Tôhoku Math. Journ.
39 (1987), 391-406.

A CLASS OF DIFFERENTIAL EQUATIONS OF FUCHSIAN TYPE

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received July 28, 1986)

1. Introduction. The pair of the period integrals

$$Y = \left(\int_\gamma \frac{dz}{w}, \int_\gamma \frac{zdz}{w}\right) \qquad \text{for a 1-cycle } \gamma$$

of the family of elliptic curves

$$w^2 = 4z^3 - xz - y ,$$

parametrized by $(x, y) \in \boldsymbol{C}^2$ with $\Delta = x^3 - 27y^2 \neq 0$, is known to satisfy the following differential equation of Fuchsian type of rank two on the complex projective plane $\boldsymbol{P}^2 = \boldsymbol{P}^2(\boldsymbol{C})$:

$$dY = Y\Omega . \tag{1.1}$$

Here Ω is a (2×2)-matrix-valued meromorphic 1-form on $\boldsymbol{P}^2$ defined by

$$\Omega = \begin{pmatrix} \dfrac{(-1/12)d\Delta}{\Delta} & \dfrac{(3/8)(xydx - (2/3)x^2dy)}{\Delta} \\ \dfrac{(-9/2)(ydx - (2/3)xdy)}{\Delta} & \dfrac{(1/12)d\Delta}{\Delta} \end{pmatrix}.$$

The differential equation (1.1) has regular singularity along $C \cup L_\infty$, where C is the closure in $\boldsymbol{P}^2$ of the affine curve $\{(x, y) \in \boldsymbol{C}^2 \mid \Delta = 0\}$ and L_∞ is the line at infinity.

For $\{\gamma_1, \gamma_2\}$ which gives rise to a $\boldsymbol{Z}$-basis for the first homology group of the elliptic curve with the intersection number $\gamma_1 \cdot \gamma_2 = 1$, the multi-valued map

$$S: \boldsymbol{P}^2 - C \cup L_\infty \to \boldsymbol{C}^2$$

which sends (x, y) to

$$(u, v) = \left(\int_{\gamma_1} \frac{dz}{w}, \int_{\gamma_2} \frac{dz}{w}\right)$$

has the single-valued inverse map

$$S^{-1}: D \to \boldsymbol{P}^2 - C \cup L_\infty ,$$

which sends (u, v) to (x, y), where

$$D = \{(u, v) \in \boldsymbol{C}^2 \mid uv \neq 0,\ \mathrm{Im}(v/u) > 0\}$$

is the image of S and $\mathrm{Im}(z)$ is the imaginary part of $z \in \boldsymbol{C}$.

S^{-1} can in fact be written as the Eisenstein series:

$$x = 60 \sum \frac{1}{(mu + nv)^4}\ , \qquad y = 140 \sum \frac{1}{(mu + nv)^6}\ ,$$

with the summation taken over all pairs (m, n) of integers with $(m, n) \neq (0, 0)$.

The purpose of this paper is to discuss a wider class of differential equations

$$dY = Y\Omega \tag{1.2}$$

of Fuchsian type of rank two on $\boldsymbol{P}^2$ with regular singularity along $C \cup L_\infty$, which contains the differential equation (1.1) as a special case.

We discuss the multi-valued map

$$S: \boldsymbol{P}^2 - C \cup L_\infty \to \boldsymbol{C}^2\ ,$$

which sends (x, y) to $(u, v) = (f_1, g_1)$, where (f_1, f_2) and (g_1, g_2) are linearly independent solutions of (1.2). We give a criterion for the single-valuedness of the inverse map S^{-1} from the image of S to $\boldsymbol{P}^2 - C \cup L_\infty$.

Finally, using (1.2) and Selberg's theorem, we give an existence theorem for finite Galois coverings $\pi: X \to \boldsymbol{P}^2$ with the branch locus $C \cup L_\infty$.

2. Differential equations of Fuchsian type. Let p be a point of a connected complex manifold M of dimension n. Let Ω be an $(r \times r)$-matrix-valued meromorphic 1-form on a neighborhood U of p in M satisfying the integrability condition

$$d\Omega + \Omega \wedge \Omega = 0\ . \tag{2.1}$$

Suppose that Ω can be written as

$$\Omega = B_1(z)dz_1 + \cdots + B_{n-1}(z)dz_{n-1} + B_n(z)dz_n/z_n\ , \tag{2.2}$$

where $z = (z_1, \cdots, z_n)$ is a local coordinate system in U with $p = (0, \cdots, 0)$, and $B_j(z)$ $(1 \leqq j \leqq n)$ are $(r \times r)$-matrix-valued holomorphic functions on U. Then we say that the differential equation

$$dY = Y\Omega \tag{2.3}$$

in an unknown vector-valued function $Y = (y_1, \cdots, y_r)$ has regular singularity along $\{z \mid z_n = 0\}$. It can be easily seen that this definition is independent of the choice of a coordinate system $(z_1, \cdots, z_n)$. In this case, the following is known:

THEOREM 1 (Gérard [2], Yoshida-Takano [7]). *There is a fundamental matrix solution $F(z)$ on U of* (2.3) *of the form*

$$F(z) = (\exp(C \log z_n))(\exp(N \log z_n))G(z) ,$$

where C is a constant matrix, N is a diagonal matrix whose components are non-negative integers and $G(z)$ is a matrix-valued holomorphic function on U with $\det G(z)$ *nowhere vanishing. Moreover, if none of the differences of the eigenvalues of $B_n(p)$ are non-zero integers, then N and C can be so chosen that $N = 0$ and C is equivalent to $B_n(p)$.*

