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INEQUALITIES OF FEJÉR-RIESZ AND HARDY-LITTLEWOOD

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Introduction. In this note, we shall derive some inequalities concerning the growth of mean values of holomorphic functions which extend classical results. Section 1 deals with the Fejér-Riesz inequality for H^p functions on the unit ball in $\boldsymbol{C}^n$ and on the generalized half-plane, and the results of [8] are extended. In Section 2, two types of Hardy-Littlewood inequalities are obtained. Section 3 concerns the weighted Bergman space on the unit ball which is closely related to the Hardy space.

1. The Fejér-Riesz inequality. Let B denote the open unit ball in $\boldsymbol{C}^n$, $n \geqq 2$, and D be the generalized half-plane defined by $\operatorname{Im} z_1 - |z'|^2 > 0$, $(z_1, z') \in \boldsymbol{C} \times \boldsymbol{C}^{n-1}$. We shall write $L_{j,k} = \boldsymbol{R}^j \times \boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\} \subset \boldsymbol{C}^n$, $1 \leqq j \leqq n$, $0 \leqq k \leqq n-j$, $L_{0,k} = \boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\}$, $1 \leqq k \leqq n$, and $L'_{j-1,k} = (i\boldsymbol{R})^{j-1} \times \boldsymbol{C}^k \times \{0\} \times \cdots \times \{0\} \subset \boldsymbol{C}^{n-1}$, $1 \leqq j \leqq n$, $0 \leqq k \leqq n-j$, where $\boldsymbol{R}$ means the real line in $\boldsymbol{C}$. dz will denote the Lebesgue measure on $L_{j,k}$.

If $c = 1$ and $j = 1$ in Theorems 1 and 2, the inequalities coincide with those of [8], except in the case $k = n$ in (2) and (4). Here we note that the method used in [8] does not work for the present situation. Theorem 1 generalizes the Fejér-Riesz inequality given in [1]. It also contains a recent result of Power's [9, Corollary] as a special case $c = 1$ and $n = j = 2$.

THEOREM 1. *Let $c \geqq 1$. Then there is a constant $C = C(n, j, k, c)$ such that the following holds for any p, $0 < p < +\infty$, and for any $f \in H^p(B)$:*

$$(1) \qquad \int_{B \cap L_{j,k}} |f(z)|^{cp} (1 - |z|^2)^{cn - (j+2k+1)/2} dz \leqq C(\|f\|_p)^{cp}, \quad 1 \leqq j \leqq n, \quad 0 \leqq k \leqq n-j.$$

There is a constant $C' = C'(n, k, c)$ such that

$$(2) \qquad \int_{B \cap L_{0,k}} |f(z)|^{cp} (1 - |z|^2)^{cn-k-1} dz \leqq C'(\|f\|_p)^{cp}, \quad 1 \leqq k \leqq n,$$

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where $c>1$ *for* $k=n$. *The exponents* $cn-2^{-1}(j+2k+1)$ *and* $cn-k-1$ *are the best possible in all cases.*

THEOREM 2. *For the same constants* C *and* C' *as in Theorem* 1, *the following hold for* $f\in H^p(D)$, $0<p<+\infty$, *where each exponent is unique*:

$$\int_0^{+\infty} dy_1 \int_{L'_{j-1,k}} |f(x_1+iy_1+i|z'|^2, z')|^{cp} y_1^{cn-(j+2k+1)/2} dz' \leqq C(\|f\|_p)^{cp}, \tag{3}$$

for any $x_1\in \boldsymbol{R}$, $1\leqq j\leqq n$, $0\leqq k\leqq n-j$.

$$\int_0^{+\infty} dy_1 \int_{L_{1,k-1}} |f(x_1+iy_1+i|z'|^2, z')|^{cp} y_1^{cn-k-1} dx_1 dz' \leqq 2C'(\|f\|_p)^{cp}, \quad 1\leqq k\leqq n, \tag{4}$$

where $c>1$ *for* $k=n$.

We shall denote by $A^p(\Omega)$ the class of holomorphic functions on $\Omega\subset \boldsymbol{C}^n$ which belong to $L^p(\Omega, dz)$. It is obvious that $H^p(B)\subset A^p(B)$, $0<p<+\infty$. The relation between these classes will be made clear in the following corollary. B_k denotes the open unit ball in $\boldsymbol{C}^k$, $1\leqq k\leqq n$. For a function g on B_k, $1\leqq k\leqq n-1$, $E_{n,k}g$ is defined by $(E_{n,k}g)(w, w')=g(w)$, $(w, w')\in B$. The statement (5) is a generalization of [3, Theorem E].

COROLLARY. *Let* $0<p<+\infty$.

