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POLARIZED PERIOD MAP FOR GENERALIZED $K3$ SURFACES AND THE MODULI OF EINSTEIN METRICS

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0. Introduction. The moduli space for marked polarized $K3$ surfaces or equivalently the moduli space for marked $K3$ surfaces with a Ricci-flat Einstein-Kähler metric is constructed in [T1] and [L]. This moduli space is isomorphic to an *open dense* subset $K\Omega^0$ of

$$K\Omega := SO_0(3, 19)/SO(2)\times SO(19) .$$

So, it is natural to ask what geometric objects correspond to the *"hole"* $K\Omega\setminus K\Omega^0$ of the moduli space. The purpose of the present paper is to make some contribution to this question from differential geometric point of view. Namely we consider the polarized period map for $K3$ surfaces with simple singular points. The flavor of our main result is most typical in the following:

THEOREM 7. *The moduli space of all Einstein metrics on a $K3$ surface, including Einstein-orbifold-metrics along simple singular points, is isomorphic to*

$$\Gamma\setminus(SO_0(3, 19)/SO(3)\times SO(19)) ,$$

where Γ is the full group of isometries of the $K3$ lattice

$$2(-E_8)\oplus 3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} .$$

For the proof of this theorem we need two main ingredients, one from algebraic geometry and the other from differential geometry. The algebro-geometric ingredient is the contribution due mainly to Todorov [T1], Looijenga [L], and the generalization of their arguments by Morrison [Mr1] which is very important in the present paper. The differential geometric ingredient is the solution of Calabi's conjecture due to Yau [Ya1] and the equivariant version of it which asserts the existence of a Ricci-flat Einstein-Kähler orbifold-metric on certain complex orbifolds. The existence of a Ricci-flat Einstein-Kähler orbifold-metric makes it possible to use the *"isometric* deformation" of Kähler structures on generalized $K3$ surfaces.

Einstein-Kähler orbifold-metrics were also used to characterize the *ball quotients* of finite volume in terms of numerical invariants of orbifolds [CY] and [Kb]. In differential geometric words, that is the criterion for the vanishing of the anti-self-dual Weyl tensor of the Einstein-Kähler orbifold-metric under consideration. In exactly the same spirit, we will obtain the criterion for the vanishing of the full curvature tensor of a Ricci-flat Einstein-Kähler orbifold-metric in terms of the numerical invariants of the orbifold:

THEOREM 9. *Let X be a complex surface with at worst simple singularities whose minimal resolution is a $K3$ surface. Then,*

$$24-\sum_{p\in \operatorname{Sing} X}\left(e(E_p)-\frac{1}{|G_p|}\right)\geqq 0\,,$$

where E_p and $|G_p|$ are the exceptional divisor for the minimal resolution for $p\in \operatorname{Sing} X$ and the order of the corresponding local fundamental group G_p, respectively. The equality holds if and only if X is obtained as the quotient of a complex two torus with respect to the discrete group of Euclidean motions.

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1. Review on the moduli of $K3$ surfaces and the formulation of the problem. A $K3$ *surface* is a compact complex surface X which is connected and simply connected and has trivial canonical bundle K_X, i.e., X has a unique (up to constant) nowhere vanishing holomorphic 2-form ω_X. The notion of a $K3$ surface is invariant under deformation, i.e., any deformation of a $K3$ surface is a $K3$ surface [Kd]. Moreover any two $K3$ surfaces are deformations of each other [Kd]. So, there exists a unique underlying differentiable manifold of $K3$ surfaces which turns out to be a smooth quartic surface in $\boldsymbol{P}_3(\boldsymbol{C})$. Hence the lattice $H^2(X;\boldsymbol{Z})$ with the cup bilinear form is the same for all $K3$ surfaces X and can be called the $K3$ *lattice*. $K3$ lattice L is the unique even unimodular lattice of rank 22 and index -16, i.e.,

$$L=2(-E_8)\oplus 3\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

where E_8 is the even unimodular positive definite lattice associated with the Dynkin diagram of type E_8.

DEFINITION. A choice of an isometry

$$\alpha: H^2(X; \boldsymbol{Z}) \xrightarrow{\sim} L$$

is called a *marking* for X. A pair (X, α) of a $K3$ surface X and a marking α is called a *marked* $K3$ surface.

Let (X, α) be a marked $K3$ surface. $H^{2,0}(X)$ is a $\boldsymbol{C}$-vector subspace of $H^2(X; \boldsymbol{C}) \cong L_C$ of dimension one generated by ω_X, which satisfies

$$\langle \omega_X, \omega_X \rangle = 0 \quad \text{and} \quad \langle \omega_X, \bar{\omega}_X \rangle > 0 .$$

We can thus associate to (X, α) a point $[\alpha_C(\omega_X)]$ in the *classical period domain*

$$\Omega = \{\omega \in L_C; \langle \omega, \omega \rangle = 0 , \quad \langle \omega, \bar{\omega} \rangle > 0\}/\boldsymbol{C}^*$$

which is an open subset of a hyperquadric in $\boldsymbol{P}_{21}(\boldsymbol{C})$.

The *classical period map* for marked $K3$ surfaces is the map sending a marked $K3$ surface (X, α) to the point $[\alpha_C(\omega_X)]$ in Ω.

Another description for Ω which we will use later is the following: $\Omega =$ {all oriented two-planes $E \subset L_R$ such that $\langle\ ,\ \rangle|_E$ is positive definite}. In this description for Ω the classical period map for marked $K3$ surfaces (X, α) is expressed as

$$(X, \alpha) \mapsto \begin{bmatrix} \alpha_R\text{-image of the two-plane } E_X \text{ in } H^2(X; \boldsymbol{R}) \text{ spanned by} \\ \operatorname{Re}\omega_X \text{ and } \operatorname{Im}\omega_X, \text{ where } \omega_X \text{ is a generator of } H^{2,0}(X) \\ \text{and the orientation of } E_X \text{ is given by } (\operatorname{Re}\omega_X, \operatorname{Im}\omega_X). \end{bmatrix} .$$

The classical period domain parametrizes effectively the local universal deformation (Kuranishi family) for any $K3$ surface X. If $p: (\mathfrak{X}, X) \to (S, 0)$ is the Kuranishi family for X, we then have a diffeomorphism $t: X \times S \to \mathfrak{X}$ such that $p \circ t = pr_2$. Once we choose a marking $\alpha: H^2(X; \boldsymbol{Z}) \to L$, we get a *marking for the family* by setting

$$\begin{array}{ccc} \alpha_s := \alpha \circ t^*: H^2(X_s; \boldsymbol{Z}) & \longrightarrow & L \\ {\scriptstyle t^*} \searrow & & \nearrow {\scriptstyle \alpha} \\ & H^2(X; \boldsymbol{Z}) . & \end{array}$$

We thus get a *marked* Kuranishi family $(\mathfrak{X} \to S, \alpha)$, which has a *period map* $\tau_S: S \to \Omega$, defined by

$$S \ni s \mapsto \tau_S(s) = [\alpha_{sC}(\omega_{X_s})] .$$

Now the local Torelli theorem due to Andreotti-Weil and Kodaira [Kd] states that the map τ_S is holomorphic and a local isomorphism at 0. Every point $x \in \Omega$ determines a Hodge structre of weight 2 on L in the following way: If $\omega \in L_C$ is a representative for x, we define

$$H^{2,0}(x) = C\omega \subset L_C$$
$$H^{0,2}(x) = C\bar{\omega} \subset L_C$$
$$H^{1,1}(x) = (H^{2,0}(x) + H^{0,2}(x))^{\perp} \subset L_C .$$