Next, let B be a hypersurface in a connected complex manifold M of dimension n. Let Ω be an $(r \times r)$-matrix-valued meromorphic 1-form on M such that

$$d\Omega + \Omega \wedge \Omega = 0 . \tag{2.4}$$

Suppose that (i) Ω is holomorphic on $M - B$ and (ii) for every point p in the set $\operatorname{Reg} B$ of all non-singular points of B, there exists a neighborhood U of p in M such that Ω has regular singularity along $B \cap U$. Then we say that the differential equation

$$dY = Y\Omega \tag{2.5}$$

in an unknown vector-valued function $Y = (y_1, \cdots, y_n)$ is of Fuchsian type. We say that the equation (2.5) has regular singularity along B.

Let p_0 be a fixed point of $M - B$. The monodromy representation

$$R: \pi_1(M - B, p_0) \to GL(r, \boldsymbol{C})$$

of the equation (2.5) is defined by

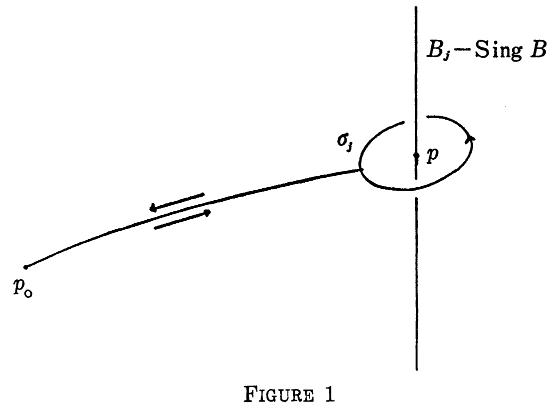

FIGURE 1

$$\sigma^*F(:= F\circ\sigma) = R(\sigma)F \quad \text{for} \quad \sigma\in\pi_1(M-B, p_o),$$

where F is a fundamental matrix solution of (2.5) in a neighborhood of p_o.

Let $B = B_1\cup\cdots\cup B_s$ be the decomposition of B into irreducible components. Let σ_j be a loop starting and terminating at p_o, encircling a point $p\in B_j - \operatorname{Sing} B$ in the positive sense as in Figure 1, where $\operatorname{Sing} B$ is the singular locus of B.

We identify σ_j with its homotopy class. Then, by Theorem 1, $R(\sigma_j)$ is equivalent to $\exp(2\pi\sqrt{-1}C)$.

3. A class of Fuchsian differential equations on $\boldsymbol{P}^2$. We now restrict ourselves to the case $M=\boldsymbol{P}^2$. For complex numbers α, β, γ ($\gamma\neq 0$), δ, ε and ε', consider the following (2×2)-matrix-valued meromorphic 1-form on $\boldsymbol{P}^2$:

$$\Omega = \begin{pmatrix} \dfrac{\alpha d\Delta}{\Delta} & \dfrac{\beta(xydx+\varepsilon x^2dy)}{\Delta} \\ \dfrac{\gamma(ydx+\varepsilon' xdy)}{\Delta} & \dfrac{\delta d\Delta}{\Delta}\end{pmatrix}, \tag{3.1}$$

which generalizes Ω appearing in (1.1), where (x, y) is an affine coordinate system and $\Delta = x^3 - 27y^2$. Ω is holomorphic on $\boldsymbol{P}^2 - C\cup L_\infty$ with C and L_∞ defined as in §1. For such Ω, consider the differential equation

$$dY = Y\Omega \tag{3.2}$$

in an unknown vector-valued function Y.

THEOREM 2. *Suppose $\beta\neq 0$ and $\gamma\neq 0$. Then the equation* (3.2) *is of Fuchsian type if and only if* (i) $\varepsilon=\varepsilon' = -2/3$ *and* (ii) $\delta = \alpha + 1/6$.

PROOF. We first examine the regular singularity condition (2.2) and then the integrability condition (2.4). Let $(X_0: X_1: X_2)$ be the homogeneous coordinate system on $\boldsymbol{P}^2$ such that

$$x = X_1/X_0 \quad\text{and}\quad y = X_2/X_0 .$$

The singular locus of $B = C\cup L_\infty$ consists of two points $(1: 0: 0)$ and $(0: 0: 1)$. (See Figure 2.)
Take a point $p = (a, b)\in C - \operatorname{Sing} B$. Then $a^3 - 27b^2 = 0$. Put

$$z_1 = x - a \quad\text{and}\quad z_2 = \Delta = x^3 - 27y^2 .$$

Then (z_1, z_2) is a local coordinate system around $p = (0, 0)$ such that, locally, $C = \{(z_1, z_2)\,|\,z_2 = 0\}$. Ω is then written as

