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POSITIVE KERNEL FUNCTIONS AND BERGMAN SPACES

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Introduction. We denote by B theo pen unit ball in C^n, $n \geqq 1$. The Poisson kernel for B is obtained from the Cauchy kernel. In the same way, we can define a positive kernel function, H_δ, from the well-known kernel which is treated in [5]. H_δ has the reproducing property for the functions in the weighted Bergman space $A^{p,\delta}(B)$, $1 \leqq p < +\infty$. Using this kernel we shall derive Hardy-Littlewood inequalities for $A^{p,\delta}(B)$, just as in $H^p(B)$, where the Poisson kernel plays an essential role ([7]). Similar results will be obtained in the setting of the generalized half plane in C^n. As an application of the inequality, we shall treat the Mackey topology of $A^{p,\delta}(B)$, $0 < p < 1$, extending the one variable result ([9]).

1. Positive kernels. $\langle z, w\rangle$ will denote the usual inner product for $z, w \in C^n$ with $|z|^2 = \langle z, z\rangle$. We fix $\delta > -1$ throughout. Let $K_\delta(z, w) = A_0(1-|w|^2)^\delta(1-\langle z, w\rangle)^{-(n+1+\delta)}$, $z, w \in B$, where

$$A_0 = \left(\int_B (1-|w|^2)^\delta dw\right)^{-1} = \frac{\Gamma(n+1+\delta)}{\Gamma(1+\delta)\pi^n}\ ;$$

here, dw denotes Lebesgue measure on R^{2n}. We define a positive kernel H_δ by

$$H_\delta(z, w) := \frac{K_\delta(z, w)K_\delta(w, z)}{K_\delta(z, z)} = \frac{A_0(1-|z|^2)^{n+1+\delta}(1-|w|^2)^\delta}{|1-\langle z, w\rangle|^{2(n+1+\delta)}}\ , \quad z, w \in B\,.$$

We shall write

$$H_\delta[f](z) = \int_B H_\delta(z, w)f(w)dw\ , \quad z \in B\ ,$$

when the integral makes sense. For $0 < p < +\infty$, $L^{p,\delta}(B)$ will denote the class of measurable functions f on B such that

$$\|f\|_{p,\delta} := \left(\int_B |f(w)|^p(1-|w|^2)^\delta dw\right)^{1/p} < +\infty\ ,$$

and $A^{p,\delta}(B)$ will mean the class of holomorphic functions which belong to

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$L^{p,\delta}(B)$. We note that the following implies $H_\delta[1](z) = 1$, $z \in B$.

THEOREM 1. *Let* $1 \leqq p < +\infty$. *Then*

(1) $f(z) = H_\delta[f](z)$, $z \in B$, *if* $f \in A^{p,\delta}(B)$.

(2) $u(z) \leqq H_\delta[u](z)$, $z \in B$, *if* $u \in L^{p,\delta}(B)$ *and* u *is plurisubharmonic.*

PROOF. It is enough to suppose $p = 1$, since $L^{p,\delta} \subset L^{1,\delta}$. Fix an arbitrary $\varepsilon > 0$. Then, for any $f \in L^{1,\delta}(B)$, we have

$$\|H_\delta[f]\|_{1,\delta+\varepsilon} \leqq A_0 \int_B \left(|f(w)|(1 - |w|^2)^\delta \int_B \frac{(1 - |z|^2)^{n+1+2\delta+\varepsilon}}{|1 - \langle z, w\rangle|^{2(n+1+\delta)}} dz \right) dw ,$$

where the inner integral is a bounded function of w on B, by [8, 1.4.10]; thus $\|H_\delta[f]\|_{1,\delta+\varepsilon} \leqq C\|f\|_{1,\delta}$. The same method as in [2, Theorem 3, (ii)] shows that if $g \in A^{p,\delta}(B)$, $0 < p < +\infty$, and $g_r(z) = g(rz)$, $0 \leqq r < 1$, then $\|g_r - g\|_{p,\delta} \to 0$ as $r \to 1$. Now K_δ has the reproducing property for the functions in $H^\infty(B)$ ([8, 7.1.2]). Hence, we can see from a standard argument that $f = H_\delta[f]$ for $f \in A(B)$, the ball algebra. Take $f \in A^{1,\delta}(B)$. Then we have $\|f_r - H_\delta[f]\|_{1,\delta+\varepsilon} \leqq C\|f_r - f\|_{1,\delta}$, so the continuous functions f and $H_\delta[f]$ coincide on the whole of B. Next we prove (2). For a fixed $z \in B$, take $\phi_z \in \mathrm{Aut}(B)$ as in [8, 2.2.1]. Then $\phi_z(0) = z$ and $\phi_z \circ \phi_z =$ identity. Since $u \circ \phi_z$ is subharmonic on B, it follows from integration in polar coordinates that