Although the local Torelli theorem is true, one *cannot* construct a universal family of $K3$ surfaces on Ω. In fact, Atiyah [At] constructed two non-isomorphic families of $K3$ surfaces with the same period map. The reason is that the classical period domain sees only Hodge structures, although the rational curves on a $K3$ surface play an essential role in the construction of the fine moduli space. The following construction due to Burns-Rapoport [BuR] clarified the importance of the rational curves on a $K3$ surface. For $x \in \Omega$, let $V^+(x)$ be one of the connected components of $V(x) = \{\kappa \in H^{1,1}(x) \cap L_R; \langle\kappa, \kappa\rangle = 1\}$, and let $\Delta(x) = \{\delta \in H^{1,1}(x) \cap L; \langle\delta, \delta\rangle = -2\}$ and $V_\Delta^+(x) = \{\kappa \in V^+(x); \langle\kappa, \delta\rangle \neq 0$ for all $\delta \in \Delta(x)\}$. Since Ω is simply connected, it is possible to make a *continuous* choice of $V^+(x)$ with respect to $x \in \Omega$. For a $K3$ surface X, we define:

$$\Delta(X) = \{\delta \in H^{1,1}(X) \cap H^2(X; \boldsymbol{Z}); \langle\delta, \delta\rangle = -2\}$$

and $\Delta^+(X) = \{$all effective $\delta \in \Delta(X)$, i.e., δ corresponds to an effective divisor on $X\}$. By the Riemann-Roch theorem, δ or $-\delta$ is effective for all $\delta \in \Delta(X)$. So, $\Delta(X) = \Delta^+(X) \cup (-\Delta^+(X))$ and if $\delta_1, \cdots, \delta_k \in \Delta^+(X)$ and $\delta = \sum n_i\delta_i$ with $\boldsymbol{Z} \ni n_i \geqq 0$ then $\delta \in \Delta^+(X)$. Let $V^+(X)$ be the connected component of $V(X) = \{\kappa \in H^{1,1}(X) \cap H^2(X; \boldsymbol{R}); \langle\kappa, \kappa\rangle = 1\}$ which contains a Kähler metric on X. We define

$$V_P^+(X) = \{\kappa \in V^+(X); \langle\kappa, \kappa\rangle = 1,\ \langle\kappa, \delta\rangle > 0 \quad \text{for all} \quad \delta \in \Delta^+(X)\}$$

for a $K3$ surface X. The half cone

$$\begin{aligned}\mathscr{C}_P^+(X) &= \boldsymbol{R}^+ \times V_P^+(X)\\ &= \{\kappa \in H^{1,1}(X) \cap H^2(X; \boldsymbol{R}); \langle\kappa, \kappa\rangle > 0,\ \langle\kappa, \delta\rangle > 0 \quad \text{for all} \quad \delta \in \Delta^+(X)\}\end{aligned}$$

over $V_P^+(X)$ is the *Kähler cone* for X. Note that every $K3$ surface admits a Kähler metric (cf. [Si]), i.e., $V_P^+(X) \neq \varnothing$. The Kähler cone $\mathscr{C}_P^+(X)$ for a Kähler surface X is originally defined by

$$(*) \qquad \mathscr{C}_P^+(X) = \begin{Bmatrix}\kappa \in H^{1,1}(X) \cap H^2(X; \boldsymbol{R}); \langle\kappa, \kappa\rangle > 0 \text{ and } \langle\kappa, \delta\rangle > 0\\ \text{for all effective classes } \delta \in H^{1,1}(X) \cap H^2(X; \boldsymbol{Z})\end{Bmatrix}.$$

But in the case of $K3$ surfaces, it is sufficient to check the property

(*) for (-2)-effective classes $d \in \Delta^+(X)$. See, for example, [LP]. Burns-Rapoport [BuR] defined the *Burns-Rapoport period domain* $\tilde{\Omega}$ in the following way. Define two fiber spaces $K\Omega^0$, $K\Omega$ over Ω by

$$K\Omega = \{(\kappa, [\omega]) \in L_R \times \Omega;\ \kappa \in V^+(x)\}\ , \quad \text{and}$$
$$K\Omega^0 = \{(\kappa, [\omega]) \in K\Omega;\ \kappa \in V_\Delta^+(x)\}\ , \quad \text{where} \quad x = [\omega]\ .$$

Let $\pi: K\Omega^0 \to \Omega = (K\Omega)^0/\sim$ be the quotient map defined by $(\kappa, \omega) \sim (\kappa', \omega')$ if and only if $\omega = \omega'$ with κ and κ' being in the same connected component of the fiber $pr_2^{-1}([\omega])$ in $K\Omega^0$. For each $x \in \Omega$, define the subgroup $W(x)$ of $\mathrm{Aut}(L) \cap SO(H^{1,1}(x) \cap L_R)$ generated by the reflections

$$s(\delta): x \mapsto x + \langle x, \delta\rangle\delta\ ,$$

where δ runs over the whole $\Delta(x)$. Since $H^{1,1}(x) \cap L_R$ has signature $(1, 19)$, $W(x)$ acts *properly discontinuously* on the hyperbolic 19-space $V^+(x)$ (cf. [Vn]). The set of fundamental domains in $V^+(x)$ is in one-to-one correspondence with the set of the partitions of $\Delta(x)$ into $\Delta^+(x)$ and $-\Delta^+(x)$ with the property that if $\delta_1, \cdots, \delta_k \in \Delta^+(x)$ and $\delta = \sum n_i\delta_i$ with $\boldsymbol{Z} \ni n_i \geqq 0$, then $\delta \in \Delta^+(x)$. For a partition $P: \Delta(x) = \Delta^+(x) \cup -(\Delta^+(x))$, the corresponding fundamental domain $V_P^+(X)$ is $\{x \in V^+(x);\ \langle x, \delta\rangle > 0 \text{ for all } \delta \in \Delta^+(x)\}$ which turns out to be a *locally finite* "polyhedron" whose sides are given by hyperplanes $H_\delta = \{\delta\}^\perp$ for $\delta \in \Delta(x)$. The *Burns-Rapoport period map* associates to each marked $K3$ surface (X, α) the point in $\tilde{\Omega}$ determined by $\pi(\alpha_R(\kappa), [\alpha_R(\omega_X)])$, where κ is a Kähler class on X. For this period map, the following is known (cf. [BuR]):

THE GLOBAL TORELLI THEOREM. *Let X and X' be two $K3$ surfaces. If there is an isometry $\phi: H^2(X'; \boldsymbol{Z}) \to H^2(X; \boldsymbol{Z})$ satisfying $\phi_C([\omega_X]) = c[\omega_{X'}]$ for some $c \in \boldsymbol{C}^*$ and $\phi_R(V_P^+(X')) = V_P^+(X)$, then there is a unique isomorphism $\phi: X \to X'$ with $\phi^* = \phi$.*

This was first proved by Prateckii-Shapiro and Shafarevich [ShP] in the algebraic case and refined in the Kählerian case by Burns-Rapoport [BaR], simplified by Looijenga-Peters [LP]. This theorem means that any two marked $K3$ surfaces having the same Burns-Rapoport periods are isomorphic in a unique way. For the surjectivity of this period map, Todorov [T1] proved:

SURJECTIVITY THEOREM. *For every $\tilde{x} \in \tilde{\Omega}$, there is a marked $K3$ surface whose Burns-Rapoport period is $\tilde{x}$.*

For the proof, he used Yau's solution to Calabi's conjecture i.e., isometric deformations of Kähler structures with respect to a Calabi-Yau metric. The same technique is used in this paper, but now for a Ricci-flat *orbifold*-metric. We can thus use the Local Torelli Theorem, the Global Torelli Theorem, and the Surjectivity Theorem to glue up marked Kuranishi families (which should be small enough to be embedded in $\tilde{\Omega}$) via the Burns-Rapoport period map to identify $\tilde{\Omega}$ with the fine moduli space for marked $K3$ surfaces. As was shown by Atiyah [At1] (see also [LP]), the space $\tilde{\Omega}$ is *not* a Hausdorff space. Moreover, $\mathrm{Aut}(L)$ *cannot* act on $\tilde{\Omega}$ in a properly discontinuous fashion. Morrison [Mr1] made a great progress in avoiding such unsatisfactory properties of $\tilde{\Omega}$ by introducing the *polarized period map* for *generalized* $K3$ *surfaces* instead of Burns-Rapoport period map for *smooth* $K3$ surfaces. The following definitions are due to Morrison [Mr1].

DEFINITION. A compact complex surface X is called a *generalized* $K3$ *surface* if X has at worst simple singular points and its minimal resolution Y is a $K3$ surface.

DEFINITION. Let X be a generalized $K3$ surface and $\rho: Y \to X$ its minimal resolution. Let $\delta_1, \cdots, \delta_k \in H^{1,1}(Y) \cap H^2(Y; \boldsymbol{Z})$ be the classes of all (-2)-curves contracted by ρ. The *root system* $R(X)$ and the *Weyl group* $W(X)$ of X are defined by

$$R(X) := \{\delta = \sum_{i=1}^{k} a_i\delta_i \in H^2(Y; \boldsymbol{Z});\ a_i \in \boldsymbol{Z},\ \langle\delta, \delta\rangle = -2\} \quad \text{and}$$

$$W(X) := \text{the group generated by } \{s(\delta);\ \delta \in R(X)\} \subset \mathrm{Isometry}(H^2(Y; \boldsymbol{Z}))\ .$$

DEFINITION. We let $I(X) := H^2(Y; \boldsymbol{Z})^{W(X)}$, i.e., the set of all classes orthogonal to $R(X)$. Note that $I(X)_C$ contains $H^{2,0}(Y)$ and so determines a Hodge structure on $H^2(Y; \boldsymbol{Z})$

DEFINITION. A metric injection

$$\alpha: I(X) \to L$$

is a *marking* of X if α is extendable to an isometry $\bar{\alpha}$ of $H^2(Y; \boldsymbol{Z})$ to L. A pair (X, α) is a *marked generalized* $K3$ *surface*.