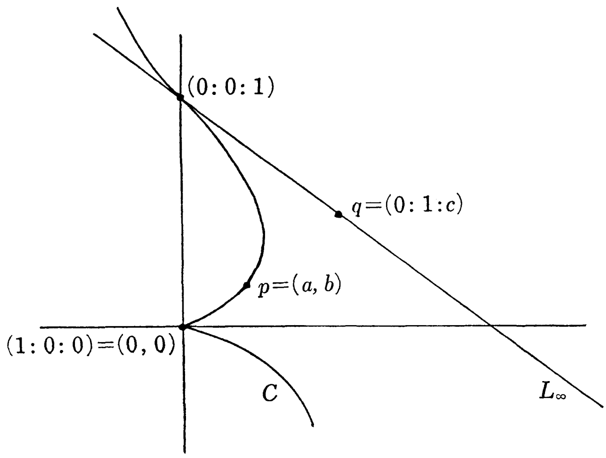

FIGURE 2

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix} ,$$

where

$$\begin{aligned}
\omega_{11} &= \alpha dz_2/z_2 , \\
\omega_{21} &= [\gamma\{(2 + 3\varepsilon')(z_1 + a)^3 - 2z_2\}dz_1 - \gamma\varepsilon'(z_1 + a)dz_2]/54hz_2 , \\
\omega_{12} &= [\beta\{(2 + 3\varepsilon)(z_1 + a)^4 - 2z_2(z_1 + a)^4\}dz_1 - \beta\varepsilon(z_1 + a)^2dz_2]/54hz_2 , \\
\omega_{22} &= \delta dz_2/z_2 , \\
h &= [((z_1 + a)^3 - z_2)/27]^{1/2} .
\end{aligned}$$

Hence Ω can be written as

$$\Omega = B_1(z)dz_1 + B_2(z)dz_2/z_2 ,$$

where $B_1(z)$ and $B_2(z)$ are (2×2)-matrix-valued holomorphic functions around p, if and only if

$$\varepsilon = \varepsilon' = -2/3 . \tag{3.3}$$

If this is the case, then

$$B_1(z) = \begin{pmatrix} 0 & \dfrac{-\beta(z_1 + a)}{27h} \\ \dfrac{-\gamma}{27h} & 0 \end{pmatrix} \tag{3.4}$$

and

$$(3.5) \qquad B_2(z) = \begin{pmatrix} \alpha & \dfrac{\beta(z_1 + a)^2}{81h} \\ \dfrac{\gamma(z_1 + a)}{81h} & \delta \end{pmatrix}.$$

In particular,

$$(3.6) \qquad B_2(p) = \begin{pmatrix} \alpha & \dfrac{\beta a^2}{81b} \\ \dfrac{\gamma a}{81b} & \delta \end{pmatrix}.$$

For a complex number c, consider a point

$$q = (0\colon 1\colon 0) \in L_\infty - \operatorname{Sing} B\ , \qquad \text{(see Figure 2)}\ .$$

Put

$$t_1 = (y/x) - c \quad \text{and} \quad t_2 = 1/x\ .$$

Then (t_1, t_2) is a local coordinate system around $q = (0, 0)$ such that, locally, $L_\infty = \{(t_1, t_2) \mid t_2 = 0\}$. Ω is written around p as

$$\Omega = C_1(t)dt_1 + C_2(t)dt_2/t_2\ , \qquad \text{with} \quad t = (t_1, t_2)\ ,$$

where

$$C_1(t) = \frac{1}{g}\begin{pmatrix} -54\alpha(t_1 + c)t_2 & \beta\varepsilon \\ \gamma\varepsilon' t_2 & -54\delta(t_1 + c)t_2 \end{pmatrix}$$

and

$$C_2(t) = \frac{1}{g}\begin{pmatrix} 54\alpha(t_1 + c)^2 t_2 - 3\alpha & -\beta(1 + \varepsilon)(t_1 + c) \\ -\gamma(1 + \varepsilon')(t_1 + c)t_2 & 54\delta(t_1 + c)^2 t_2 - 3\delta \end{pmatrix}$$

with $g = 1 - 27t_2(t_1 + c)^2$. Hence $C_1(t)$ and $C_2(t)$ are (2×2)-matrix-valued holomorphic functions around p. In particular,

$$(3.7) \qquad C_2(q) = \begin{pmatrix} -3\alpha & -\beta(1 + \varepsilon)c \\ 0 & -3\delta \end{pmatrix}.$$

Next, by simple calculation, we obtain

$$d\Omega + \Omega \wedge \Omega = \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix} \frac{dx \wedge dy}{\Delta^2}\ ,$$

where

$$\xi_{11} = \beta\gamma(\varepsilon' - \varepsilon)x^2 y\ ,$$
$$\xi_{21} = \gamma(\varepsilon' - 1 + 3\varepsilon'(\delta - \alpha - 1))x^3 + 27\gamma(1 - \varepsilon' + 2(\delta - \alpha - 1))y^2$$
$$\xi_{12} = \beta(2\varepsilon - 1 + 3\varepsilon(\alpha - \delta - 1))x^4 + 27\beta(1 - 2\varepsilon + 2(\alpha - \delta - 1))xy^2\ ,$$
$$\xi_{22} = \beta\gamma(\varepsilon - \varepsilon')x^2 y\ .$$

Since $\beta \neq 0$ and $\gamma \neq 0$, we have $d\Omega + \Omega \wedge \Omega = 0$ if and only if $\varepsilon = \varepsilon' = -2/3$ and $\delta - \alpha = 1/6$. q.e.d.

If $\beta = 0$, then the above proof also shows:

THEOREM 2′. *The differential equation* (3.2), *where* Ω *is as defined in* (3.1) *with* $\beta = 0$ *and* $\gamma \neq 0$, *is of Fuchsian type if and only if* $\varepsilon' = -2/3$ *and* $\delta - \alpha = 1/6$.