$$\int_B (u \circ \phi_z)(\xi)(1 - |\xi|^2)^\delta d\xi \geqq u(z)|\partial B| \int_0^1 r^{2n-1}(1 - r^2)^\delta dr .$$

Making the change of variable $\xi = \phi_z(w)$, $w \in B$, we have, by [8, 2.2.2 and 2.2.6],

$$1 - |\xi|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \langle z, w\rangle|^2} , \qquad d\xi = \left(\frac{1 - |z|^2}{|1 - \langle z, w\rangle|^2} \right)^{n+1} dw .$$

(2) follows from these and the proof is completed.

We denote by D the domain $\{(z_1, z') \in \boldsymbol{C} \times \boldsymbol{C}^{n-1} \mid \mathrm{Im}\, z_1 - |z'|^2 > 0\}$. This is the upper half plane if $n = 1$. The Cayley transform Ψ, defined by $\Psi(z_1, \cdots, z_n) = (w_1, \cdots, w_n)$ with $w_1 = (z_1 - i)(z_1 + i)^{-1}$ and $w_j = 2z_j(z_1 + i)^{-1}$, $2 \leqq j \leqq n$, maps D onto B biholomorphically. We have $1 - \langle \Psi(z), \Psi(w)\rangle = 2\rho(z, w)((z_1 + i)(\overline{w_1 + i}))^{-1}$, z, $w \in D$, where $\rho(z, w) = i(\bar{w}_1 - z_1) - 2\langle z', w'\rangle$. The Jacobian of Ψ is $2^{2n}|z_1 + i|^{-2n-2}$. For ζ, $\xi \in B$, put $\zeta = \Psi(z)$, $\xi = \Psi(w)$, z, $w \in D$. Then

$$H_\delta(\zeta, \xi) d\xi = \frac{2^{n-1} A_0 \rho(z, z)^{n+1+\delta} \rho(w, w)^\delta}{|\rho(z, w)|^{2(n+1+\delta)}} dw ;$$

the kernel occurring on the right side will be denoted by $H_\delta^*(z, w)$. If

$g(\zeta)$ is a measurable function on B, then we can write $H_\delta[g](\Psi(z)) = H_\delta^*[g \circ \Psi](z)$, $z \in D$. In particular, we have $H_\delta^*[1](z) = 1$ for any $z \in D$. We denote by $L^{p,\delta}(D)$, $0 < p < +\infty$, the class of measurable functions f on D such that

$$\|f\|_{p,\delta} := \left(\int_D |f(z)|^p \rho(z, z)^\delta dz\right)^{1/p} < +\infty .$$

$A^{p,\delta}(D)$ will denote the class of holomorphic $L^{p,\delta}(D)$-functions. Take $f \in A^{p,\delta}(D)$. Then from $|z_1 + i| > 1$, $z \in D$, we see that

$$\int_B |(f \circ \Psi^{-1})(w)|^p (1 - |w|^2)^\delta dw < 2^{2n+\delta}(\|f\|_{p,\delta})^p < +\infty ,$$

i.e., $f \circ \Psi^{-1} \in A^{p,\delta}(B)$. Thus, for $1 \leqq p < +\infty$, we have $f(z) = H_\delta[f \circ \Psi^{-1}](\Psi(z)) = H_\delta^*[f](z)$, $z \in D$. Similarly, if u is plurisubharmonic and $u \in L^{p,\delta}(D)$, $1 \leqq p < +\infty$, then we have