DEFINITION. For a generalized $K3$ surface X, we let

$$I(X) \supset V_P^+(X) := \left\{\begin{matrix}\kappa \in \overline{V_P^+(Y)};\ \text{for all } \delta \in H^{1,1}(Y) \cap H^2(Y; \boldsymbol{Z}) \text{ with} \\ \langle\delta, \delta\rangle = -2,\ \langle\kappa, \delta\rangle = 0 \text{ if and only if } \delta \in R(X)\end{matrix}\right\},$$

where $V_P^+(Y) = \{\kappa \in H^{1,1}(Y) \cap H^2(Y; \boldsymbol{R});\ \langle\kappa, \kappa\rangle = 1 \text{ and } \langle\kappa, \delta\rangle > 0$ for all effective (-2)-classes δ on $Y\}$ as before. The *Kähler cone* $\mathscr{C}_P^+(X)$ is defined by

$$\mathscr{C}_P^+(X) := \boldsymbol{R}^+ \times V_P^+(X)\ .$$

DEFINITION. An element $\phi \in V_P^+(X)$ is called a *polarization* on X. A triple (X, ϕ, α) is a marked *polarized generalized* $K3$ *surface*. The *polarized period map* p for marked polarized generalized $K3$ surfaces sends (X, ϕ, α) to $p(X, \phi, \alpha) = (\alpha_R(\phi), [\alpha_C(\omega_Y)] \in K\Omega$.

For this map, Morrison [Mr] proved the Polarized Global Torelli Theorem for marked polarized generalized $K3$ surfaces:

THEOREM A (cf. [Mr1]). *Let (X, ϕ) and (X', ϕ') be two polarized generalized $K3$ surfaces and let $\rho: Y \to X$ and $\rho': Y' \to X'$ be their minimal resolutions. Suppose $\gamma: I^2(X') \to I^2(X)$ is an isometry such that $\gamma_C(H^{2,0}(Y')) = H^{2,0}(Y)$, $\gamma_R(\phi') = \phi$, and extends to an isometry $\bar{\gamma}: H^2(Y', \boldsymbol{Z}) \to H^2(Y; \boldsymbol{Z})$. Then there is a unique isomorphim $\phi: X \to X'$ such that $\Phi^* = \gamma$. Φ comes from a unique isomorphism $\bar{\Phi}: Y \to Y'$ which induces isomorphisms of exceptional sets for ρ and ρ'.*

For the surjectivity of this period map, there is a strong result due to Looijenga [L] (see also [T1] and [Na]):

THEOREM B (cf. [L]). *For every $(\kappa, x) \in K\Omega^0$, there is a marked polarized $K3$ surface (X, ϕ, α) such that $p(X, \phi, \alpha) = (\kappa, x)$ and the polarization ϕ contains a Kähler metric.*

So it may be natural to ask what geometric objects correspond to the hole $K\Omega \setminus K\Omega^0$ of the moduli space of marked polarized generalized $K3$ surfaces. Morrison proved the following weak version of the Surjective Theorem:

THEOREM C (cf. [Mr]). *For every $(\kappa, x) \in K\Omega$, there is a marked polarized generalized $K3$ surface (X, ϕ, α) such that $p(X, \phi, \alpha) = (\kappa, x)$.*

Yau's solution to Calabi's conjecture tells us that $K\Omega^0$ is the moduli space for marked Einstein-Kähler $K3$ surfaces. On the other hand a point in the hole $K\Omega \setminus K\Omega^0$ corresponds to a $K3$ surface Y and a class $\phi \in H^{1,1}(Y)$ with $\langle \phi, \phi \rangle = 1$ such that the area of some effective curves are zero. So, the problem is to find the *singular* Ricci-flat Einstein-Kähler metric corresponding to ϕ. This question was asked by several authors (cf. [Be], [Mr1]). We shall solve this problem in the following sections.

2. Ricci-flat orbifold-metrics on generalized $K3$ surfaces.

In [Ya2], Yau presented some results for the existence of a sigular Ricci-flat Kähler metric on certain complex manifolds. Since [Ya2] is not published yet as far as the authors know, we include the proof of the equivariant version of the Calabi-Yau theorem in this section. There may be many ways arranging the material involved in Yau's proof of Calabi's conjecture. Here, we shall prove the simplest version sufficient for our purposes, namely filling the "holes" of the moduli space of Einstein metrics on a $K3$ surface.

THEOREM 1. *Let X be a compact complex surface with at worst isolated quotient singularities. Let $Y \to X$ be the minimal resolution and $D = \sum_i D_i$ its exceptional sets decomposed into irreducible components. Choose non-negative rational numbers μ_i less than one such that $K_Y + \sum_i \mu_i D_i \sim 0$ near D (such μ_i's are uniquely determined). Assume that some tensor power of $K_Y + \sum_i \mu_i D_i$ is a trivial line bundle over Y. Then for any Kähler form ϕ in the sense of Fujiki-Moišezon (if exists) on X, we can find a unique real-valued orbifold-smooth function U on X up to additive constants such that $\phi + \sqrt{-1}\partial\bar{\partial} U$ is a Ricci-flat Einstein-Kähler orbifold-metric form on X. Moreover $\phi + \sqrt{-1}\partial\bar{\partial} U$ defines a closed current on Y cohomologous to ϕ in $H^2(Y; \boldsymbol{R})$.*

For the proof, we follow Yau's proof [Ya] of Calabi's conjecture partially simplified by Bourguignon [Bo2], namely the simple proof for the C^0-estimate. It is easy to see that there exists a real valued orbifold-smooth function U_0 such that $\phi + \sqrt{-1}\partial\bar{\partial} U_0 =: \phi_0$ is an orbifold-Kähler form. Since the resolution of quotient singularities involve only polynomial functions and there exist nonnegative rational numbers μ_i such that $K_Y + \sum_i \mu_i D_i$ is trivial near D, the following estimates hold:

$$dz \approx \mathcal{O}(\gamma^{\varepsilon-1}) \ , \quad dw \approx \mathcal{O}(\gamma^{\varepsilon-1}) \ ,$$

where ε is a small positive number, (z, w) are holomorphic local coordinates near D and $\gamma = (|\lambda|^2 + |\mu|^2)^{1/2}$ is the distance function on the local uniformization $\boldsymbol{B}^2: (\lambda, \mu)$ of the quotient singularity corresponding to D. It follows that for any orbifold-smooth function U on X the current $\phi + \sqrt{-1}\partial\bar{\partial} U$ defines the same cohomology class as $\phi \in H^2(Y; \boldsymbol{R})$. Indeed, we have only to show that

$$\int_Y dd^c u \wedge \Psi = 0 \ ,$$

for any smooth closed 2-form Ψ on Y. This is equivalent to showing that

$$\lim_{r\to 0} \int_{S^3(r)} d^c\tilde{u} \wedge \tilde{\Psi} = 0 \ ,$$

where $\sim$ means form the lifting to the local uniformization and $S^3(r)$ is the sphere of radius r centered at the origin. But this is clear from the above estimates. Since some tensor power of $K_Y + \sum_i \mu_i D_i$ is trivial on Y, there exists a Ricci-flat volume form V on X which is orbifold-smooth. Thus we have

$$\phi_0^2 = e^{-f} V$$

for some orbifold-smooth function f on X. To find U in Theorem 1, we solve the following Monge-Ampère equation:

$$(1) \qquad (\phi_0 + \sqrt{-1}\partial\bar{\partial} U)^2 = e^f \phi_0^2 \quad \text{on} \quad X .$$

The proof for the uniqueness of the orbifold-smooth U is exactly the same as in [Ya] and [Bo2]. To solve the equation (1) we use the continuity method.