Henceforth, we only consider the differential equation (3.2), where

$$\Omega = \begin{pmatrix} \dfrac{\alpha d\Delta}{\Delta} & \dfrac{\beta(xydx - (2/3)x^2dy)}{\Delta} \\ \dfrac{\gamma(ydx - (2/3)xdy)}{\Delta} & \dfrac{(\alpha + 1/6)d\Delta}{\Delta} \end{pmatrix} \tag{3.8}$$

with $\gamma \neq 0$. This equation is of Fuchsian type by Theorems 2 and 2′.

Let p_o be a fixed point of $\boldsymbol{P}^2 - C \cup L_\infty$. Let σ (resp. σ', resp. τ) be a loop starting and terminating at p_o, encircling the point $(x, y) = (3, -1) \in C$ (resp. $(x, y) = (3, 1) \in C$, resp. $(X_0 \colon X_1 \colon X_2) = (0 \colon 1 \colon 1) \in L_\infty$) in the positive sense as in Figure 3.

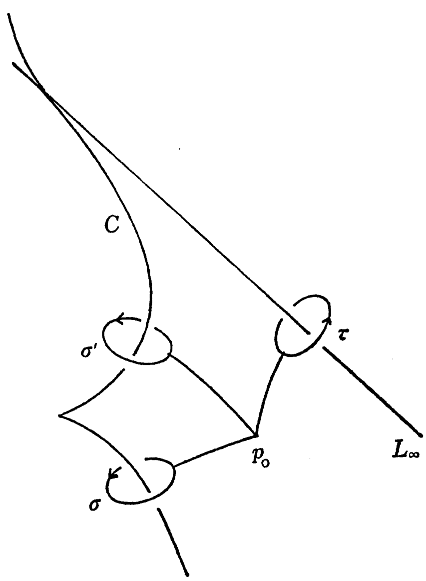

FIGURE 3

Then it is known (see Van Kampen [6]) that $\pi_1(\boldsymbol{P}^2 - C \cup L_\infty, p_o)$ is generated by σ, σ' and τ with the relations

$$\sigma\sigma'\sigma = \sigma'\sigma\sigma' = \tau^{-1} .$$

Note that σ and σ' are conjugate, since $\sigma' = (\sigma\sigma')\sigma(\sigma\sigma')^{-1}$. Let

$$R: \pi_1(\boldsymbol{P}^2 - C \cup L_\infty, p_o) \rightarrow GL(2, \boldsymbol{C})$$

be the monodromy representation of the differential equation (3.2). Then, by (3.6), (3.7) and Theorem 1, $R(\sigma)$ and $R(\sigma')$ are both equivalent to

$$\text{(3.9)} \qquad \exp 2\pi\sqrt{-1}B_2(p) = \exp 2\pi\sqrt{-1}\begin{pmatrix} \alpha & \dfrac{\beta a^2}{81b} \\ \dfrac{\gamma a}{81b} & \alpha + 1/6 \end{pmatrix},$$

unless $2\sqrt{D}$ is a non-zero integer, where

$$D = (1/12)^2 + \beta\gamma/243 ,$$

while $R(\tau)$ is equivalent to

$$\text{(3.10)} \qquad \exp 2\pi\sqrt{-1}C_2(q) = \exp 2\pi\sqrt{-1}\begin{pmatrix} -3\alpha & -\beta c/3 \\ 0 & -3\alpha - 1/2 \end{pmatrix}.$$

4. The single-valuedness of the inverse map. Let (f_1, f_2) and (g_1, g_2) be linearly independent solutions of the equation (3.2), where Ω is given by (3.8). Consider the multi-valued map

$$S: \boldsymbol{P}^2 - C \cup L_\infty \rightarrow \boldsymbol{C}^2$$

which sends (x, y) to $(f_1(x, y), g_1(x, y))$.

LEMMA 1. *If $\alpha \neq 0$ and $\gamma \neq 0$, then S is locally biholomorphic.*

PROOF. Put $f_{1x} = \partial f_1/\partial x$, etc. Since

$$df_1 = f_1\alpha(d\Delta/\Delta) + f_2\gamma(ydx - (2/3)xdy)/\Delta$$

and

$$dg_1 = g_1\alpha(d\Delta/\Delta) + g_2\gamma(ydx - (2/3)xdy)/\Delta ,$$

we have

$$f_{1x} = (3\alpha x^2 f_1 + \gamma y f_2)/\Delta , \quad f_{1y} = (-54\alpha y f_1 - (2/3)\gamma x f_2)/\Delta ,$$
$$g_{1x} = (3\alpha x^2 g_1 + \gamma y g_2)/\Delta , \quad g_{1y} = (-54\alpha y g_1 - (2/3)\gamma x g_2)/\Delta .$$

Hence

$$\begin{vmatrix} f_{1x} & f_{1y} \\ g_{1x} & g_{1y} \end{vmatrix} = \frac{-2\alpha\gamma}{\Delta}\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} \neq 0 .$$

q.e.d.

Henceforth, we assume $\alpha \neq 0$ and $\gamma \neq 0$. The image

$$W = S(\boldsymbol{P}^2 - C \cup L_\infty)$$

is an open set of $\boldsymbol{C}^2$. Consider the inverse map

$$S^{-1}: W \to \boldsymbol{P}^2 - C \cup L_\infty .$$

In general, S^{-1} is also a multi-valued map.