$$u(z) \leqq H_\delta^*[u](z) , \quad z \in D . \tag{3}$$

2. Hardy-Littlewood inequalities for $A^{p,\delta}(B)$. For a continuous function f on B, $1 \leqq k \leqq n$, and $0 \leqq r < 1$, we define means $M_q(f, k; r)$, $0 < q \leqq +\infty$, as follows:

$$M_\infty(f, k; r) = \max_{\zeta' \in \partial B_k} |f(r\zeta', 0'')| ,$$

$$M_q(f, k; r) = \left(\int_{\partial B_k} |f(r\zeta', 0'')|^q d\sigma_k(\zeta')\right)^{1/q} , \quad 0 < q < +\infty ,$$

where B_k and $d\sigma_k$ denote, respectively, the unit ball in $\boldsymbol{C}^k$ and the surface measure on ∂B_k. We shall simply write $d\sigma$ instead of $d\sigma_n$. Also, $M_q(f; r)$ will mean $M_q(f, n; r)$.

LEMMA 1. *Let $1 \leqq p < +\infty$ and put $u = H_\delta[h]$ for $h \in L^{p,\delta}(B)$. Let $\sigma = p^{-1}(n + 1 + \delta) - q^{-1}n$ for $p \leqq q \leqq +\infty$. Then*

$$M_q(u; r) \leqq A(n, p, q, \delta)\|h\|_{p,\delta}(1 - r)^{-\sigma} , \quad 0 \leqq r < 1 . \tag{4}$$

If $1 < p < q \leqq +\infty$, $p \leqq \lambda < +\infty$, then

$$\left(\int_0^1 M_q(u; r)^\lambda (1 - r)^{\lambda\sigma - 1} dr\right)^{1/\lambda} \leqq A(n, p, q, \delta, \lambda)\|h\|_{p,\delta} . \tag{5}$$

PROOF. (4): Suppose $q = +\infty$. Let $\zeta \in \partial B$, $0 \leqq r < 1$. Since $H_\delta[1](z) = 1$ for any $z \in B$, Jensen's inequality shows that $|u(r\zeta)|^p < 2^{n+1+\delta}A_0(1 - r)^{-(n+1+\delta)}(\|h\|_{p,\delta})^p$, and (4) is clear. Suppose $p \leqq q < +\infty$. Then we have, by (4) with $q = +\infty$,

$$M_q(u; r)^q \leqq (C\|h\|_{p,\delta}(1 - r)^{-(n+1+\delta)/p})^{q-p} \int_{\partial B} |u(r\zeta)|^p d\sigma(\zeta) .$$

In the second factor, we see that

$$\int_{\partial B}|u(r\zeta)|^p d\sigma(\zeta)$$
$$\leqq A_0(1-r^2)^{n+1+\delta}\int_B\left(|h(w)|^p(1-|w|^2)^\delta\int_{\partial B}\frac{d\sigma(\zeta)}{|1-\langle r\zeta, w\rangle|^{2(n+1+\delta)}}\right)dw\ .$$

The inner integral, $I(w, r)$, is $\approx (1-|rw|^2)^{-(n+2+2\delta)}$ as $|rw|\to 1$, by [8, 1.4.10], hence $I(w, r)\leqq C(1-r^2)^{-(n+2+2\delta)}$, $0\leqq r<1$. Thus (4) follows from these estimates. (5): We define a measure $d\nu$ on $[0, 1)$ by $d\nu(r)=(1-r)^{n+\delta}dr$. Let $1\leqq p\leqq q$, if $q<+\infty$, and $1\leqq p<+\infty$, if $q=+\infty$. For $h\in L^{p,\delta}(B)$, putting $u=H_\delta[h]$, we define $(Th)(r)=M_q(u; r)(1-r)^{-n/q}$, $0\leqq r<1$. The rest of the proof is quite similar to the case $k=n$ in [7, (17)].

For a function g on B_k, let $(E_{n,k}g)(w', w'')=g(w')$, $(w', w'')\in B$. For a function f on B, let $(R_{k,n}f)(w')=f(w', 0'')$, $w'\in B_k$.