Let $C^{k,\alpha}(X)$ be the Banach space of all C^k functions on X whose k-th derivative are Hölder continuous of exponent α. This means that any element f in $C^{k,\alpha}(X)$ is of class $C^{k,\alpha}$ on local uniformations. The norm on $C^{k,\alpha}(X)$ is defined in a completely analogous way as in the usual Hölder space. Consider the one-parameter family of equations:

$$(1)_t \qquad (\phi_0 + \sqrt{-1}\partial\bar{\partial} U)^2 = e^{tf}\left(\frac{\int_X \phi_0^2}{\int_X e^{tf}\phi_0^2}\right)\phi_0^2 .$$

$(1)_0$ has a solution 0 and $(1)_1$ is what we want to solve. We show that the non-empty set $A = \{t \mid t \in [0, 1]$ and $(1)_t$ has a solution in $C^{k,\alpha}(X)\}$ is open and closed. Let E be defined by $E = \left\{u \mid u \in C^{k,\alpha}(X) \text{ and } \int_X u\phi_0^2 = 0\right\}$, a closed subspace of $C^{k,\alpha}(X)$. Suppose $u \in C^{k,\alpha}(X)$ is a solution of (1) and let $\tilde{\phi} := \phi_0 + \sqrt{-1}\partial\bar{\partial} U$. Define

$$H := \left\{h \mid h \in C^{k-2,\alpha}(X) \quad \text{and} \quad \int_X h\tilde{\phi}^2 = 0\right\} .$$

The openness of A follows from:

LEMMA. *$\Delta_{\tilde{\phi}}: E \to H$ is an isomorphism.*

PROOF. Since $\int_X v\Delta_{\tilde{\phi}} v = \int_X |dv|^2_{\tilde{\phi}}$ is true in the orbifold category, the map $\Delta_{\tilde{\phi}}: E \to H$ is injective. To show the surjectivity it suffices to construct the Green function on the Riemannian orbifold $(X, \tilde{\phi})$. The standard technique to construct the Green functions works in our case. See, for example, [Au]. q.e.d.

The closedness follows from the a-priori C^0-estimate for U. For the complex Monge-Ampère equation $(\phi + \sqrt{-1}\partial\bar{\partial} U)^n = e^F \phi^n$ on the compact Kähler manifold, where ω and F are given, a C^0-bound for U is obtained in terms of the C^0-bound for F. The main ingredients are: (a) the construction of the Green function, and (b) the Sobolev inequality for L_1^2-function f with $\int f = 0$

$$c\left(\int |f|^{2n/(n-2)}\right)^{(n-2)/2n} \leqq \left(\int |df|^2\right)^{1/2}$$

on a compact Riemannian manifold. But (a) is carried out without difficulty in our case (see [Au]) and (b) is clearly true for some constant $c > 0$. We thus get:

LEMMA. *There is a C^0-bound for U in terms of the C^0-bound for F.*

Since C^2 and $C^{2,\alpha}$ estimates are carried out by local calculations and the classical maximum principle, exactly the same arguments as those in [Ya1] gives us the C^2 and $C^{2,\alpha}$ bounds for U in our case, i.e., in the orbifold category. Now the proof of Theorem 1 is complete.

In particular, any generalized $K3$ surface X with a Fujiki-Moišezon-Kähler form ϕ admits a unique Einstein-Kähler orbifold-metric form $\tilde{\phi}$ such that $[\tilde{\phi}] = [\phi]$ in $H^2(Y; \boldsymbol{R})$.

3. Isometric deformations.

THEOREM 2. *Suppose that X is a generalized $K3$ surface and $g_{\alpha\bar{\beta}}$ is an Einstein-Kähler orbifold-metric (necessarily Ricci-flat) on X. Then*

(1) *$X \times S^2$ has a complex structure $\mathfrak{X}$ such that*

(a) *the projection $\pi: \mathfrak{X} \to S^2 \cong \boldsymbol{P}_1(\boldsymbol{C})$ is a holomorphic map and fibers are generalized $K3$ surfaces. From $\mathfrak{X} \to \boldsymbol{P}_1(\boldsymbol{C})$ we can obtain a family of non-singular $K3$ surfaces $\tilde{\mathfrak{X}} \to \boldsymbol{P}_1(\boldsymbol{C})$;*

(b) *if (X, α) is a marked generalized $K3$ surface, then α induces an isomorphism of local systems (in fact trivial systems) $\alpha: R^2\pi_* \boldsymbol{Z}_{\tilde{\mathfrak{X}}} \xrightarrow{\sim} \boldsymbol{P}_1(\boldsymbol{C}) \times L$;*

(c) *for each $t \in \boldsymbol{P}_1(\boldsymbol{C})$ the period in Ω of $X_t = \pi^{-1}(t)$ is an oriented two-plane in the three dimensional space $E \subset L_{\boldsymbol{R}}$ spanned by $(\mathrm{Re}\,\omega_X, \mathrm{Im}\,\omega_X, \mathrm{Im}\, g_{\alpha\bar{\beta}})$ or more intrinsicaly three linearly independent parallel self-dual 2-forms with respect to the Ricci-flat orbifold metric.*

(2) *For each $t \in \boldsymbol{P}_1(\boldsymbol{C})$ the Ricci-flat Riemannian orbifold-metric (determined by $g_{\alpha\bar{\beta}}$) on X_t is orbifold-Kählerian with respect to the corresponding complex structure.*

(3) *The base space $\boldsymbol{P}_1(\boldsymbol{C})$ parametrizes all complex structures with respect to which g is Kähler.*

PROOF. The proof is based on the following two lemmas and the

Andreotti-Weil remark.

LEMMA 1. *The Kähler orbifold-metric $g_{\alpha\bar\beta}$ is Ricci-flat if and only if for a positive constant $c \in \boldsymbol{R}_+$ we have an equality*

$$\phi \wedge \phi = c\omega_X \wedge \bar\omega_X$$

of differentiable 4-forms, where $\phi = \operatorname{Im} g_{\alpha\bar\beta}$ is the Kähler form of g and ω_X is the holomorphic 2-form on X without zeros and poles.

Its proof is clear.

LEMMA 2. *Let (X, g) be as in Theorem 2. Let ρ be a (closed) 2-form written as*

$$\rho = a\omega_X + b\bar\omega_X + e\phi$$

where $a, b, e \in \boldsymbol{C}$. Then $\rho \wedge \rho = 0$ as a form if and only if $[\rho] \wedge [\rho] = 0$ as a cohomology class in $H^4(X; \boldsymbol{C})$ where $[\rho]$ denotes the cohomology class of ρ in $H^2(X; \boldsymbol{C})$.

PROOF. Since $\rho \wedge \rho = (2ab + e^2c)\omega_X \wedge \bar\omega_X$, $\rho \wedge \rho = 0 \leftrightarrow 2ab + e^2c = 0 \leftrightarrow [\rho]\wedge[\rho] = 0$. The last equivalence is because of $\int_X \omega_X \wedge \bar\omega_X > 0$. q.e.d.

ANDREOTTI-WEIL REMARK [W]. Let Y be an oriented (possibly non-compact) differentiable manifold of dimension 4. If there is a $\boldsymbol{C}$-valued 2-form ρ on X such that (a) $\rho \wedge \rho = 0$, (b) $\rho \wedge \bar\rho > 0$ everywhere, and (c) $d\rho = 0$, then Y admits a unique complex structure such that ρ is a holomorphic 2-form.

PROOF OF THEOREM 2. S^2 parametrizes all oriented two-planes in the three dimensional space $E \subset L_{\boldsymbol{R}}$ spanned by $\{\operatorname{Re}\omega_X, \operatorname{Im}\omega_X, \operatorname{Im} g_{\alpha\bar\beta}\}$. We may assume that $\{\operatorname{Re}\omega_X, \operatorname{Im}\omega_X, \operatorname{Im} g_{\alpha\bar\beta}\}$ is an orthonormal basis with respect to $\langle\ ,\ \rangle|_E$. Let E_t be any oriented 2-plane in E and let α, β be an orthonormal basis in E_t. Then we define

$$\omega_t = \alpha + i\beta\ .$$

Clearly $\omega_t \wedge \omega_t = 0$ and $\omega_t \wedge \bar\omega_t > 0$, since $\{\alpha, \beta\}$ is an orthonormal basis in E_t. So $\omega_t = a\omega_X + b\bar\omega_X + e \operatorname{Im} g_{\alpha\bar\beta}$ and it defines a new complex structure on X. It is known that if x is a simple singular point on X and U is a pseudo-convex neighborhood of x then $U = V/G$, where $V \subset \boldsymbol{C}^2$ and $G \subset SU(2)$. Let $\pi\colon V\setminus\{0\} \to V\setminus\{0\}/G \cong U\setminus\{x\}$. Then $\pi^*(\omega_t|V\setminus\{x\})$ can be prolonged to a 2-form on V invariant under the action of G. So in such a way we get a complex analytic family $\mathfrak{X} = \bigcup_{t\in S^2} X_t$. Here, we have used the Andreotti-Weil remark twice: first on $X\setminus\operatorname{Sing} X$ to show that ω_t defines a complex structure there: then on V (where $U = V/G$) to show that this extends to an orbifold-complex structure at the singular point of X. In fact the complex structure $\mathfrak{X}$ is nothing but the orbifold-twistor space for a half-conformally flat Riemannian 4-manifold [AtHS]. (Recall that any Kähler metric with vanishing Ricci tensor is anti-self-dual.) Now let ℓ_t be a vector orthogonal to E_t in E. Let $\alpha, \beta, \gamma \in \boldsymbol{R}$ be such that $\ell_t = \alpha \operatorname{Re} \omega_X + \beta \operatorname{Im} \omega_X + \gamma \operatorname{Im} g_{\alpha\bar{\beta}}$. Suppose that $\{\operatorname{Re} \omega_t, \operatorname{Im} \omega_t, \ell_t\}$ defines the same orientation on $E \subset \Gamma(X, \Lambda^+)$ as $\{\operatorname{Re} \omega_X, \operatorname{Im} \omega_X, \operatorname{Im} g_{\alpha\bar{\beta}}\}$. ℓ_t is a closed form of type (1, 1). $\det(\ell_t)$ vanishes nowhere on $X \setminus \operatorname{Sing} X$ and $\det(\ell_t) = c\omega_t \wedge \bar{\omega}_t = c\omega_X \wedge \bar{\omega}_X$. On the other hand, for each $x \in X \setminus \operatorname{Sing} X$ we can find $A \in SO(4)$ such that