THEOREM 3. *Let p be a non-zero integer and q be either $+\infty$ or an integer greater than one. If $\alpha = 1/6p$ and $\beta\gamma = 27(36 - q^2)/16q^2$, (while $\gamma \neq 0$ and $\beta\gamma = -27/16$ if $q = +\infty$), then S^{-1} is single-valued.*

PROOF. Consider the following coordinate transformation:

$$(x, y) \mapsto (t, \lambda) = (x^3/27y^2, y/x) ,$$

where

$$x \neq 0, \; y \neq 0, \; x^3 \neq 27y^2$$

and so

$$t \neq 0, \; \lambda \neq 0, \; t \neq 1 .$$

Using the new coordinate system (t, λ), the (2×2)-matrix-valued 1-form Ω is written as

$$\text{(4.1)} \qquad \Omega = \begin{pmatrix} \dfrac{6\alpha d\lambda}{\lambda} + \dfrac{\alpha(3t^2 - 2t)dt}{t^3 - t^2} & \dfrac{\beta\lambda dt}{3(t-1)} \\ \dfrac{\gamma dt}{81\lambda(t^2 - t)} & (\alpha + 1/6)\Big(\dfrac{6d\lambda}{\lambda} + \dfrac{(3t^2 - 2t)dt}{t^3 - t^2}\Big) \end{pmatrix} .$$

The restriction Ω_λ of Ω to the line

$$L_\lambda = \{(t, \lambda) \mid \lambda \text{ is constant}\}$$

is written as

$$\text{(4.2)} \qquad \Omega_\lambda = \begin{pmatrix} \dfrac{\alpha(3t^2 - 2t)dt}{t^3 - t^2} & \dfrac{\beta\lambda dt}{3(t-1)} \\ \dfrac{\gamma dt}{81\lambda(t^2 - t)} & (\alpha + 1/6)\Big(\dfrac{(3t^2 - 2t)dt}{t^3 - t^2}\Big) \end{pmatrix} .$$

For an unknown vector-valued function $\widetilde{Y} = (\widetilde{h}_1(t), \widetilde{h}_2(t))$, consider the differential equation

$$\text{(4.3)} \qquad d\widetilde{Y} = \widetilde{Y}\Omega_\lambda .$$

Eliminating $\widetilde{h}_2$ from (4.3), we get the following ordinary differential equation of second order for $\widetilde{h}_1$:

$$(4.4)\quad \frac{d^2\tilde{h}_1}{dt^2} + \frac{(-6\alpha + 3/2)t + (4\alpha - 2/3)}{t(t-1)}\left(\frac{d\tilde{h}_1}{dt}\right) + \frac{(9\alpha^2 - (3/2)\alpha)t^2 + (-12\alpha^2 + \alpha - \beta\gamma/243)t + 4\alpha^2 + (2/3)\alpha}{t^2(t-1)^2}\tilde{h}_1 = 0 .$$

Note that the equation (4.4) does not involve λ. Hence, using the symbol of Riemann-Papperitz (see Hochstadt [3]), we can write $\tilde{h}_1$ as

$$\tilde{h}_1(t) = P\begin{bmatrix} 0 & 1 & \infty & \\ 2\alpha & \alpha + (1/12) - \sqrt{D} & (1/2) - 3\alpha & t \\ (1/3) + 2\alpha & \alpha + (1/12) - \sqrt{D} & -3\alpha & \end{bmatrix},$$

where

$$D = (1/12)^2 + \beta\gamma/243 .$$

By a well-known transformation, we get

$$\tilde{h}_1(t) = t^{2\alpha}(1-t)^{\alpha + (1/12) + \sqrt{D}} P\begin{bmatrix} 0 & 1 & \infty & \\ 0 & 0 & (7/12) + \sqrt{D} & t \\ 1/3 & -2\sqrt{D} & (1/12) + \sqrt{D} & \end{bmatrix}.$$

Hence a pair of linearly independent solutions of (4.4) is given by

$$(4.5)\quad \begin{aligned} \tilde{h}_1(t) &= \varphi(t)F((7/12) + \sqrt{D},\ (1/12) + \sqrt{D},\ 2/3; t) , \\ \tilde{k}_1(t) &= \psi(t)F((11/12) + \sqrt{D},\ (5/12) + \sqrt{D},\ 4/3; t) \end{aligned}$$

in terms of Gauss' hypergeometric function $F(a, b, c; t)$ and

$$\varphi(t) = t^{2\alpha}(1-t)^{\alpha + (1/12) + \sqrt{D}} ,$$
$$\psi(t) = t^{2\alpha + (1/3)}(1-t)^{\alpha + (1/12) + \sqrt{D}} .$$