LEMMA 2. *Let $f\in A^{p,\delta}(B)$, $0<p<+\infty$, and $1\leqq k\leqq n$. Then*

$$\left(\int_{B_k}|f(z', 0'')|^p(1-|z'|^2)^{n+\delta-k}dz'\right)^{1/p}\leqq A(n, k, p, \delta)\|f\|_{p,\delta}\ . \tag{6}$$

Moreover, for $1\leqq k\leqq n-1$, $E_{n,k}$ becomes a linear isometry of $A^{p,n+\delta-k}(B_k)$ into $A^{p,\delta}(B)$, and $R_{k,n}$ is a norm-decreasing operator of $A^{p,\delta}(B)$ onto $A^{p,n+\delta-k}(B_k)$.

PROOF. Suppose $1\leqq k\leqq n-1$. We write L_k for the space $C^k\times\{0\}\times\cdots\times\{0\}\subset C^n$ and consider measures on B: $d\mu_k(z)=(1-|z'|^2)^{n+\delta-k}dz'$, $z=(z', 0'')\in B\cap L_k$, and $d\mu_\delta(z)=(1-|z|^2)^\delta dz$, $z\in B$. For $\xi\in\partial B$, let $K(\xi, r)=\{z\in B\,|\,|1-\langle z, \xi\rangle|<r^2\}$. It is enough to see that there is a constant C, independent of ξ, r, such that $\mu_k(K(\xi, r))\leqq C\mu_\delta(K(\xi, r))$, since this implies (6) by [1] or [6]. First suppose $0<r\leqq 2^{-1/2}$. We shall show that $\mu_k(K(\xi, r))\leqq Cr^{2(n+1+\delta)}$, $\xi\in\partial B$, just as in [7, Theorem 1, (2)]. Put $\alpha=n+\delta-k$ and $t=|\xi'|$, where $\xi=(\xi', \xi'')$ with $\xi'\in C^k$. Then

$$I_k(r):=\mu_k(K(\xi, r))=C(n, k, \delta)\int_{G''}(1-|w_1|^2)^{\alpha+k-1}dw_1\ ,$$

where $G''=\{w_1\in B_1\,|\,|1-tw_1|<r^2\}$, and then we get $I_k(r)\leqq Cr^{2(n+1+\delta)}$ by the change of variable $w_1=\phi(\lambda)=t^{-1}(1-r^2\lambda^{-1})$, $\lambda\in C-\{0\}$. Next, letting $E=\{w_1\in B_1\,|\,|1-w_1|<r^2\}$, we have

$$\mu_\delta(K(\xi, r))=C(n, \delta)\int_E(1-|w_1|^2)^{n+\delta-1}dw_1$$
$$=Cr^{2(n+1+\delta)}\int_{E'}(2\,\mathrm{Re}\,\lambda-r^2)^{n+\delta-1}|\lambda|^{-2(n+1+\delta)}d\lambda$$

with $E'=\{\lambda\in C\,|\,|\lambda|>1,\ \mathrm{Re}\,\lambda>2^{-1}r^2\}$. Since $n\geqq 2$, the above integral exceeds the integral of $(2\operatorname{Re}\lambda - 2^{-1})^{n+\delta-1}|\lambda|^{-2(n+1+\delta)}$ over the domain $\{|\lambda| > 1, \operatorname{Re}\lambda > 4^{-1}\}$, thus showing that $\mu_\delta(K(\xi, r)) \geqq Cr^{2(n+1+\delta)}$. If $r > 2^{-1/2}$, then $\mu_k(K(\xi, r)) \leqq \mu_k(B \cap L_k) \leqq C\mu_\delta(K(\xi, 2^{-1/2})) \leqq C\mu_\delta(K(\xi, r))$ for any $\xi \in \partial B$, hence the desired inequality holds for $r > 0$. Next, let $g \in A^{p,n+\delta-k}(B_k)$. Then, by Fubini's theorem,

$$\int_B |(E_{n,k}g)(w)|^p(1 - |w|^2)^\delta dw = C\int_{B_k} |g(w')|^p(1 - |w'|^2)^{n+\delta-k}dw' .$$

$R_{k,n}$ is continuous by (6) and onto, since $R_{k,n} \circ E_{n,k} =$ identity.