(5) $H^p(B)\subset A^{(n+1)p/n}(B)$, *and* $H^p(B)\not\subset A^q(B)$ *for* $q>n^{-1}(n+1)p$.

(6) $H^p(B_k)$ *is imbedded in* $H^{np/k}(B)$, *by the operator* $E_{n,k}$,

where $k^{-1}np$ *is the best possible.*

(7) $H^p(D)\subset A^{(n+1)p/n}(D)$, *and* $H^p(D)\not\subset A^q(D)$ *for* $q\neq n^{-1}(n+1)p$.

In (5) *and* (7), H^p *is properly contained in* $A^{(n+1)p/n}$.

PROOF OF THEOREM 1. For $\xi\in\partial B$ and $r>0$, let $K(\xi, r)=\{z\in B \mid |1-\langle z, \xi\rangle|<r^2\}$. Let μ be a positive finite measure on B. Suppose that, for $c\geqq 1$, there are positive numbers A, δ such that

$$\mu(K(\xi, r))\leqq Ar^{2cn} \tag{8}$$

for every $\xi\in\partial B$ and $0<r<\delta$. If μ is a measure supported on $\{z\in \boldsymbol{C}^n \mid 2^{-1}\leqq |z|<1\}$, then by the same argument as in [9], we can see that, for $f\in L^1(\partial B)$ and $\lambda>0$, $\mu(\{|P[f]|>\lambda\})\leqq (C\lambda^{-1}\|f\|_1)^c$, where $P[f]$ denotes the Poisson integral of f. The Marcinkiewicz interpolation theorem then shows that $\|P[f]\|_{L^{2c}(\mu)}\leqq C'(n, \mu, c)\|f\|_2$, $f\in L^2(\partial B)$. Let μ be supported on B. Take $f\in H^p(B)$, $0<p<+\infty$. Then there is an $h\in L^2(\partial B)$

such that $h \geqq 0$, $(\|h\|_2)^2 = (\|f\|_p)^p$, and $|f|^{p/2} \leqq P[h]$. It follows that

$$\int_B |f|^{cp} d\mu \leqq \int_{|z|<1/2} (P[h^2])^c d\mu + \int_{1/2\leqq|z|<1} (P[h])^{2c} d\mu \leqq C(n, \mu, c)(\|f\|_p)^{cp} .$$

First, we shall prove (1). It suffices to see that the measure $d\mu(z) = (1-|z|^2)^\alpha dz$, $z \in B \cap L_{j,k}$, satisfies (8) for $0 < r < 1$, where $\alpha = cn - 2^{-1}(j + 2k + 1)$. Put $K = K(\xi, r)$ and $K' = \{z \in B \,|\, 1 - \mathrm{Re}\langle z, \xi\rangle < r^2\}$. Clearly, $K \subset K'$. Suppose that $K \cap L_{j,k} \neq \emptyset$. Using real coordinates for $\boldsymbol{C}^n$, we write $\xi = (a', b', a'', b'', a''', b''')$, where $(a', b') = (a_1, b_1, \cdots, a_j, b_j) \in \boldsymbol{R}^{2j}$, $z_l = a_l + ib_l$, $1 \leqq l \leqq j$. Similarly, (a'', b'') and (a''', b''') represent points of $\boldsymbol{C}^k$ and $\boldsymbol{C}^{n-j-k}$, respectively. The inner product in $\boldsymbol{R}^m$ will be denoted by $[x, y]$, $x, y \in \boldsymbol{R}^m$. Now take $z \in K' \cap L_{j,k}$. Then $z = (x', 0', x'', y'', 0''', 0''')$ with $|x'|^2 + |x''|^2 + |y''|^2 < 1$ and $1 - \mathrm{Re}\langle z, \xi\rangle = 1 - [(x', x'', y''), (a', a'', b'')] < r^2$. Writing $a = (a', a'', b'')$ and $G = \{x = (x', x'', y'') \in B_{j,k} \,|\, 1 - [x, a] < r^2\}$, where $B_{j,k}$ denotes the open unit ball in $\boldsymbol{R}^{j+2k}$, we see that

$$I_{j,k}(r) := \mu(K) \leqq \int_{K' \cap L_{j,k}} (1 - |z|^2)^\alpha dz = \int_G (1 - |x|^2)^\alpha dx .$$