$$d\ell^+(A)(g_{\alpha\bar{\beta}})^t d\ell^+(A) = \ell_t \quad \text{in} \quad T_x X ,$$

where ℓ^+ is the homomorphism of $SO(4)$ to $SO(3)$ determined by the decomposition of the second exterior representation of $SO(4)$ into irreducible subspaces. So ℓ_t is positive definite and is an Einstein-Kähler metric on $X \setminus \operatorname{Sing} X$ Riemannian equivalent to $g_{\alpha\bar{\beta}}$. Since $\operatorname{Re} \omega_X$, $\operatorname{Im} \omega_X$ and $\operatorname{Im} g_{\alpha\bar{\beta}}$ are smooth differential forms on X in the sense of orbifolds, ℓ_t defines an Einstein-Kähler orbifold-metric with respect to the new complex structure corresponding to ω_t. Notice that in the family $\mathfrak{X} \to S^2 \cong \boldsymbol{P}_1(\boldsymbol{C})$ we can resolve the singularities and get a family of non-singular $K3$ surfaces $\tilde{\mathfrak{X}} \to \boldsymbol{P}_1(\boldsymbol{C})$. The desingularization can be done by successive blow ups. Although there is no isomorphism $\tilde{\mathfrak{X}} \cong \boldsymbol{P}_1(\boldsymbol{C}) \times Y$ of C^∞-manifolds where Y is the minimal resolution of X, $\tilde{\mathfrak{X}}$ has the same additive cohomology as $\boldsymbol{P}_1(\boldsymbol{C}) \times Y$ (cf. [At2]). Hence a marking exists on $\tilde{\mathfrak{X}} \to \boldsymbol{P}_1(\boldsymbol{C})$. See also the arguments in Section 5. q.e.d.

In [Va], Varouchas proved the following:

FACT. Let X be an analytic variety admitting an open covering $\{U_i\}$ and a family of functions $\psi_j \colon U_j \to \boldsymbol{R}$ which are *continuous* and strictly plurisubharmonic such that $\psi_j - \psi_k$ is pluriharmonic on $U_j \cap U_k$. Then X is Kählerian in the sense of Fujiki-Moišezon.

Let (X, ϕ) be a generalized $K3$ surface with a Ricci-flat Kähler orbifold-metric form ϕ. In this situation the proof of Varouchas shows that we can find a Fujiki-Moišezon-Kähler form $\tilde{\phi}$ in the same cohomology class as ϕ.

4. Surjectivity of the polarized period map for generalized $K3$ surfaces. In this section, we prove the strong version of Morrison's Surjectivity Theorem (Theorem C). Namely, we show that every polarization ϕ in Theorem C contains a Kähler form on X in the sense of Fujiki-

Moišezon ([F], [Mo]). Let $\mathscr{M}$ be the set of all isomorphic classes of marked polarized generalized $K3$ surfaces under the following equivalence: $(X, \phi, \alpha) \sim (X', \phi', \alpha')$ if and only if there is an isomorphism $f\colon Y' \to Y$ which induces isomorphisms on the exceptional sets, such that $f^*(\phi) = \phi'$ and the diagram

$$\begin{array}{ccc} H^2(Y; \boldsymbol{Z}) & & \\ & \searrow^{\bar{\alpha}} & \\ f^* \downarrow & & L \\ & \nearrow_{\bar{\alpha}'} & \\ H^2(Y'; \boldsymbol{Z}) & & \end{array}$$

is commutative. Thus Theorems A and C are unified in the following:

THEOREM A+C. *The polarized period map* $p(X, \phi, \alpha) = (\alpha_{\boldsymbol{R}}(\phi), [\alpha_{\boldsymbol{C}}(\omega_X)])$ *descends to a bijection* $\tau\colon \mathscr{M} \to K\Omega$.

Now we define a subset $\mathscr{M}_1$ of $\mathscr{M}$ in the following way: the equivalence class of (X, ϕ, α) is an element of $\mathscr{M}_1$ if and only if ϕ is a Kähler class on X in the sense of Fujiki-Moišezon. The main result in this paper is:

STRONG SURJECTIVITY THEOREM. *The map* $\tau\colon \mathscr{M}_1 \to K\Omega$ *is surjective.*

Combining this with Theorem A, we have:

THEOREM 3. $\mathscr{M} = \mathscr{M}_1$, *i.e., every polarization* ϕ *for any generalized* $K3$ *surfaces contains a Kähler metric in the sense of Fujiki-Moišezon, and the map* $\tau\colon \mathscr{M}_1 = \mathscr{M} \to K\Omega$ *is bijective.*

PROOF OF STRONG SURJECTIVITY THEOREM. Let

$$\pi\colon K\Omega \to G_3^+(L_{\boldsymbol{R}})$$

be defined by $\pi(\kappa, [\omega]) = P_\omega + \boldsymbol{R} \cdot \kappa$, where P_ω is an oriented positive 2-plane in $L_{\boldsymbol{R}}$ whose oriented basis is $\{\operatorname{Re}\omega, \operatorname{Im}\omega\}$. $G_3^+(L_{\boldsymbol{R}})$ is the moduli space for oriented positive definite 3-planes in $L_{\boldsymbol{R}}$ which turns out to be the Riemannian symmetric space

$$SO_0(3, 19)/SO(3) \times SO(19) .$$

Using the isometric deformation of generalized $K3$ structures with respect to the Ricci-flat orbifold metric, we get the following:

LEMMA. *The image of* $\tau\colon \mathscr{M}_1 \to K\Omega$ *consists of fibers of* $\pi\colon K\Omega \to G_3^+(L_{\boldsymbol{R}})$.

PROOF OF LEMMA. Let $(\kappa, [\omega]) \in K\Omega$ be in the image of τ. We find a generalized $K3$ surface X and a Kähler form ϕ such that $\alpha_R([\phi]) = \kappa$ and $[\alpha_C(\omega_X)] = [\omega]$ for some marking α. From Theorem 1, we can find a unique Ricci-flat Einstein-Kähler orbifold-metric in the form of $\phi + \sqrt{-1}\partial\bar{\partial}U$ for an orbifold-smooth function U. The cohomology class of $\phi + \sqrt{-1}\partial\bar{\partial}U$ (in the sense of currents) is the same as $[\phi]$. Now by Theorem 2, there is an isometric family of generalized $K3$ structures parametrized by $\boldsymbol{P}_1(\boldsymbol{C})$ and the period of these structures is exactly the fiber $\pi^{-1}(\pi(\kappa, [\omega])) \cong \boldsymbol{P}_1(\boldsymbol{C})$. By Varouchas [Va], the cohomology class of each Einstein-Kähler orbifold-metric form contains a Kähler form in the sense of Fujiki-Moišezon. This completes the proof of Lemma.

Just as in the proof of the Surjectivity Theorem for smooth Einstein-Kähler $K3$ surfaces in [L], the remaining part of Theorem 1 is divided into three steps.

