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C^1 FOLIATIONS WHICH CANNOT BE APPROXIMATED BY C^2 FOLIATIONS

Dedicated to Professor Nagayoshi Iwahori on his sixtieth birthday

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Introduction. In this paper we construct C^1 foliated pieces, the foliation of which cannot be approximated by C^2 foliations. Plugging such a piece into a given manifold and extending it to the whole manifold, we show that in any homotopy class of plane fields of dimension greater than one on a given manifold, there exists a foliation of class C^1 which cannot be approximated by foliations of class C^2.

Our foliated piece is a cut off from an example of Pixton [8] and is similar to that of Rosenberg-Thurston [9] (see also [13]). In the case of codimension one, the argument is based on Kopell [6] and Pixton [8] and the result holds for C^0 approximations. In the case of codimension greater than one, we use the invariant manifold theorem due to Hirsch et al. [5] and reduce the problem to the case of codimension one. In this case, we can only show the result for C^1 approximations, because the invariant manifold theorem is valid only for such approximations.

Let M be a closed manifold and τ be a homotopy class of codimension k plane fields on M $(1 \leqq k \leqq \dim M - 1)$. We define $\mathrm{Fol}_\tau^r(M)$ to be the space of the codimension k foliations of class C^r of M whose tangent bundles are in τ. In this paper we deal with two spaces $\mathrm{Fol}_\tau^1(M) \supset \mathrm{Fol}_\tau^2(M)$, and the topology on $\mathrm{Fol}_\tau^1(M)$ we consider is Hirsch's C^0 or C^1 topology (see Hirsch [4] and Epstein [2]).

THEOREM 1. *Let M be a manifold of dimension greater than two and with the Euler characteristic zero, and let τ be any homotopy class of codimension one plane fields on M. Then there exists an open subset U of $\mathrm{Fol}_\tau^1(M)$ with respect to the C^0 topology such that any element of U is not topologically conjugate to a C^2 foliation. In particular, $\mathrm{Fol}_\tau^2(M)$ is not dense in $\mathrm{Fol}_\tau^1(M)$ with respect to the C^0 topology.*

Here two foliations are *topologically conjugate* if there exists a foliation preserving homeomorphism.

THEOREM 2. *Let M be a manifold of dimension greater than two* *and τ a homotopy class of plane fields of dimension greater than one on M. Then* $\mathrm{Fol}_\tau^2(M)$ *is not dense in* $\mathrm{Fol}_\tau^1(M)$ *with respect to the C^1 topology.*

The first example of foliations non-approximable by smoother ones was obtained by Rosenberg-Thurston [9] as an $\boldsymbol{R}^2$-action on T^3, and Ennis et al. [1] gave examples of another type, containing foliations of codimension greater than one. On the other hand, Pixton [8] proved that generic C^1 actions of $\boldsymbol{R}^d$ on S^1 with $d \geqq 2$ are not approximable by C^2 actions. While the method of [1] is "global" and does not apply to, for example, simply connected manifolds, our plugging pieces yield examples in any manifold.

Note that Hirsch's topology we use of the space of foliations is coarser than that defined by plane fields which is used in [9] and [1].

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1. Codimension one. For the proof of Theorem 1, we need the following two lemmas, the former due to Kopell [6] and the latter essentially to Pixton [8].

LEMMA 1. *Let $f: [0, 1] \to [0, f(1)]$ and $g: [0, 1] \to [0, 1]$ be commuting C^2 diffeomorphisms such that f is a contraction and g fixes* 1. *Then* $g = \mathrm{id}$.

LEMMA 2. *There exist commuting diffeomorphisms $f, g: [0, 1] \to [0, 1]$ which satisfy*

(i) $f|_{[0,1)}$ *is a contraction, i.e., $f(t) < t$ for all $t \in (0, 1)$, and satisfies* $(df/dt)(0) = (df/dt)(1) = 1$,

(ii) *g is not the identity but has a contracting fixed point in* $(0, 1)$, *and*

(iii) *f and g are of class C^1.*

PROOF. It is shown by Pixton [8] (see also Tsuboi [12]) that there exist commuting C^1 diffeomorphisms of $[0, 1]$ one of which is as (i) above and the other is not the identity but fixes some point in $(0, 1)$. Then by the trick used in the proof of Lemma 3.4 in [13], we get this lemma. □

PROPOSITION 1. *There exists a C^1 foliation $\mathscr{F}_0$ of codimension one of $T^n \times I$, $n = 2, 3, \cdots$ such that*

(1) *$\mathscr{F}_0$ coincides with the product foliation $T^n \times \{*\}$ near the boundary, and*

(2) *any foliation of $T^n \times I$ which is sufficiently C^0 close to $\mathscr{F}_0$ is*

not topologically conjugate to a C^2 foliation (even if it is not tangent to the boundary).

PROOF. We only deal with $T^2 \times I$ because the same proof applies to the other cases. By Lemma 2, there exist commuting C^1 diffeomorphisms f_0, g_0: $[0, 1] \to [0, 1]$ satisfying

(i) f_0 and g_0 are the identity near 0 and 1, and

(ii) there are $0<a<b<1$ such that $f_0|_{[a,b]}$ and $g_0|_{[a,b]}$ are as in Lemma 2 and a and b are contracting fixed points of f_0.