If we put $|a| = t$, then $0 < t \leqq 1$, since $G \neq \emptyset$. Take $P \in O(j + 2k)$ so that $Pe = t^{-1}a$, where $e = (1, 0, \cdots, 0) \in \boldsymbol{R}^{j+2k}$. Let $G' = \{x \in B_{j,k} \,|\, 1 - tx_1 < r^2\}$ and $G'' = \{x \in B_{j,k} \,|\, 1 - r^2 < x_1 < 1\}$. Then $P(G') = G$ and $G' \subset G''$. Thus, by integration over G'' instead of G and by Fubini's theorem in the case $j + 2k \geqq 2$, we get $I_{j,k}(r) \leqq C(n, j, k, c) r^{2cn}$. To verify (2), let $\alpha = cn - k - 1$. Note that $\alpha > -1$ in all cases. We shall show that μ satisfies (8) for $0 < r < 2^{-1/2}$. We write $\xi = (\xi', \xi'')$ with $\xi' \in \boldsymbol{C}^k$ and put $|\xi'| = t$. Suppose that $K \cap L_{0,k} \neq \emptyset$. Then $2^{-1} < t \leqq 1$. Take $U \in U(k)$ so that $Ue = t^{-1}\xi'$, where $e = (1, 0, \cdots, 0) \in \boldsymbol{C}^k$. Let $G = \{w \in \boldsymbol{C}^k \,|\, |w| < 1,\ |1 - \langle w, \xi'\rangle| < r^2\}$ and $G' = \{w \in \boldsymbol{C}^k \,|\, |w| < 1,\ |1 - tw_1| < r^2\}$. Then $U(G') = G$, and

$$I_k(r) := \mu(K) = \int_{G'} (1 - |w|^2)^\alpha dw .$$

Using Fubini's theorem when $2 \leqq k \leqq n$, we have

$$I_k(r) = C(k, \alpha) \int_{G''} (1 - |w_1|^2)^{cn-2} dw_1 ,$$

where $G'' = \{w_1 \in \boldsymbol{C} \,|\, |w_1| < 1,\ |1 - tw_1| < r^2\}$. Modifying the change of variables made in [10, 5.1.4], we define ϕ: $w_1 = \phi(\lambda) = t^{-1}(1 - r^2\lambda^{-1})$, $\lambda \in \boldsymbol{C} - \{0\}$. Since $\phi^{-1}(G'') \subset \{\lambda \,|\, \mathrm{Re}\,\lambda > 0,\ |\lambda| > 1\}$ and $1 - |\phi(\lambda)|^2 < 2t^{-2} r^2 |\lambda|^{-2} \mathrm{Re}\,\lambda$, it is seen that

$$\int_{G''} (1 - |w_1|^2)^{cn-2} dw_1 \leqq C(n, c) r^{2cn} .$$

Suppose that $\alpha < cn - 2^{-1}(j+2k+1)$. Then, for $b = (2c)^{-1}(2\alpha + j + 2k + 1)$, the function $(1 - z_1)^{-b}$ belongs to $H^1(B)$ and it is easily seen that

$$\int_{B\cap L_{j,k}} |1 - z_1|^{-bc}(1 - |z|^2)^{\alpha} dz = +\infty .$$

If $-1 < \alpha < cn - k - 1$, then just as in [8], the integral in (2) becomes $+\infty$ for $f(z) = (1 - z_1)^{-b}$ with $b = c^{-1}(\alpha + k + 1)$.

PROOF OF THEOREM 2. This is very similar to the proof of [8, Theorem 2]. Let $w = \Psi(z)$, where $z = (iy_1, \cdots, iy_j, z_{j+1}, \cdots, z_{j+k}, 0, \cdots, 0)$, $y_1 > y_2^2 + \cdots + y_j^2 + |z_{j+1}|^2 + \cdots + |z_{j+k}|^2$. Then Ψ transforms $D\cap L'_{j,k}$ onto $B\cap L_{j,k}$ and the Jacobian determinant is $2^{j+2k}(y_1 + 1)^{-(j+2k+1)}$, so that the inequality (3) follows. Suppose that $\alpha \neq cn - 2^{-1}(j + 2k + 1)$ and put $b = (2c)^{-1}(2\alpha+j+2k+1)$. If $\alpha > cn-2^{-1}(j + 2k + 1)$ then $(z_1 + i)^{-b} \in H^1(D)$, and if $\alpha < cn - 2^{-1}(j + 2k + 1)$ then $z_1^{-b}(z_1 + i)^{-2n+b} \in H^1(D)$. A simple computation shows that the integrals in (3), with y_1^{α}, become $+\infty$ for these functions. The inequality (4), as well as the uniqueness of the exponent, can similarly be verified.