We put

$$a = (7/12) + \sqrt{D} , \quad b = (1/12) + \sqrt{D} , \quad c = 2/3 .$$

Then

$$1 - c = 1/3 , \quad c - a - b = -2\sqrt{D} , \quad b - a = -1/2 .$$

Hence, by Schwarz' theory, the inverse of the multi-valued map

$$\tilde{S}: \boldsymbol{C} - \{0, 1\} \to \boldsymbol{C}$$

which sends t to $\tilde{k}_1(t)/\tilde{h}_1(t)$ is single-valued, if (and only if) $2\sqrt{D}$ is written as

$$2\sqrt{D} = \pm 1/q ,$$

where q is either $+\infty$ or an integer greater than one. This last condition means

$$\beta\gamma = \frac{27(36 - q^2)}{16q^2}. \tag{4.6}$$

Note that the functions $\tilde{h}_2(t)$ and $\tilde{k}_2(t)$ appearing in the linearly independent solutions $(\tilde{h}_1, \tilde{h}_2)$ and $(\tilde{k}_1, \tilde{k}_2)$ of (4.3), where $\tilde{h}_1$ and $\tilde{k}_1$ are given above, can be given by

$$\tilde{h}_2(t) = 81\lambda(t^2 - t)\gamma^{-1}(d\tilde{h}_1/dt) - 81\alpha\lambda(3t - 2)\gamma^{-1}\tilde{h}_1,$$
$$\tilde{k}_2(t) = 81\lambda(t^2 - t)\gamma^{-1}(d\tilde{k}_1/dt) - 81\alpha\lambda(3t - 2)\gamma^{-1}\tilde{k}_1.$$

Next, for an unknown vector-valued function Y, consider the differential equation

$$dY = Y\Omega, \tag{4.7}$$

where Ω is given by (4.1). We show that linearly independent solutions (h_1, h_2) and (k_1, k_2) of the equation (4.7) are given by

$$\begin{aligned} h_1(t, \lambda) &= \varphi(t, \lambda)F((7/12) + \sqrt{D}, (1/12) + \sqrt{D}, 2/3; t), \\ h_2(t, \lambda) &= 81\lambda(t^2 - t)\gamma^{-1}(\partial h_1/\partial t) - 81\alpha\lambda(3t - 2)\gamma^{-1}h_1, \\ k_1(t, \lambda) &= \psi(t, \lambda)F((11/12) + \sqrt{D}, (5/12) + \sqrt{D}, 4/3; t), \\ k_2(t, \lambda) &= 81\lambda(t^2 - t)\gamma^{-1}(\partial k_1/\partial t) - 81\alpha\lambda(3t - 2)\gamma^{-1}k_1, \end{aligned} \tag{4.8}$$

where

$$\varphi(t, \lambda) = \lambda^{6\alpha}t^{2\alpha}(1 - t)^{\alpha + (1/12) + \sqrt{D}},$$
$$\psi(t, \lambda) = \lambda^{6\alpha}t^{2\alpha + (1/3)}(1 - t)^{\alpha + (1/12) + \sqrt{D}}.$$

Indeed, the vector-valued 1-form $d(h_1, h_2) - (h_1, h_2)\Omega$ vanishes on every line L_λ, since $h_1|L_\lambda = \tilde{h}_1$, $h_2|L_\lambda = \tilde{h}_2$ and $\Omega|L_\lambda = \Omega_\lambda$. On the other hand, this vector-valued 1-form vanishes on every line $L'_t = \{(t, \lambda)\,|\,t \text{ is constant}\}$, since

$$\Omega'_t = \Omega\,|\,L'_t = \begin{pmatrix} \dfrac{6\alpha d\lambda}{\lambda} & 0 \\ 0 & \dfrac{(6\alpha + 1)d\lambda}{\lambda} \end{pmatrix}.$$

Hence we identically have $d(h_1, h_2) = (h_1, h_2)\Omega$. In a similar way, (k_1, k_2) is also a solution of (4.7), which clearly is linearly independent of (h_1, h_2).

Now, consider the multi-valued map

$$S'\colon \boldsymbol{C}^2 - \{(t, \lambda) \in \boldsymbol{C}^2\,|\,t \neq 0, \lambda \neq 0, t \neq 1\} \to \boldsymbol{C}^2$$

which sends (t, λ) to $(k_1(t, \lambda)/h_1(t, \lambda), h_1(t, \lambda))$. We show that the inverse S'^{-1} of S' is single-valued if (4.6) is satisfied and $\alpha = 1/6p$ for a non-zero integer p. Suppose the contrary. Then we may assume that, for distinct points (t, λ) and (t', λ'),

$$(k_1(t, \lambda)/h_1(t, \lambda),\ h_1(t, \lambda)) = (k_1(t', \lambda')/h_1(t', \lambda'),\ h_1(t', \lambda')) .$$

Note that the function $k_1/h_1 = \tilde{k}_1/\tilde{h}_1$ is independent of λ, (see (4.8) and (4.5)). By the assumption (4.6), the equality $\tilde{k}_1(t)/\tilde{h}_1(t) = \tilde{k}_1(t')/\tilde{h}_1(t')$ implies $t = t'$. Then we have $h_1(t, \lambda) = h_1(t, \lambda')$. By (4.8), this implies $\lambda^{6\alpha} = \lambda'^{6\alpha}$. If $\alpha = 1/6p$ for a non-zero integer p, then $\lambda^{1/p} = \lambda'^{1/p}$. Hence $\lambda = \lambda'$, a contradiction. Hence S'^{-1} is single-valued.