Let $\mathscr{F}_0$ be a foliated I-product over T^2 the holonomies of which are generated by f_0 and g_0, and consider a foliation $\mathscr{F}'$ sufficiently C^0 close to $\mathscr{F}_0$. Let f' and g' be the holonomies of $\mathscr{F}'$ corresponding to f_0 and g_0. Then there is an invariant interval $[a', b']$ close to $[a, b]$ such that $f'|_{[a',b')}$ is a contraction and g' fixes some point in (a', b'), and f' and g' commute by an argument in Hirsch [4]. Since these properties about f' and g' are preserved by a topological conjugacy, the foliation $\mathscr{F}'$ is not topologically conjugate to a C^2 foliation by Lemma 1. □

To embed the foliated piece above into a given manifold, we need the following lemma, the proof of which is easy and is omitted.

LEMMA 3. *Let $n \geqq 2$ and $k \geqq 1$. Then there exists an embedding ι: $T^n \to D^n \times D^k$ such that the normal bundle of $\iota(T^n)$ extends to a k-plane field on $D^n \times D^k$, which is homotopic to the trivial plane field $\tau(\{*\} \times D^k)$ relative to the boundary.*

PROOF OF THEOREM 1. In the case of $\dim M = 3$, modifications along closed transversals yield leaves diffeomorphic to the torus without changing the homotopy class of the plane fields tangent to the foliations. Hence replacing the toral leaf by the foliation of $T^2 \times I$ in Proposition 1, we get the required result.

Let $m = \dim M \geqq 4$. Together with Lemma 3, Thurston's existence theorem in [11] says that in any homotopy class of codimension one plane fields on M, there exists a foliation with a compact leaf diffeomorphic to T^{m-1}. Replacing this T^{m-1} by the foliation of $T^{m-1} \times I$ in Proposition 1, we get the theorem. □

The diffeomorphisms f and g in Lemma 2 can be taken arbitrarily close to the identity. Hence by the existence of Reeb components due to Novikov [7] (see Hector-Hirsch [3] for a C^0 version), we get the following:

COROLLARY. *Let M be a closed three-dimensional manifold and suppose either that M is nonprime or that the fundamental group of M*

is finite. Then for any homotopy class τ of codimension one plane fields, there exists a subset U of $\mathrm{Fol}^1_\tau(M)$ *satisfying*

(i) *U is open and dense with respect both to the C^0 and to the C^1 topology, and*

(ii) *each element of U is not topologically conjugate to a C^2 foliation.*

The above statement does not hold for general three-dimensional manifolds: The product foliation of $S^2\times S^1$ and Anosov foliations have neighborhoods consisting of foliations topologically conjugate to themselves.

2. Codimension greater than one. As Theorem 1 is reduced to Proposition 1, Theorem 2 is reduced to Proposition 2 below by the existence of C^1 foliations due to Thurston [10] and Tsuboi [12], together with Lemma 3. We use, as in Ennis et al. [1], the invariant manifold theorem of Hirsch et al. [5, p. 39] in the proof of Proposition 2.

PROPOSITION 2. *There exists a C^1 foliation $\mathscr{F}_0$ of codimension $l+1$ of $T^n\times D^l\times I$, $n=2, 3, \cdots$ such that*

(1) *$\mathscr{F}_0$ coincides with the product foliation $T^n\times\{*\}$ near the boundary $T^n\times\partial(D^l\times I)$, and*

(2) *any foliation of $T^n\times D^l\times I$ which is sufficiently C^1 close to $\mathscr{F}_0$ is not a C^2 foliation (even if it is not tangent to the boundary).*

PROOF. For $0<\lambda<1$, let $h_\lambda\colon D^l\to D^l$ be a diffeomorphism such that

(i) $h_\lambda=\mathrm{id}$ on some neighborhood of ∂D^l, and

(ii) $h(x)=\lambda x$ on some neighborhood of 0.

Using the diffeomorphisms f_0 and g_0 of $I=[0, 1]$ as in the proof of Proposition 1, we define C^1 diffeomorphisms F and G of $D^l\times I$ by $F=h_\lambda\times f$ and $G=h_\lambda\times g$, and put $\mathscr{F}_0$ to be the foliated $D^l\times I$-product over T^n defined by F and G. Then for small λ, $\{0\}\times I$ is an invariant manifold with respect both to F and to G in the sense of Hirsch et al. [5].

Consider a foliation $\mathscr{F}'$ sufficiently C^1 close to $\mathscr{F}_0$, and let F' and G' be the holonomies of $\mathscr{F}'$ corresponding to F and G. Then F' and G' are C^1 close to F and G, respectively, and thus by the invariant manifold theorem, there exist one-dimensional manifolds $N_{F'}$ and $N_{G'}$ invariant under F' and G', respectively, for small λ. Since F' and G' commute (see Hirsch [4]), we have $N_{F'}=N_{G'}$ by the uniqueness of invariant manifolds. Put $N=N_{F'}=N_{G'}$ and suppose that $\mathscr{F}'$ is of class C^2. Then F' and G' are of class C^2 and thus N is a C^2 submanifold of $D^l\times I$. Hence $F'|_N$ and $G'|_N$ are commuting C^2 diffeomorphisms of N which are C^1 close to F and G. This contradicts Lemma 1 by the same argument as in the proof of Proposition 1. □

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