PROOF OF COROLLARY. (5): By (2), the identity mapping of $H^p(B)$ into $A^{(n+1)p/n}(B)$ is continuous. If $q>n^{-1}(n+1)p$, then $(1-z_1)^{-(n+1)/q} \in H^p(B)$ and $\notin A^q(B)$. (6): From the relation $H^p(B_k)\subset A^{(k+1)p/k}(B_k)$ and [10, 7.2.4, (a)], it follows that $H^p(B_k)$ is imbedded in $H^{(k+1)p/k}(B_{k+1})$ by the operator $E_{k+1,k}$. This procedure gives (6). (7): $A^p(B)$ is a complete, linear metric space, as will be seen from (19) with $q = +\infty$, $k = n$. Now assume that $H^p(B) = A^q(B)$, $q = n^{-1}(n + 1)p$. The open mapping theorem would imply that, if $\{f_j\}$ is a sequence of holomorphic functions on B, bounded in L^q, then it is also bounded in $H^p(B)$. Let $g_j(z) = z_1^{2j}$, $z \in B$, $j = 1, 2, \cdots$. Then

$$I_j := \int_{\partial B} |g_j(\zeta)|^p d\sigma(\zeta) = \frac{2\pi^n\Gamma(pj + 1)}{\Gamma(pj + n)} ,$$

$$J_j := \int_{B} |g_j(z)|^p dz = \frac{\pi^n\Gamma(pj + 1)}{\Gamma(pj + n + 1)} .$$

Here, by Stirling's formula, $I_j \approx j^{-n+1}$ and $J_j \approx j^{-n}$ as $j \to \infty$. Putting $f_j(z) = j^{a(n)}g_j(z)$, $a(n) = ((n + 1)p)^{-1}n^2$, we see that $\|f_j\|_p \to \infty$ as $j \to \infty$, while $\|f_j\|_{L^q}$ are bounded. Next, (4) implies that $H^p(D)\subset A^{(n+1)p/n}(D)$. Put $b = q^{-1}(n + 1)$. If $q < n^{-1}(n + 1)p$, then $(z_1 + i)^{-b} \in H^p(D)$ and $\notin A^q(D)$. If $q > n^{-1}(n + 1)p$, then $z_1^{-b}(z_1 + i)^{-(2n/p)+b} \in H^p(D)$ and $\notin A^q(D)$. Now, with $q = n^{-1}(n + 1)p$, we define Ψ^* by $(\Psi^*g)(z) = 2^{2n/q}g(\Psi(z))(z_1 + i)^{-(2n+2)/q}$, $z \in D$, for $g \in A^q(B)$. Since the Jacobian determinant of Ψ is $2^{2n}|z_1 + i|^{-2n-2}$, we have $\Psi^*g \in A^q(D)$. It is clear that Ψ^* is an isometric isomorphism of

$A^q(B)$ onto $A^q(D)$. If Ψ^* is restricted to $H^p(B)$, then this induces the isometric isomorphism of $H^p(B)$ onto $H^p(D)$, due to [13], up to a constant multiple ([8], (8)). Thus, the rest of the assertion follows.

2. Hardy-Littlewood inequalities. (11) and (12) in the following Theorem 3 generalize a theorem of Hardy and Littlewood ([5], [6]) and are immediate consequences of Theorem 1, (2). The fact that these are the best possible can be seen by reduction to the one variable case ([2], [12]), where Corollary, (6) plays an essential role. Theorem 1, (2) will again be used to complete the proof of Theorem 4. Related results are contained in [4] and [7], in the case $k = n$.

For a continuous function f on B and for k, $1 \leqq k \leqq n$, we define means $M_q(f, k; r)$, $0 \leqq r < 1$, $0 < q \leqq +\infty$, as follows:

$$M_\infty(f, k; r) = \max_{\zeta \in \partial B_k} |f_r(\zeta, 0')| ,$$

$$M_q(f, k; r) = \left(\int_{\partial B_k} |f_r(\zeta, 0')|^q d\sigma_k(\zeta)\right)^{1/q} , \quad 0 < q < +\infty ,$$

where σ_k denotes the surface measure on ∂B_k. In the case $q = +\infty$, [10, 7.2.5] implies that, if $f \in H^p(B)$, $0 < p < +\infty$, and $1 \leqq k \leqq n$, then

$$(9) \qquad M_\infty(f, k; r) = o((1 - r)^{-n/p}) \quad \text{as} \quad r \to 1 ,$$

$$(10) \qquad M_\infty(f, k; r) \leqq A(n, p)\|f\|_p(1 - r)^{-n/p} , \quad 0 \leqq r < 1 .$$

In the case $k = n$, (11) and (12) follow from (9) and (10), since $M_q(f, n; r)^q \leqq M_\infty(f, n; r)^{q-p} M_p(f, n; r)^p$. Let $(R_{k,n}f)(w) = f(w, 0')$, $w \in B_k$, $1 \leqq k \leqq n - 1$, for a function f on B. If $R_{k,n}f \in H^{kp/n}(B_k)$ for $f \in H^p(B)$, then (11) and (12) would follow from the same argument. But this is not the case, because $H^{kp/n}(B_k) \subsetneqq R_{k,n}(H^p(B))$, which will be seen in Section 3.