It is clear that if S'^{-1} is single-valued, then so is S^{-1} on the set

$$S(\boldsymbol{P}^2 - C \cup L_\infty - \{(x, y) \in \boldsymbol{C}^2 \mid xy = 0\}) .$$

By Lemma 1, S is locally biholomorphic. If there exist distinct points (x, y) and (x', y') in $\boldsymbol{P}^2 - C \cup L_\infty$ such that $S(x, y) = S(x', y')$, then there must exist disjoint neighborhoods U and U' of (x, y) and (x', y') in $\boldsymbol{P}^2 - C \cup L_\infty$, respectively, such that (i) $S(U) = S(U')$ and (ii) $S\colon U \to S(U)$ and $S\colon U' \to S(U)$ are biholomorphic. Since the set $\{(x, y) \in \boldsymbol{C}^2 \mid xy = 0\}$ is nowhere dense in $\boldsymbol{P}^2$, there must exist a point (x_1, y_1) in U (resp. (x'_1, y'_1) in U') with $x_1 y_1 \neq 0$ (resp. $x'_1 y'_1 \neq 0$) such that $S(x_1, y_1) = S(x'_1, y'_1)$. Thus S^{-1} is single-valued on $S(\boldsymbol{P}^2 - C \cup L_\infty)$, if S'^{-1} is single-valued. q.e.d.

Under the assumption of Theorem 3, we write

$$S^{-1}\colon (u, v) \mapsto (x, y) = (x(u, v), y(u, v)) .$$

Then the functions $x(u, v)$ and $y(u, v)$ are automorphic with respect to the monodromy group. That is, putting

$$R(\gamma) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for $\gamma \in \pi_1(\boldsymbol{P}^2 - C \cup L_\infty, p_o)$, where $R\colon \pi_1(\boldsymbol{P}^2 - C \cup L_\infty, p_o) \to GL(2, \boldsymbol{C})$ is the monodromy representation of the equation (3.2) with Ω as in (3.8), we have

$$x(au + bv, cu + dv) = x(u, v) \quad \text{and} \quad y(au + bv, cu + dv) = y(u, v) .$$

For example, if

$$\alpha = 1/6,\ \beta = 0 \ \text{ and } \ \gamma = 9/2 , \qquad (\text{i.e., } p = 1,\ q = 6) ,$$

then, for a suitable choice of linearly independent solutions (f_1, f_2) and (g_1, g_2) of the equation (3.2) with Ω as in (3.8), the functions $x(u, v)$ and $y(u, v)$ satisfy

$$x(\zeta u,\ -u + \zeta^2 v) = x(-u, v) = x(u, v) ,$$
$$y(\zeta u,\ -u + \zeta^2 v) = y(-u, v) = y(u, v) ,$$

where

$$\zeta = \exp(2\pi\sqrt{-1}/6) .$$

(See (3.9) and (3.10).)

5. Branched finite Galois coverings. A branched finite covering of a connected compact complex manifold M is, by definition, an irreducible normal complex space X together with a surjective proper finite holomorphic map $\pi\colon X \to M$. The sets

$$R_\pi = \{p \in X \mid \pi^*\colon \mathcal{O}_{M,\pi(p)} \to \mathcal{O}_{X,p} \text{ is not isomorphic}\} ,$$
$$B_\pi = \pi(R_\pi) ,$$

where $\mathcal{O}_{X,p}$ is the local ring of germs at p of holomorphic functions, are hypersurfaces of X and M, called the ramification locus and the branch locus of π, respectively. For a non-singular point q of B_π, every point p in $\pi^{-1}(q)$ is non-singular as a point of both $\pi^{-1}(B_\pi)$ and X. Choosing suitable local coordinate systems $(z_1, \cdots, z_n)$ around $p = (0, \cdots, 0)$ and $(w_1, \cdots, w_n)$ around $q = (0, \cdots, 0)$ such that

$$\pi^{-1}(B_\pi) = \{(z_1, \cdots, z_n) \mid z_n = 0\} ,$$
$$B_\pi = \{(w_1, \cdots, w_n) \mid w_n = 0\} ,$$

locally, we can write the map π locally as

$$\pi\colon (z_1, \cdots, z_n) \to (w_1, \cdots, w_n) = (z_1, \cdots z_{n-1}, z_n^e)$$

for a positive integer e, which is constant on each irreducible component C of $\pi^{-1}(B_\pi)$ and is called the ramification index of π along C. For any irreducible hypersurface C' of X which is not contained in $\pi^{-1}(B_\pi)$, the ramification index of π along C' is defined to be one.

For branched finite coverings $\pi\colon X \to M$ and $\pi'\colon X' \to M$, a morphism (resp. isomorphism) of π to π' is a surjective holomorphic (resp. biholomorphic) map

$$\varphi\colon X \to X'$$

such that $\pi = \pi' \circ \varphi$. The group G_π of all isomorphisms of π to itself is called the covering transformation group. $\pi\colon X \to M$ is said to be a Galois covering if G_π acts transitively on every fiber of π.

Let D_j $(1 \leqq j \leqq s)$ be distinct irreducible hypersurfaces of M. For positive integers e_j $(1 \leqq j \leqq s)$, put

$$B = D_1 \cup \cdots \cup D_s \qquad \text{(a hypersurface of } M) ,$$
$$D = e_1 D_1 + \cdots + e_s D_s \qquad \text{(a positive divisor on } M) .$$

A branched finite covering $\pi: X \to M$ is said to branch along D (resp. at most along D) if (i) $B_\pi = B$ (resp. $B_\pi \subset B$) and (ii) for every irreducible component C of $\pi^{-1}(B_j)$, the ramification index of π along C is e_j (resp. divides e_j) for $1 \leqq j \leqq s$.