Step 1. Suppose $(\kappa, [\omega]) \in K\Omega$ is such that $(P_\omega + \boldsymbol{R}\cdot\kappa) \cap L$ contains a primitive rank two lattice M. By Lemma, we may replace $(\kappa, [\omega])$ by any other element in $\pi^{-1}\pi(\kappa, [\omega])$. We may thus assume that $M \subset P_\omega$. By the weak version of the Surjectivity Theorem due to Morrison, we can find a marked polarized generalized $K3$ surface (X, ϕ, α) such that $\alpha_R(\phi) = \kappa$ and $[\alpha_C(\omega_X)] = [\omega]$. Since $I^2(X)$ is the orthogonal complement of integral classes, $I^2(X) \otimes_Z R = I(X)_R$ is a linear subspace of $H^2(Y; \boldsymbol{R})$ defined over $\boldsymbol{Q}$. Since $M(\subset P_\omega \subset I^2(X)_R)$ is defined over $\boldsymbol{Z}$, the orthogonal complement of P_ω in $I^2(X)_R$ is defined over $\boldsymbol{Q}$. So the elements $\swarrow$ which are defined over $\boldsymbol{Q}$ are dense in $\mathscr{C}_P^+(X)$. By the theorem of Mayer [Ma], such $\swarrow$ contains a Kähler metric on X in the sense of Fujiki-Moišezon. Since $\mathscr{C}_P^+(X)$ is a convex cone, ϕ is a linear combination of rational points in $\mathscr{C}_P^+(X)$ with positive coefficients. So, ϕ is a Kähler class on X in the sense of Fujiki-Moišezon.

Step 2. Suppose $(\kappa, [\omega]) \in K\Omega$ is such that $(P_\omega + \boldsymbol{R}\cdot\kappa) \cap L$ contains a primitive rank one lattice L. $I^2([\omega]) = L_C^{W([\omega])}$ is defined over $\boldsymbol{Q}$. $V^+([\omega])$ is partitioned into chambers by reflection hypersurfaces H_δ for $\delta \in \Delta([\omega])$. Let K be the chamber containing κ. If $\eta \in K$ is such that $(P_\omega + \boldsymbol{R}\cdot\eta) \cap L$ contains a primitive rank two lattice, then $(\eta, [\omega]) \in \operatorname{Im}\tau$, i.e., there is a $(X_\eta, \phi_\eta, \alpha_\eta)$ with $\alpha_{\eta R}(\phi_\eta) = \eta$ and $[\alpha_{\eta C}(\omega_{X_\eta})] = [\omega]$. It is shown in the proof of the weak version of the Surjectivity Theorem (see pp. 326–327 of [Mr1]) that the isomorphism class of X_η is independent of $\eta \in K$. Such η with the property as above are dense in an open convex subcone K of $V^+([\omega])$. So, we can find a marked polarized generalized $K3$ surface (X, ϕ, α) such that $\alpha_R(\phi) = \kappa$, $[\alpha_C(\omega_X)] = [\omega]$ and ϕ contains a Kähler metric in the sense of Fujiki-Moišezon.

Step 3. Let $(\kappa, [\omega]) \in K\Omega$ be an arbitrary point, and K the chamber of $V^+([\omega])$ with respect to the action of the Weyl group $W([\omega])$ containing κ. Since $I^2([\omega])$ is defined over $\boldsymbol{Q}$, the η's such that $(P_\omega + \boldsymbol{R}\cdot\eta)\cap L$ contains a primitive rank one lattice form a dense subset in K. For such an η, we can find a $(X_\eta, \phi_\eta, \alpha_\eta)$ such that $\alpha_{\eta R}(\phi_\eta) = \eta$, $[\alpha_{\eta C}(\omega_{X_\eta})] = [\omega]$ and ϕ_η contains a Kähler metric, by Step 2. The isomorphism class of X_η is independent of the choice of $\eta \in K$. Since K is a convex cone, κ contains a Kähler metric in the sense of Fujiki-Moišezen. q.e.d.

It is shown in [Vn] that the action of the automorphism group Γ of L on $K\Omega \cong SO_0(3, 19)/SO(2)\times SO(19)$ is discrete and properly discontinuous. We thus have a moduli space for the isomorphism classes of polarized generalized $K3$ surfaces:

COROLLARY 4. *The coarse moduli space for the following objects are all isomorphic to*

$$\Gamma\backslash K\Omega = \Gamma\backslash(SO_0(3, 19)/SO(2)\times SO(19))$$

under the correspondence induced by the polarized period map:

(i) *The isomorphism classes of polarized generalized $K3$ surfaces.*

(ii) *The isomorphism classes of polarized generalized $K3$ surfaces whose polarization comes from a Kähler form in the sense of Fujiki-Moišezon.*

(iii) *The isomorphism classes of Einstein-Kähler generalized $K3$ surfaces with volume 1.*

PROOF. The bijection (ii) $\rightleftarrows$ (iii) is given by Theorem. There is a natural injection (ii) $\to$ (i). Theorem A means that there is an injection (i) $\to \Gamma\backslash K\Omega$ induced from the period map. Theorem 1 means (ii) $\to \Gamma\backslash K\Omega$ is surjective. q.e.d.

REMARK. Einstein-Kähler generalized $K3$ surfaces *with* simple singularities correspond to points in the fixed point set Fix(W) of the group $W\subset\Gamma$ generated by all reflections

$$s_\delta(v) = v + \langle v, \delta\rangle\delta\ , \quad \text{where} \quad \delta\in L \quad \text{and} \quad \langle\delta, \delta\rangle = -2\ .$$

Fix(W) is a countable union of submanifolds of real codimension 3 ([Mr1]).

5. Moduli of Einstein metrics on a $K3$ surface. In this section we define the period map for Ricci-flat orbifolds diffeomorphic to generalized $K3$ surfaces and study its properties. We begin with some standard facts on 4-dimensional Riemannian geometry [AtHS]. Let (M, g) be a 4-dimensional Riemannian manifold with a metric g and $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$ the decomposition of 2-forms into self-dual and anti-self-dual parts. The Riemannian curvature tensor defines a self-adjoint transformation $R\colon \Lambda^2 \to \Lambda^2$

expressed as $R(e_i \wedge e_j) = (1/2)\sum_{i,j,k,l} R_{ijkl} e_k \wedge e_l$, where $\{e_i\}$ is a local orthonormal basis of 1-forms. If we write

$$R = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$$

relative to the decomposition $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, the decomposition of the curvature tensor into irreducible pieces under $SO(4)$ is given by

$$R \to (\mathrm{tr}\, A, B, W_+, W_-)$$

where $\mathrm{tr}\, A = \mathrm{tr}\, C = (1/4)$ scalar curvature, $B =$ the traceless Ricci tensor, and $W_+ = -(1/3)\mathrm{tr}\, A$, $W_- = C - (1/3)\mathrm{tr}\, C$, the Weyl tensors. If the metric is Kählerian with vanishing Ricci-tensor, then $R \wedge \omega \equiv 0$, where ω is the Kähler form. This means that R is anti-self-dual with vanishing Ricci tensor:

$$R = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

$\mathrm{tr}\, C = 0$. For any Einstein metric over 4-manifolds, Hitchin [H] showed an inequality $2e(g) \geqq -P_1(g)$ between the Euler form $e(g)$ and the Pontrjagin form $P_1(g)$. The equality occurs if and only if the curvature R is anti-self-dual and Ricci-flat. In particular any Ricci-flat Riemannian metric on a $K3$ surface is anti-self-dual.

Let X be a real four dimensional differentiable orbifold which is orbifold-diffeomorphic to a generalized $K3$ surface X'. Suppose X admits a Ricci-flat-metric g. Then we have:

THEOREM 5. *Let (X, g) be as above. Then the bundle of self-dual 2-forms (in the sense of orbifolds) is a flat trivial bundle with respect to the Levi-Civita connection.*

PROOF. As in the proof of Lemma 12 in [Kb] we get

$$(*) \qquad \int_X e(X, g) = e(Y) - \sum_{p \in \mathrm{Sing}X} (e(E_p) - \left(\frac{1}{|G_p|}\right),$$

where $e(X, g)$ is the Euler form for the Levi-Civita connection of g, $e(E_p)$ is the Euler number of the exceptional set E_p for the simple singularity $p \in X$ and $|G_p|$ is the order of the corresponding finite subgroup G_p of $SU(2)$. Let g_1 and g_2 be two Riemannian orbifold metrics on X and $P_1(g_1)$, $P_1(g_2)$ the corresponding Pontrjagin forms, respectively. Then $P_1(g_1) - P_1(g_2) = d\eta$, where η is an orbifold-3-form on X. So, we have $\int_X P_1(g_1) - \int_X P_1(g_2) = \int_X d\eta = 0$. Now in a small neighborhood of simple singularities of X we can introduce a canonical orbifold-complex structure, such as $\boldsymbol{B}^2/G$ where G is a finite subgroup of $SU(2)$ and $\boldsymbol{B}^2$ is an open ball in $\boldsymbol{C}^2$. We can thus deform g to be Kähler-orbifold metric around simple singularities with respect to the above complex structure. If the metric g is Kählerian then $P_1(g) = c_1(g)^2 - 2c_2(g)$. Just as in the proof of Lemma 12 in [Kb] we have

$$(**) \qquad \frac{1}{3}\int_X P_1(g) = \operatorname{sign}(Y) + \frac{2}{3}\sum_{p \in \operatorname{Sing} X}\left(e(E_p) - \frac{1}{|G_p|}\right).$$