THEOREM 2. *Let $f \in A^{p,\delta}(B)$, $0 < p < +\infty$. Put $\sigma = p^{-1}(n + 1 + \delta) - q^{-1}k$ for $p \leqq q \leqq +\infty$, $1 \leqq k \leqq n$. Then, for $p \leqq \lambda < +\infty$,*

$$(7) \qquad \left(\int_0^1 M_q(f, k; r)^\lambda(1 - r)^{\lambda\sigma-1}dr\right)^{1/\lambda} \leqq A(n, k, p, q, \delta, \lambda)\|f\|_{p,\delta} .$$

σ is the best possible exponent. (7) *does not hold, when $0 < q < p$.*

PROOF. It is sufficient to assume $k = n$, because the other cases can be settled by Lemma 2. First suppose $p < q \leqq +\infty$, $p \leqq \lambda < +\infty$. Since $|f|^{p/2} \in L^{2,\delta}(B)$, we have $|f(z)|^{p/2} \leqq H_\delta[|f|^{p/2}](z) =: u(z)$, $z \in B$, by (2), hence $M_q(f; r)^\lambda \leqq M_{(2q/p)}(u; r)^{2\lambda/p}$. Taking 2, $p^{-1}(2q)$, and $p^{-1}(2\lambda)$, respectively, for p, q, and λ in (5), we get (7). In the case $p = q = \lambda$, we can derive (7) from the definition of $\|f\|_{p,\delta}$, as we have obtained (13) from (14) in [7, Theorem 4]. If $p = q < \lambda$, then (7) follows from [7, (19)]. Now the function $(1 - z_1)^{-\beta}$, $\beta > 0$, belongs to $A^{p,\delta}(B)$ if and only if $\beta < p^{-1}(n + 1 + \delta)$. Let $0 < \alpha < \sigma$, $0 < p \leqq q \leqq +\infty$. Then $f(z) := (1 - z_1)^{-\alpha-(k/q)} \in A^{p,\delta}(B)$ and $M_q(f, k; r) \approx (1 - r^2)^{-\alpha}$ as $r \to 1$. Thus the integral in (7), with σ replaced by α, becomes $+\infty$. The last assertion can be verified by taking the functions $z_1^{2^j}$, $j = 1, 2, \cdots$, as in the proof of [7, Theorem 4].

3. Hardy-Littlewood inequalities for $A^{p,\delta}(D)$. We denote by G_r the domain $\{(z_1, z') \in D \mid \operatorname{Im} z_1 - |z'|^2 > r\}$, $r > 0$.

LEMMA 3. *Let u be plurisubharmonic on D, $u \geqq 0$, and $u \in L^{p,\delta}(D)$, $1 \leqq p < +\infty$. Then $u(z) \to 0$ as $|z_1| \to +\infty$, uniformly on $\bar{G}_r$ for any $r > 0$.*

PROOF. We can suppose $p = 1$, since u^p is plurisubharmonic. Let $d\mu(w) = \rho(w, w)^\delta u(w)dw$, $w \in D$. Then, in view of (3), it is enough to verify the assertion for the function v defined by

$$v(z) = \int_D \rho(z, z)^{n+1+\delta}|\rho(z, w)|^{-2(n+1+\delta)}d\mu(w) , \quad z \in D .$$

Pick $r > 0$ and fix $\varepsilon > 0$. Let $Q_m = \{(y_1 + is + i|w'|^2, w') \in C\times C^{n-1} \mid |y_1| < m, 0 < s < m, |w'| < m\}$. We can take m so that $\mu(D\setminus Q_m) < \varepsilon$ and, then,

T, S, and R so that $T^{-(n+1+\delta)}\mu(Q_m) < \varepsilon$, $T^{n+1+\delta}(S-m)^{-4(n+1+\delta)}\mu(Q_m) < \varepsilon$ with $S > m$, and $T^{n+1+\delta}(R - m(1+2S))^{-2(n+1+\delta)}\mu(Q_m) < \varepsilon$ with $R > m(1+2S)$. Now take an arbitrary $z = (z_1, z') \in \bar{G}_r$, $z_1 = x_1 + it + i|z'|^2$, such that $|z_1|^2 > R^2 + (T+S^2)^2$. Then we have, for any $w = (y_1 + is + i|w'|^2, w') \in D$,