Denote also by σ_j $(1 \leqq j \leqq s)$ the homotopy classes of the loops σ_j defined in § 2. (See Figure 1.) Let

$$J = \langle \sigma_1^{e_1}, \cdots, \sigma_s^{e_s} \rangle^{\pi_1}$$

be the smallest normal subgroup of $\pi_1(M-B, p_o)$ which contains $\sigma_1^{e_1}, \cdots, \sigma_s^{e_s}$. For the proof of the following theorem, see Namba [4].

THEOREM 4. *There is a one-to-one correspondence* $\pi \mapsto N = N(\pi)$ *between the set of all isomorphism classes of finite Galois coverings* $\pi: X \to M$ *which branch at most along* D *and the set of all normal subgroups* N *of* $\pi_1(M-B, p_o)$ *of finite index such that* $J \subset N$. *The correspondence satisfies* (i) $G_\pi \simeq \pi_1(M-B, p_o)/N(\pi)$ *and* (ii) π *branches along* D *if and only if, for every* j $(1 \leqq j \leqq s)$, *the following condition for* $N(\pi)$ *is satisfied*:

$$\sigma_j^d \in N(\pi) \quad \textit{if and only if} \quad d \equiv 0 \pmod{e_j} .$$

We recall the following theorem of Selberg [5], (see also Borel [1]):

THEOREM 5 (Selberg). *For any finitely generated subgroup* $\Gamma(\neq\{1\})$ *of* $GL(r, \boldsymbol{C})$, *there exists a normal torsion free subgroup* H $(\neq \Gamma)$ *of* Γ *of finite index.*

Combining Theorems 4 and 5, we have:

THEOREM 6. *Assume that* $\pi_1(M-B, p_o)$ *is finitely generated. Suppose that there exists a homomorphism* $R: \pi_1(M-B, p_o) \to GL(r, \boldsymbol{C})$ *such that* $R(\sigma_j)$ *has order* e_j *for* $1 \leqq j \leqq s$. *Then we have a finite Galois covering* $\pi: X \to M$ *which branches along* $D = e_1 D_1 + \cdots + e_s D_s$.

REMARK. If $M = \boldsymbol{P}^n$, then $\pi_1(\boldsymbol{P}^n - B, p_o)$ is generated by $\sigma_1, \cdots, \sigma_s$ and a finite number of their conjugates. M. Oka informed us that $\pi_1(M-B, p_o)$ is finitely generated in general, if M is a projective manifold.

Now we apply Theorem 6 to the monodromy represention

$$R: \pi_1(\boldsymbol{P}^2 - C \cup L_\infty, p_o) \to GL(2, \boldsymbol{C})$$

of the differential equation (3.2), where Ω is given by (3.8) and satisfies the condition of Theorem 3. Suppose that $q \neq +\infty$.

By (3.9) and (3.10), the orders of $R(\sigma)$ and $R(\tau)$ are given by

$$\operatorname{ord} R(\sigma) = \operatorname{ord} R(\sigma') = m_0 \ , \quad \operatorname{ord} R(\tau) = 2|p| \ ,$$

where m_0 is the smallest among positive integers m such that $m/6p + m/12 \pm m/2q$ are integers. In particular, putting $\beta = 0$ (i.e., $q = 1/6$), we have $\operatorname{ord} R(\sigma) = 6|p|$. Thus we have:

THEOREM 7. *For any positive integer k, there exists a finite Galois covering $\pi: X \to \boldsymbol{P}^2$ which branches along $6kC + 2kL_\infty$.*

6. A generalization. For positive integers a and b with $a \geqq 2$ and $a \geqq b$, let C be the closure in $\boldsymbol{P}^2$ of the affine curve

$$\{(x, y) \mid f(x, y) = x^a - y^b = 0\} .$$

For non-negative integers k and l, consider the following differential equation

$$dY = Y\Omega \ , \tag{6.1}$$

where

$$\Omega = \begin{pmatrix} \alpha df/f & \beta x^k \omega/f \\ \gamma x^l \omega/f & \delta df/f \end{pmatrix}$$

for complex numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and $\omega = ydx + xdy$. Then Ω is holomorphic on $\boldsymbol{P}^2 - C \cup L_\infty$. As in Theorem 2, we have:

THEOREM 8. *The equation* (6.1) *is of Fuchsian type if and only if* (i) $\varepsilon = -b/a$, (ii) $\beta(1 - (k+1)/a - 1/b - \alpha + \delta) = 0$, (iii) $\gamma(1 - (l+1)/a - 1/b + \alpha - \delta) = 0$, (iv) $k \leqq a - 2$ *if* $\beta \neq 0$ *and* (v) $l \leqq a - 2$ *if* $\gamma \neq 0$.

In particular, let us assume

$$\beta = 0, \ \gamma \neq 0, \ a > b, \ l = a - 2 \quad \text{and} \quad \alpha = 1/em \ ,$$

where e is the least common multiple of a and b, and m is a positive integer. Then we have the following generalization of Theorem 7.

THEOREM 9. *Assume $a > b$. Then, for any positive integer m, there exists a finite Galois covering $\pi: X \to \boldsymbol{P}^2$ which branches along $em(C_1 + \cdots + C_s) + (e/a)mL_\infty$, where e is the least common multiple of a and b, and $C = C_1 \cup \cdots \cup C_s$ is the irreducible decomposition of C.*

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