mathscr{X}) \to \mathscr{O}_{\mathscr{U}} \to \mathscr{O}_{\mathscr{X}} \to 0 .$$

Since $\mathscr{O}_{\mathscr{U}}$ and $\mathscr{O}_{\mathscr{X}}$ are ω-flat, so is $\mathscr{O}_{\mathscr{U}}(-\mathscr{X})$. Let $\omega_c := \omega|_{\{\varphi<c\}}$, $c \in \boldsymbol{R}$. Since $H^1(\mathscr{U}_0, \mathscr{O}_{\mathscr{U}_0}(-\mathscr{X})) = H^1(U, \mathscr{O}_U(-X)) = 0$, the canonical restriction mapping

$$(\omega_{c*}\mathscr{O}_{\mathscr{X}})_{s_0} \to (\omega_{c*}\mathscr{O}_{\mathscr{X}_0})_{s_0}$$

is surjective for every $c > c_*$. From the semi-continuity of $\dim_{\boldsymbol{C}} H^1(\mathscr{U}_s, \mathscr{O}_{\mathscr{U}_s}(-\mathscr{X}_s))$, $s \in S$, and the vanishing of $H^1(\mathscr{U}_0, \mathscr{O}_{\mathscr{U}_0}(-\mathscr{X}_0))$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{U}}(-\mathscr{X}))_{s_0} & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{U}})_{s_0} & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{X}})_{s_0} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{U}_0}(-\mathscr{X}_0))_{s_0} & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{U}_0})_{s_0} & \longrightarrow & (\omega_{c*}\mathscr{O}_{\mathscr{X}_0})_{s_0} & \longrightarrow & 0 . \end{array}$$

In the above diagram, two horizontal rows are exact, and the left and right vertical arrows are both surjective. Hence the middle arrow is surjective. From this and [R1, Theorem 1] we see that the fiber $\mathscr{V}_0 := \rho^{-1}(s_0)$ is the Remmert quotient of $\mathscr{U}_0$, i.e., $(\mathscr{V}_0, v_0)$ is isomorphic to (V, p) as germs of complex spaces.

Next we need to show that $\mathscr{O}_{\mathscr{V}}$ is ρ-flat. Since ρ is a Stein morphism and the ρ-flatness of $\mathscr{O}_{\mathscr{V}}$ is equivalent to the flatness of $\rho_*\mathscr{O}_{\mathscr{V}}$ over $\mathscr{O}_S$ (cf. [Hn, Theorem 1.3]), it is enough to prove that $\rho_*\mathscr{O}_{\mathscr{V}}$ is flat over $\mathscr{O}_S$. Since $\mathscr{X}_0 \cong X$ is reduced and connected, the natural morphism $\mathscr{O}_{S', s_0} \to (\omega_*\mathscr{O}_{\mathscr{X}})_{s_0}$ is an isomorphism. In particular, $\omega_*\mathscr{O}_{\mathscr{X}}$ is flat over $\mathscr{O}_S$ at s_0. By shrinking $\mathscr{U}$ and S if necessary, we may assume that $\omega_*\mathscr{O}_{\mathscr{X}}$ is flat over $\mathscr{O}_S$ because of the openness of the flat locus ([Fs]). From the vanishing of $H^1(\mathscr{U}_0, \mathscr{O}_{\mathscr{U}_0}(-\mathscr{X}_0))$, we can show that $\tau_*\mathscr{O}_{\mathscr{U}}(-\mathscr{X})$ is ρ-flat as in the proof of [R2, Theorem 2]. Hence we see that $\rho_*\tau_*\mathscr{O}_{\mathscr{U}}(-\mathscr{X}) = \omega_*\mathscr{O}_{\mathscr{U}}(-\mathscr{X})$ is flat over $\mathscr{O}_S$. Consider the exact sequence

$$0 \to \omega_*\mathscr{O}_{\mathscr{U}}(-\mathscr{X}) \to \omega_*\mathscr{O}_{\mathscr{U}} \to \omega_*\mathscr{O}_{\mathscr{X}} \to 0 .$$

Since $\omega_*\mathscr{O}_{\mathscr{U}}(-\mathscr{X})$ and $\omega_*\mathscr{O}_{\mathscr{X}}$ are both flat over $\mathscr{O}_S$, so is $\omega_*\mathscr{O}_{\mathscr{U}} = \rho_*\tau_*\mathscr{O}_{\mathscr{U}} = \rho_*\mathscr{O}_{\mathscr{V}}$. q.e.d.

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