$$(8)\qquad \frac{\rho(z, z)^{n+1+\delta}}{|\rho(z, w)|^{2(n+1+\delta)}} = \frac{2^{n+1+\delta}t^{n+1+\delta}}{[(x_1 - y_1 + 2\,\mathrm{Im}\langle z', w'\rangle)^2 + (t + s + |z' - w'|^2)^2]^{n+1+\delta}} \leqq (2r^{-1})^{n+1+\delta} =: M\,,$$

hence

$$v(z) \leqq \int_{Q_m} + M\varepsilon\,.$$

Suppose $t > T$. Then

$$v(z) \leqq \int_{Q_m} 2^{n+1+\delta}t^{-(n+1+\delta)}d\mu(w) + M\varepsilon < (2^{n+1+\delta} + M)\varepsilon\,.$$

Suppose $r \leqq t \leqq T$. If $|z'| > S$, then

$$v(z) \leqq \int_{Q_m} \frac{2^{n+1+\delta}T^{n+1+\delta}}{|z' - w'|^{4(n+1+\delta)}}d\mu(w) + M\varepsilon < (2^{n+1+\delta} + M)\varepsilon\,.$$

If $|z'| \leqq S$, then $|x_1| > R$, hence from

$$v(z) \leqq \int_{Q_m} \frac{2^{n+1+\delta}T^{n+1+\delta}}{|x_1 - y_1 + 2\,\mathrm{Im}\langle z', w'\rangle|^{2(n+1+\delta)}}d\mu(w) + M\varepsilon\,,$$

we have $v(z) < (2^{n+1+\delta} + M)\varepsilon$, completing the proof.

Let f be a complex-valued function on D such that $|f|$ is upper semi-continuous. We define means $M_q(f, k; t)$, $t > 0$, for $0 < q \leqq +\infty$ and $1 \leqq k \leqq n$ as follows:

$$M_\infty(f, k; t) = \sup_{(x_1, z') \in \boldsymbol{R}\times\boldsymbol{C}^{k-1}} |f(x_1 + it + i|z'|^2, z', 0'')|\,,$$

$$M_q(f, k; t) = \left(\int_{\boldsymbol{R}\times\boldsymbol{C}^{k-1}} |f(x_1 + it + i|z'|^2, z', 0'')|^q dx_1 dz'\right)^{1/q}\,,$$

for $0 < q < +\infty$. $M_q(f, k; t)$ is an extended real-valued function on $(0, +\infty)$. $M_q(f; t)$ will mean $M_q(f, n; t)$.

LEMMA 4. *Let u be plurisubharmonic on D, $u \geqq 0$, and $u \in L^{p,\delta}(D)$, $1 \leqq p < +\infty$. Then, for $p \leqq q \leqq +\infty$, $M_q(u; t)$ is a real-valued decreasing function of t.*

PROOF. Suppose $q = +\infty$. By Lemma 3, the maximum principle for subharmonic functions holds on the domain $\bar{G}_r$ and $M_\infty(u; r)$ is identical with the supremum of $u(z)$ taken over $\bar{G}_r$. This proves the assertion. Suppose $p \leqq q < +\infty$. The fact that $M_q(u; t) < +\infty$ will be seen from (9) in the next Lemma 5, so we show that M_q is decreasing on $(0, +\infty)$. For a fixed $z' \in \boldsymbol{C}^{n-1}$, put $u_{z'}(x_1 + it) = u(x_1 + it + i|z'|^2, z')^q$, $(x_1, t) \in \boldsymbol{R} \times (0, +\infty)$. Then $u_{z'}$ is subharmonic on $\boldsymbol{R} \times (0, +\infty)$ and we can write

$$M_q(u; t)^q = \int_{C^{n-1}} dz' \int_R u_{z'}(x_1 + it) dx_1 .$$

Lemma 3 implies that $u_{z'}(x_1 + it) \to 0$ as $|x_1 + it| \to +\infty$, uniformly on $\boldsymbol{R} \times [r, +\infty)$ for any $r > 0$. It follows from [3, Theorem 1] that the inner integral is an extended real-valued, decreasing function of t, so that $M_q(u; t)^q$ is decreasing. This completes the proof.