THEOREM 3. *Suppose $f \in H^p(B)$, $0<p<+\infty$. Let $p \leqq q < +\infty$ ($p<q$ when $k = n$) and put $\alpha = p^{-1}n - q^{-1}k$, $1 \leqq k \leqq n$. Then*

$$(11) \qquad M_q(f, k; r) = o((1 - r)^{-\alpha}) \quad \textit{as} \quad r \to 1 ,$$

$$(12) \qquad M_q(f, k; r) \leqq A(n, k, p, q)\|f\|_p(1 - r)^{-\alpha} , \quad 0 \leqq r < 1 .$$

The exponent α cannot be replaced by any smaller value. Moreover, (9), (10), (11), *and* (12) *are the best possible in the sense that for any function $\phi(r)$, $0 \leqq r < 1$, such that $\phi(r) > 0$ and $\phi(r) \to 0$ as $r \to 1$, there exists $f \in H^p(B)$ with $M_q(f, k; r) \neq O(\phi(r)(1 - r)^{-\alpha})$ as $r \to 1$, $1 \leqq k \leqq n$.*

THEOREM 4. *Suppose $f \in H^p(B)$, $0<p<+\infty$. Let $p \leqq q \leqq +\infty$ ($p<q$ when $k = n$) and put $\alpha = p^{-1}n - q^{-1}k$, $1 \leqq k \leqq n$. Let $p \leqq \lambda < +\infty$. Then*

$$\left(\int_0^1 M_q(f, k; r)^\lambda (1 - r)^{\lambda\alpha - 1} dr\right)^{1/\lambda} \leqq A\|f\|_p , \tag{13}$$

where $A = A(n, k, p, q, \lambda)$. *The exponent* α *is the best possible. If* $0 < q < p$, *then* (13) *does not hold.*

PROOF OF THEOREM 3. We write $M(r)$ for $M_q(f, k; r)^q$, temporarily. Let $c = p^{-1}q$ and $\beta = cn - k - 1$. Then, by integration in polar coordinates, (2) becomes

$$\int_0^1 M(r)(1 - r^2)^\beta r^{2k-1} dr \leqq C(\|f\|_p)^q .$$

Since $M(r)$ is an increasing function, we can find a constant $A(\beta, k)$, depending only on β and k, such that

$$\int_0^1 M(r)(1 - r)^\beta dr \leqq A(\beta, k)\int_0^1 M(r)(1 - r)^\beta r^{2k-1} dr .$$

Hence we have

$$\int_0^1 M_q(f, k; r)^q (1 - r)^\beta dr \leqq C(\|f\|_p)^q . \tag{14}$$

Now, as in [3, (1.3)], we have

$$\int_r^1 M_q(f, k; t)^q (1 - t)^\beta dt \geqq (\beta + 1)^{-1} M_q(f, k; r)^q (1 - r)^{\beta+1} , \quad 0 \leqq r < 1 ,$$

whence (11) and (12) follow. Next, we prove that (9) and (10) are the best possible. Let U be the unit disc in $\boldsymbol{C}$. Take an arbitrary function $\phi(r)$, $0 \leqq r < 1$, with the property that $\phi(r) > 0$ and $\phi(r) \to 0$ as $r \to 1$. Then [12, Theorem 1′] shows that, for $\phi(r)^{1/2}$, there exists $g \in H^{p/n}(U)$ such that $|g(r_j)| \geqq C\phi(r_j)^{1/2}(1 - r_j)^{-n/p}$, $j = 1, 2, \cdots$, where C is a constant and $\{r_j\}$ is a sequence: $r_1 < r_2 < \cdots$, $r_j \to 1$ as $j \to \infty$. Put $f = E_{n,1}g$. Then $f \in H^p(B)$, by (6), and we see that $M_\infty(f, k; r_j) \geqq C\phi(r_j)^{-1/2}\phi(r_j)(1 - r_j)^{-n/p}$, $j = 1, 2, \cdots$, $1 \leqq k \leqq n$. This means that $M_\infty(f, k; r) \neq O(\phi(r)(1 - r)^{-n/p})$ as $r \to 1$. The case $0 < q < +\infty$ will be settled after [2], as follows. Taking an $f \in H^p(B)$, as above, for the function $\phi(r^{1/2})$, we see that $M_\infty(f, k; r_j^2) \geqq C\phi(r_j)(1 - r_j)^{-n/p}$, $j = 1, 2, \cdots$, $1 \leqq k \leqq n$. The Cauchy formula implies that, for $0 \leqq r < 1$,

$$f_r(w, 0') = C(k)\int_{\partial B_k} (1 - \langle w, \zeta\rangle)^{-k} f_r(\zeta, 0') d\sigma_k(\zeta) , \quad w \in B_k .$$