Formulas (*) and (**) are valid for any Riemannian orbifold metric. Now we assume that g is an orbifold-metric with vanishing Ricci tensor. From (*) and (**) we have

$$\int_X 2e(X, g) + P_1(X, g) = 0 .$$

Applying the same argument as in [H] we see that the Ricci-flat orbifold-metric g is anti-self-dual:

$$R = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix} \quad \text{with} \quad \operatorname{tr} C = 0 .$$

For any oriented Riemannian four-manifold the curvature of the connection on the bundle Λ^+ of self-dual 2-forms induced from the Levi-Civita connection is given by $A + B^* \in \operatorname{Hom}(\Lambda^+, \Lambda^2)$. In fact, the bundle Λ^2 of 2-forms is the adjoint bundle associated with the orthonormal frame bundle and the second exterior power representation λ^2 of $SO(4)$ splits into two irreducible subspaces $\lambda^2 = \lambda^+ \oplus \lambda^-$. The representation λ^+ defines a homomorphism $\swarrow^+: SO(4) \to SO(3)$ which gives rise a principal $SO(3)$-bundle whose adjoint bundle is Λ^+. So, in our case, the bundle Λ^+ with the induced connection is flat. Since the metric g is an orbifold-metric and the minimimal resolution of X is simply connected, the bundle Λ^+ is falt and trivial, i.e., Λ^+ has three linearly independent parallel sections. q.e.d.

REMARK. From the above proof one sees that there exists an orbifold-complex structure J on X such that the metric g is a Kähler orbifold-metric. Since the metric is an Einstein-Kähler orbifold metric with vanishing Ricci tensor and the canonical bundle on the minimal resolution Y descends to an orbifold-holomorphic line bundle on holomorphic orbifold-2-forms, Y must have trivial canonical bundle. So, Y is a $K3$ surface with the given complex structure J.

Let (X, g) be as in Theorem 5 and $\alpha: I^2(X) \to L$ a marking, i.e., a metric injection which extends to an isometry $\bar{\alpha}: H^2(Y) \to L$. The triple

(X, g, α) is a marked $K3$-orbifold with a Ricci-flat metric g. We define the period map p of all equivalence classes of marked $K3$-orbifold with a Ricci-flat metric to $G_3^+(L_R) \cong SO_0(3, 19)/SO(3) \times SO(19)$ in the following way: $p(X, g, \alpha)$ is the oriented three-plane in L_R generated by the α_R-image of the oriented basis (three ordered linearly independent parallel self-dual 2-forms on X) of the space of parallel self-dual 2-forms. Here, if $\{e_1, e_2, e_3, e_4\}$ is an oriented basis for $\boldsymbol{R}^4$, then $\{e_1 \wedge e_2 + e_3 \wedge e_4, e_1 \wedge e_3 + e_4 \wedge e_2, e_1 \wedge e_4 + e_2 \wedge e_3\}$ gives the induced orientation on $\Lambda^+(\boldsymbol{R}^4) \subset \Lambda^2(\boldsymbol{R}^4)$. Two marked Ricci-flat $K3$-orbifolds (X, g, α) and (X', g', α') are said to be equivalent if there exists a diffeomorphism $f: Y' \to Y$ which descends to an orbifold-diffeomorphism of X' to X and $f^*g = g'$, $\bar{\alpha}' \circ f^* = \bar{\alpha}$. Write $\mathscr{N}$ for the set of all equivalence classes of marked $K3$-orbifolds with a Ricci-flat metric.

THEOREM 6. *The period map* $p(X, g, \alpha) \in G_3^+(L_R)$ *descends to the bijection* $\sigma: \mathscr{N} \to G_3^+(L_R)$.

PROOF. Suppose $p(X, g, \alpha) = p(X', g', \alpha') = x \in G_3^+(L_R)$. Pick a point $(\kappa, [\omega]) \in K\Omega$ in the fiber $\pi^{-1}(x)$, where $\pi: K\Omega \to G_3^+(L_R)$ is the natural projection with the fiber S^2. There are marked generalized $K3$ surfaces (X, ϕ, α) and (X', ϕ', α') such that g and g' are Einstein-Kähler orbifold-metric. By using the isometric deformations, with respect to the Ricci-flat orbifold-metric, we may assume that the polarized periods are the same for (X, ϕ, α) and (X', ϕ', α'). From Theorem A, there exists a unique isomorphism $\bar{\Phi}: Y' \to Y$ which descends to a unique isomorphism $\Phi: X' \to X$ such that $\Phi^* = \bar{\gamma} = \bar{\alpha}' \circ \bar{\alpha}$ ($\Phi^* = \alpha'^{-1} \circ \alpha$ on $I^2(X)$ and $I^2(X')$). Φ, which is an orbifold-diffeomorphism of X' to X, is an isometry with respect to g and g'. So, (X, g, α) and (X', g', α') are equivalent, i.e., σ is injective. To show the surjectivity of σ we pick a point $x \in G_3^+(L_R)$. Choose any $(\kappa, [\omega])$ in the fiber $\pi^{-1}(x) \subset K\Omega$. From the strong version of the Surjectivity Theorem, there exists a marked generalized Einstein-Kähler $K3$ surface (X, ϕ, α) with its period $(\kappa, [\omega])$. If we forget the complex structure of (X, ϕ, α) and look at it only as a Ricci-flat marked $K3$-orbifold, then its period is $\pi(\kappa, [\omega]) = x$. q.e.d.

Γ acts on both $\mathscr{N}$ and $G_3^+(L_R)$. The action of Γ on $G_3^+(L_R)$ is discrete and properly discontinuous by [Vn]. The following is a generalization of the corresponding results in [Bo1] and [T2].

THEOREM 7. *The set of all isomorphism classes of Ricci-flat K3-orbifolds is isomorphic to*

$$\Gamma \diagdown (SO_0(3, 19)/SO(3) \times SO(19)) .$$

The Ricci-flat K3-orbifold with simple singularities correspond to the points in Fix(W), *which is a countable union of submanifold of codimension* 3.

PROOF. The last statement follows from the arguments in pp. 311–317 of [Mr1]. q.e.d.

For the convergence of non-singular Ricci-flat metrics to an orbifold-metric we can show the following:

THEOREM 8. *Let* $\{E_t\}$ *be a differentiable family of three dimensional subspaces in* L_R *where* t *runs over the unit punctured disk* $\Delta^* = \Delta - \{0\}$ *such that*

(a) $\langle\ ,\ \rangle$ *on each* E_t *is positive definite,*

(b) *for every* $\delta \in L$ *with* $\langle \delta, \delta \rangle = -2$, $s_\delta(E_t) \neq E_t$,

(c) $\lim_{t\to 0} E_t = E_0$ *in* $G_3^+(L_R)$, *where* $\langle\ ,\ \rangle$ *on* E_0 *is positive definite and there exists* $\delta \in L$ *such that* $\langle \delta, \delta \rangle = -2$ *and* $s_\delta(E_0) = E_0$,

(d) *let* $\{g(t)\}$ *be a differentiable family of Einstein metrics corresponding to* $\{E_t\}$ *on the underlying differentiable four-manifold* M *of a* $K3$ *surface with a fixed marking* α *and suppose that* $\mathrm{Vol}(g_{ij}(t)) = 1$ *for all* t. *Then* "$\lim_{t\to 0} g(t) = g(0)$ *exists*" *and* $g(0)$ *is an Einstein-Kähler orbifold-metric with respect to any generalized* $K3$ *structure* X_0 *corresponding to a two dimensional oriented subspace* $F_0 \subset E_0$. *Here, the meaning of the existence of* $\lim_{t\to 0} g_{(t)}$ *is as follows: Suppose* $\{F_t\}$ *be a differentiable family of two-dimensional subspaces in* E_t *such that* $\lim_{t\to 0} F_t = F_0$ *exists and* F_0 *is a two-dimensional subspace in* E_0. *Let* X_t *be a differentiable family of* $K3$ *structures on* (M, α) *corresponding to* $\{F_t\}$, $t \in \Delta$. *Write* D *for the exceptional set for the generalized* $K3$ *structure determined by* (E_0, F_0). *Let* $(U, (x^i))$ *be any open subset of* $X_0 - D$ *with local coordinates* $(x^i)_{1\leq i\leq 4}$. *Let* $g_{ij}(t)$ *be the components of* $g(t)$ *with respect to* $(U, (x^i))$. *Then* $\lim_{t\to 0} g_{ij}(t) = g_{ij}(0)$ *exists.*