Put $w = r\xi$, $\xi \in \partial B_k$. If $1 < q < +\infty$, then by Hölder's inequality,

$$|f(r^2\xi, 0')| \leqq CM_q(f, k; r)\left(\int_{\partial B_k} |1 - \langle r\xi, \zeta\rangle|^{-kq'} d\sigma_k(\zeta)\right)^{1/q'} .$$

The above integral is $\approx (1-r^2)^{-(kq'-k)}$, by [10, 1.4.10], and hence $M_\infty(f, k; r^2) \leqq CM_q(f, k; r)(1-r)^{-k/q}$. It follows that $M_q(f, k; r_j) \geqq C\phi(r_j)(1-r_j)^{-\alpha}$, $j = 1, 2, \cdots$. Similarly, this inequality is seen to hold for $q = 1$. Finally, let $0 < q < 1$. If we take $f \in H^p(B)$, for $\phi(r)^q$, so that $M_1(f, k; r_j) \geqq C\phi(r_j)^q(1-r_j)^{-(n/p)+k}$, $j = 1, 2, \cdots$, then, since $M_1(f, k; r) \leqq M_\infty(f, k; r)^{1-q}M_q(f, k; r)^q \leqq C(1-r)^{(-n/p)(1-q)}M_q(f, k; r)^q$, by (10), the desired result follows.

PROOF OF THEOREM 4. Suppose first that $1 \leqq p < +\infty$. If $u = P[h]$, $h \in L^p(\partial B)$, then as in (10), we have

$$M_\infty(u, k; r) \leqq A(n, p)\|h\|_p(1-r)^{-n/p}, \quad 0 \leqq r < 1, \quad 1 \leqq k \leqq n. \tag{15}$$

We are going to show that, for $p \leqq q < +\infty$, $1 \leqq k \leqq n$,

$$M_q(u, k; r) \leqq A(n, k, p, q)\|h\|_p(1-r)^{-\alpha}, \quad 0 \leqq r < 1. \tag{16}$$

By (15), we have

$$M_q(u, k; r)^q \leqq (A\|h\|_p(1-r)^{-n/p})^{q-p}\int_{\partial B_k}|u(r\zeta, 0')|^p d\sigma_k(\zeta).$$

Here, with $z = (r\zeta, 0')$,

$$\int_{\partial B_k}|u(r\zeta, 0')|^p d\sigma_k(\zeta) \leqq \int_{\partial B}\left(|h(\eta)|^p\int_{\partial B_k}P(z, \eta)d\sigma_k(\zeta)\right)d\sigma(\eta),$$

where $P(z, \eta)$ denotes the Poisson kernel for B. Putting $\eta = (\xi, \xi')$, $\xi \in \boldsymbol{C}^k$, we see that

$$\begin{aligned} P((r\zeta, 0'), (\xi, \xi')) &= C(n)(1-r^2)^n|1-\langle r\zeta, \xi\rangle|^{-2n} \\ &\leqq C(n, k)(1-r)^{-n+k}((1-|r\xi|^2)|1-\langle r\xi, \zeta\rangle|^{-2})^k. \end{aligned}$$

Since $|\partial B_k|^{-1}((1-|w|^2)|1-\langle w, \zeta\rangle|^{-2})^k$, $w \in B_k$, $\zeta \in \partial B_k$, is the Poisson kernel for B_k, we get

$$M_q(u, k; r)^q \leqq (A\|h\|_p(1-r)^{-n/p})^{q-p}C(1-r)^{-n+k}(\|h\|_p)^p.$$

Next, following [3], we shall show that, for $1<p<q\leqq +\infty$, $p \leqq \lambda < +\infty$, and $u = P[h]$ with $h \in L^p(\partial B)$,

$$\left(\int_0^1 M_q(u, k; r)^\lambda(1-r)^{\lambda\alpha-1}dr\right)^{1/\lambda} \leqq C\|h\|_p, \quad 1 \leqq k \leqq n, \tag{17}$$

where $C = C(n, k, p, q, \lambda)$. Suppose, for the moment, that $1 \leqq p<q \leqq +\infty$. Fix k, $1 \leqq k \leqq n$. We define a measure ν by $d\nu(r) = (1-r)^{n-1}dr$, $0 \leqq r<1$. Let $(Th)(r) = M_q(u, k; r)(1-r)^{-k/q}$, $h \in L^p(\partial B)$. Then the operator T is subadditive and, by (15) and (16), $(Th)(r) \leqq A\|h\|_p(1-r)^{-n/p}$, $0 \leqq r < 1$. Hence, for any $s \geqq A\|h\|_p$, $G := \{r \in [0, 1) \mid (Th)(r)>s\} \subset \{r \mid 1-(A\|h\|_p s^{-1})^{p/n} < r < 1\} =: E$. If $0 < s < A\|h\|_p$, then $E = [0, 1)$. Thus

$$\nu(G) \leqq \int_E (1-r)^{n-1} dr \leqq (C\|h\|_p s^{-1})^p .$$

The Marcinkiewicz interpolation theorem shows that $\|Th\|_{L^p(\nu)} \leqq C(n, k, p, q)\|h\|_p$ for $1 < p < q$. This means that (17) is valid in the case $p = \lambda$. Let $p < \lambda$. Then, since $M_q(u, k; r)^\lambda \leqq (A\|h\|_p(1-r)^{-\alpha})^{\lambda-p} M_q(u, k; r)^p$ by (15) and (16), we obtain (17). Now let $f \in H^p(B)$, $0 < p < +\infty$, and take $h \in L^2(\partial B)$ with the property that $|f|^{p/2} \leqq P[h]$, $(\|h\|_2)^2 = (\|f\|_p)^p$. Let q, λ be such that $p < q \leqq +\infty$, $p \leqq \lambda < +\infty$. Then $M_q(f, k; r)^\lambda \leqq M_{(2q)/p}(u, k; r)^{(2\lambda)/p}$, where we put $2p^{-1}q = +\infty$ when $q = +\infty$. Taking 2, $2p^{-1}q$, and $2p^{-1}\lambda$ in place of p, q, and λ in (17), we can get (13). Finally, let $p = q \leqq \lambda < +\infty$, $1 \leqq k \leqq n-1$. Then, putting $c = 1$ in (14), we obtain (13) with $p = \lambda$. In the case $p < \lambda$, (13) follows from (12). To see that α is the best possible, let $0 < \beta < \alpha$. Then $f(z) := (1 - z_1)^{-\beta-(k/q)} \in H^p(B)$, and $M_q(f, k; r) \approx (1-r)^{-\beta}$ as $r \to 1$. Thus, the integral in (13) becomes $+\infty$, if α is replaced by β. Suppose $0 < q < p$. It is enough to assume that $1 \leqq k \leqq n-1$ and $q^{-1}(n-1) < p^{-1}n$. Putting $g_j(z) = z_1^{2j}$, as in the proof of the Corollary, we have

$$I_j := \left(\int_0^1 M_q(g_j, k; r)^\lambda (1-r)^{\lambda\alpha-1} dr\right)^{1/\lambda}$$
$$= \left(\frac{2\pi^k \Gamma(qj+1)}{\Gamma(qj+k)}\right)^{1/q} \left(\frac{\Gamma(2\lambda j+1)\Gamma(\lambda\alpha)}{\Gamma(2\lambda j+1+\lambda\alpha)}\right)^{1/\lambda} .$$

Also, $\|g_j\|_p = (2\pi^n \Gamma(pj+1)(\Gamma(pj+n))^{-1})^{1/p}$. We can write $I_j(\|g_j\|_p)^{-1} = C\Delta(j) j^{(1/q)-(1/p)}$, where $\Delta(j) \to 1$ as $j \to \infty$.

3. The weighted Bergman space. This is the class of holomorphic functions f on B such that

$$\|f\|_{p,\delta} := \left(\int_B |f(z)|^p (1-|z|^2)^\delta dz\right)^{1/p} < +\infty ,$$

where $p > 0$ and $\delta > -1$, and will be denoted by $A^{p,\delta}(B)$. Note that (2) implies $H^p(B) \subset A^{cp,cn-n-1}(B)$ for $c > 1$, with $\|f\|_{cp,cn-n-1} \leqq C\|f\|_p$, $f \in H^p(B)$. We can see that this inclusion is proper, as in the proof of the Corollary, (7).

THEOREM 5. *Suppose $f \in A^{p,\delta}(B)$. Let $p \leqq q \leqq +\infty$ and put $\sigma = p^{-1}(n+1+\delta) - q^{-1}k$, $1 \leqq k \leqq n$. Then*

(18) $$M_q(f, k; r) = o((1-r)^{-\sigma}) \quad as \quad r \to 1 ,$$

(19) $$M_q(f, k; r) \leqq A(n, k, p, q, \delta)\|f\|_{p,\delta}(1-r)^{-\sigma} , \quad 0 \leqq r < 1 .$$

These are the best possible; namely, for any $\phi(r)$, $0 \leqq r < 1$, such that

$\phi(r) > 0$ and $\phi(r) \to 0$ as $r \to 1$, there exists $f \in A^{p,\delta}(B)$ with $M_q(f, k; r) \neq O(\phi(r)(1-r)^{-\sigma})$ as $r \to 1$, $1 \leqq k \leqq n$.