PROOF. Let $F_t \subset E_t$ be a sequence of two dimensional subspaces in E_t such that $\lim_{t\to 0} F_t = F_0$ exists and F_0 is a two dimensional subspace in E_0. From Surjectivity Theorem and Global Torelli Theorem, we see that the sequence $\{F_t\}$ corresponds to a unique sequence of $K3$ surfaces (X_t, α) with a fixed marking α such that $\lim_{t\to 0}(X_t, \alpha) = (X_0, \alpha)$. It is possible to choose a holomorphic 2-form ω_t on X_t with $\int_{X_t} \omega_t \wedge \bar{\omega}_t = 1$ such that $\lim_{t\to 0} \omega_t = \omega_0$ exists and $\int_{X_t} \omega_0 \wedge \bar{\omega}_0 = 1$. Now let F_t' be the two dimensional subspace in E_t defined by $\mathrm{Re}\,\omega_t$ and $\mathrm{Im}\, g_{\alpha\bar{\beta}}(t)$, where $g_{\alpha\bar{\beta}}(t)$ is the Einstein-Kähler metric on X_t corresponding to $\alpha^{-1}(\kappa_t)$, where $\kappa_t \perp F_t$ in E_t. Since we may suppose that $\langle \kappa_t, \kappa_t \rangle = 1$ we get from $E_t \to E_0$ and

$F_t \to F_0$ that

$$\lim_{t\to 0} \kappa_t = \kappa_0 \in E_0 \quad \text{and} \quad \kappa_0 \perp F_0 .$$

So $\lim_{t\to 0} F_t' = F_0'$ exists and repeating the same arguments as for F_t we see that there exists a unique family of $K3$ with $\lim_{t\to 0}(X_t', \alpha) = (X_0', \alpha)$. From the theory of isometric deformations of $K3$ structures with respect to the Calabi-Yau metric (see Section 3), we see that if ω_t' is a holomorphic 2-form on X_t' such that $\int_{X_t'} \omega_t' \wedge \bar{\omega}_t' = 1$, then

$$\omega_t' = \operatorname{Re} \omega_t + i \operatorname{Im} g_{\alpha\bar{\beta}}(t) .$$

Since $\lim_{t\to 0} \omega_t' = \omega_0'$ exists, we see that $\lim_{t\to 0} \operatorname{Im} g_{\alpha\bar{\beta}}(t)$ exists. Now it is easy to see that $\lim_{t\to 0} \operatorname{Im} g_{\alpha\bar{\beta}}(t)$ is an Einstein-Kähler orbifold-metric form on X_0. This is so because for each point $x \in X$, $\operatorname{vol}(g_{\alpha\bar{\beta}}(t)) = \omega_t \wedge \bar{\omega}_t$ and ω_0' is an orbifold-holomorphic 2-form in a neighborhood of some root systems of (-2)-curves on X_0. Note that if we express the complex structure tensor J_0 of X_0 in terms of the local coordinates (x'^i, y'^i) where $(z'^i) = (x'^i + \sqrt{-1}y'^i)$ are holomorphic coordinates for X_0', then it is smooth off the exceptional set of X_0'. But it is "orbifold-smooth". q.e.d.

If $F_t \equiv F_0$ in Theorem 8, then $\{g_{\alpha\bar{\beta}}(t)\}_{t\in\Delta^*}$ is a family of Einstein-Kähler metrics on a fixed $K3$ surface X_0 corresponding to F_0. In this case, $\lim_{t\to 0} \operatorname{Im} g_{\alpha\bar{\beta}}(t) = \operatorname{Im} g_{\alpha\bar{\beta}}(0)$ exists and defines a current on X_0.

6. The number of quotient singularities. In Section 2 we have proved the existence of a Ricci-flat Einstein-Kähler orbifold-metric on some orbifolds. We use this metric to estimate the maximal possible number of what occurs in case the maximal number is attained. The following is a generalization of Theorem 1 in [Ni].

THEOREM 9. *Let X be a compact complex surface with at worst isolated quotient singularities which admits a Kähler form in the sense of Fujiki-Moišezon. Let $\bar{X}$ be the minimal resolution for X and D its exceptional sets. Let μ_i be the nonnegative rational numbers such that $K_{\bar{X}} + \sum_i \mu_i D_i \sim 0$ near D as $\mathbf{Q}$-divisors (such μ_i's are uniquely determined), where $D = \sum_i D_i$ is the decomposition into irreducible components. Suppose that some tensor power of $K_{\bar{X}} + \sum_i \mu_i D_i$ is a trivial line bundle. Then we have the following inequality*:

$$e(\bar{X}) - \sum_{p\in \operatorname{Sing} X} \left(e(D_p) - \frac{1}{|G_p|} \right) \geqq 0 ,$$

where D_p is the exceptional set for the minimal resolution of p, G_p is the corresponding local fundamental group around p. The equality *occurs if and only if $X = \Gamma \diagdown T^2$, where T^2 is a complex 2-torus and Γ is a group of Euclidean motions acting on T^2 discretely and properly discontinuously with only isolated fixed points.*

PROOF. From Theorem 1, there exists a Ricci-flat Einstein-Kähler orbifold-metric on X. Using the same arguments as in [Kb] we see that the integral of the Euler form with respect to the Levi-Civita connection of the orbifold-metric is equal to $e(\bar{X}) - \sum_{p \in \mathrm{Sing}X}(e(D_p) - 1/|G_p|)$. On the other hand, since our metric is a Ricci-flat Einstein-Kähler metric, only the anti-self-dual Weyl tensor W_- remains in the decomposition of the curvature tensor (see Section 5). So, the Euler form is equal to $(1/8\pi^2)|W_-|^2 * 1$ and thus we get

$$0 \leqq (1/8\pi^2)\int_X |W_-|^2 * 1 = e(\bar{X}) - \sum_{p \in \mathrm{Sing}X}(e(D_p) - (1/|G_p|)) .$$

The equality occurs if and only if $W_- \equiv 0$ for our Ricci-flat Einstein-Kähler orbifold-metric. Since every compact flat orbifold is uniformized by a torus, the equality occurs if and only if X is uniformized by a torus with the covering transformation group consisting of Euclidean motions. q.e.d.

COROLLARY 10. *For generalized $K3$ surfaces X, we have*

$$24 - \sum_{p \in \mathrm{Sing}X}(e(D_p) - (1/|G_p|)) \geqq 0 ,$$

where the equality occurs if and only if $X = \Gamma \diagdown T^2$, with T^2 a complex 2-torus and Γ a group Euclidean motions.

Theorem 9 was first proved by Miyaoka [Mi] by means of algebraic geometry. The advantage of our method is that it gives informations on the equality case. More recently Morrison [Mr2] proved Corollary 10 in an algebraic way. His method gives a precise description of the corresponding complex crystallographic groups.

The Kummer surface with the (-2)-curves collapsed is the simplest example for the above equality: $24 - 16 \times (3/2) = 0$. Ivinskis [I] found a non-trivial example which is as follows. Consider the double covering branched over a sexitic curve in $\boldsymbol{P}_2(\boldsymbol{C})$ with simple singularities. The double covering X is a generalized $K3$ surface. If $\sum_{p \in \mathrm{Sing}X}(e(D_p) - (1/|G_p|)) = 24$, then $X = \Gamma \diagdown T^2$. For the sexitic curve $x_0x_1x_2(x_0 - x_1) \times (x_0 - x_2)(x_1 - x_2) = 0$ the double cover X has four D_4 and three A_1 singularities. So,

$$\sum_{p \in \mathrm{Sing}X}(e(D_p) - (1/|G_p|)) = 4 \times (5 - 8^{-1}) + 3 \times (2 - 2^{-1}) = 24 .$$

For the dual sexitic curve of a smooth one, X has nine A_2 singularities. So,

$$\sum_{p \in \mathrm{Sing} X} (e(D_p) - (1/|G_p|)) = 9 \times (2 - 3^{-1}) = 24 .$$

The sexitics with the above property are classified in [I]. For the classification of complex crystallographic groups, see [YoKT].

These examples show that the equality case in Theorem 9 is not void. As a final remark, we mention the degeneration of Riemannian metrics. The convergence in Theorem 8 is the simplest example of the degeneration of Riemannian metrics with bounded Ricci curvature and volume. Namely, the following occurs: there exist certain submanifoids $((-2)$-rational curves) such that the "area" goes to zero and the Riemannian sectional curvature concentrates along these, and the formal Euler number $\int$(Euler form) decreases (in a "quantized" way in our case) at the limit. In the above examples, the curvature tensor concentrates so completely that the limit metric is a *flat* orbifold-metric.

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