PROOF. Suppose first that f is a holomorphic function on B such that $M_p(f, n; r) \leqq C(1-r)^{-\beta}$, $0 \leqq r < 1$, with constants $\beta, C > 0$. Then, for $1 \leqq k \leqq n$, $p \leqq q \leqq +\infty$,

$$M_q(f, k; r) \leqq K(n, k, p, q, \beta)C(1-r)^{-\alpha-\beta}\ , \quad 0 \leqq r < 1\ , \tag{20}$$

where $\alpha = p^{-1}n - q^{-1}k$. Indeed, since $f_r \in H^p(B)$ with $\|f_r\|_p \leqq C(1-r)^{-\beta}$, $0 < r < 1$, (10) implies that $M_\infty(f_r, k; \rho) \leqq A(n, p)C(1-r)^{-\beta}(1-\rho)^{-n/p}$, $0 \leqq \rho < 1$, hence, letting $\rho = r$, we have $M_\infty(f, k; r^2) \leqq A(n, p, \beta)C(1-r^2)^{-(n/p)-\beta}$, proving the case $q = +\infty$. The case $q < +\infty$ is similar, by (12). Next, we can derive (18) and (19) when $p = q$ and $k = n$, following [11, Theorem B]. Take $f \in A^{p,\delta}$. It is enough to assume that $2^{-1} \leqq r < 1$. From

$$\begin{aligned}(\|f\|_{p,\delta})^p &\geqq \int_r^1 M_p(f, n; t)^p(1-t^2)^\delta t^{2n-1}dt \\ &\geqq C(n, \delta)M_p(f, n; r)^p(1-r)^{1+\delta}\end{aligned}$$

it follows that

$$M_p(f, n; r) = o((1-r)^{-(1+\delta)/p}) \quad \text{as} \quad r \to 1\ , \tag{21}$$

$$M_p(f, n; r) \leqq C\|f\|_{p,\delta}(1-r)^{-(1+\delta)/p}\ , \quad 0 \leqq r < 1\ . \tag{22}$$

Let $1 \leqq k \leqq n$ and $p \leqq q \leqq +\infty$. Then, combining (20) with (22), we obtain (19). Finally, from (21), (10), and (12), we can see that $M_q(f_r, k; \rho) \leqq A\varepsilon(r)(1-r)^{-(1+\delta)/p}(1-\rho)^{-\alpha}$, $0 < \rho < 1$, where $\varepsilon(r) \to 0$ as $r \to 1$, whence we get (18). To see that (18) and (19) are the best possible, take an arbitrary $\phi(r)$. Then Theorem 3 shows that there is $f \in H^{(np)/(n+1+\delta)}(B)$ such that $M_q(f, k; r) \neq O(\phi(r)(1-r)^{-\sigma})$ as $r \to 1$. Since $H^{(np)/(n+1+\delta)}(B) \subset A^{p,\delta}(B)$, the proof is completed.

We have mainly been concerned with restrictions of H^p functions from B to B_k. In this respect, H^p and $A^{p,\delta}$ are closely connected in the following manner. The case $k = n-1$ is in [10, 7.2.4].

The operator $E_{n,k}$ defines a linear isometry of $A^{p,n-k-1}(B_k)$ into $H^p(B)$, $1 \leqq k \leqq n-1$, and $R_{k,n}$ is a continuous operator of $H^p(B)$ onto $A^{p,n-k-1}(B_k)$. The latter contains $H^{kp/n}(B_k)$ properly. Indeed, taking $g \in A^{p,n-k-1}(B_k)$, we can see from [8, (7)] that

$$\begin{aligned}\int_{\partial B}|(E_{n,k}g)_r(\zeta, \zeta')|^p d\sigma(\zeta, \zeta') &= |\partial B_{n-k}|\int_{B_k}|g_r(w)|^p(1-|w|^2)^{n-k-1}dw \\ &= |\partial B_{n-k}|r^{-2k}\int_{|w|<r}|g(w)|^p(1-r^{-2}|w|^2)^{n-k-1}dw\ ,\end{aligned}$$

where the integral converges to $(\|g\|_{p,n-k-1})^p$, increasingly, as $r \to 1$. On the other hand, it follows that $R_{k,n}$: $H^p(B) \to A^{p,n-k-1}(B_k)$ is continuous and onto, from (2) and the relation $R_{k,n} \circ E_{n,k} =$ identity.

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