The Poisson kernel $P(z, \eta)$ for the domain D is given by

$$P(z, \eta) = \frac{2^{n-2}\Gamma(n)}{\pi^n} \frac{\rho(z, z)^n}{|\rho(z, \eta)|^{2n}} , \quad z \in D , \quad \eta \in \partial D .$$

$H_{n-1} := \boldsymbol{R} \times \boldsymbol{C}^{n-1}$ becomes the Heisenberg group under the group operation, $x \cdot y = (x_1 + y_1 + 2\,\mathrm{Im}\langle z', w'\rangle, z' + w')$ for $x = (x_1, z')$, $y = (y_1, w') \in H_{n-1}$. If we put $x \cdot w = (x_1 + w_1 + 2i\langle w', z'\rangle + i|z'|^2, z' + w')$ for $w = (w_1, w') \in \boldsymbol{C}^n$, we can write $(x_1 + it + i|z'|^2, z') = x \cdot ite$, with $e = (1, 0, \cdots, 0) \in \boldsymbol{C}^n$. Since $P(x \cdot ite, y \cdot 0) = P(ite, x^{-1} \cdot y \cdot 0)$, we have

$$\int_{H_{n-1}} P(x \cdot ite, y \cdot 0) dx = \int_{H_{n-1}} P(ite, u \cdot 0) du = 1 , \quad t > 0 , \quad y \in H_{n-1} .$$

LEMMA 5. *Put* $u = H_\delta^*[h]$ *for* $h \in L^{p,\delta}(D)$, $1 \leqq p < +\infty$. *Let* $\sigma = p^{-1}(n + 1 + \delta) - q^{-1}n$ *for* $p \leqq q \leqq +\infty$. *Then*

$$M_q(u; t) \leqq A(n, p, q, \delta)\|h\|_{p,\delta} t^{-\sigma} , \quad t > 0 . \tag{9}$$

If $p < q \leqq +\infty$, *then*

$$M_q(u; t) = o(t^{-\sigma}) \quad as \quad t \to 0^+ . \tag{10}$$

If $1 < p < q \leqq +\infty$, $p \leqq \lambda < +\infty$, *then*

$$\left(\int_0^{+\infty} M_q(u; t)^\lambda t^{\lambda\sigma - 1} dt\right)^{1/\lambda} \leqq A(n, p, q, \delta, \lambda)\|h\|_{p,\delta} . \tag{11}$$

PROOF. (9): Suppose $q = +\infty$. For $z = (x_1 + it + i|z'|^2, z') \in D$, we have $H_\delta^*(z, w) \leqq C(n, \delta) t^{-(n+1+\delta)} \rho(w, w)^\delta$, $w \in D$, by (8), so $|u(z)|^p \leqq C(n, \delta) t^{-(n+1+\delta)} (\|h\|_{p,\delta})^p$ and (9) follows. Suppose $p \leqq q < +\infty$. Note that $M_q(u; t)^q \leqq (C\|h\|_{p,\delta} t^{-(n+1+\delta)/p})^{q-p} M_p(u; t)^p$. For $z = (x_1 + it + i|z'|^2, z')$ and $w = (y_1 + is + i|w'|^2, w') \in D$, we see that

$$\begin{aligned}&\frac{\rho(z, z)^{n+1+\delta}}{|\rho(z, w)|^{2(n+1+\delta)}}\\ &\qquad \leqq 2^{n+1+\delta}t^{-(1+\delta)}\frac{t^n}{[(x_1 - y_1 + 2\,\mathrm{Im}\langle z', w'\rangle)^2 + (t + |z' - w'|^2)^2]^n}\\ &\qquad = C(n, \delta)t^{-(1+\delta)}P(z, \eta)\ ,\end{aligned}$$

where we have put $\eta = (y_1 + i|w'|^2, w') \in \partial D$. It follows that

$$|u(z)|^p \leqq C(n, \delta)t^{-(1+\delta)}\int_D P(z, \eta)\rho(w, w)^\delta |h(w)|^p dw\ ,$$

hence $M_p(u; t)^p \leqq C(n, \delta)t^{-(1+\delta)}(\|h\|_{p,\delta})^p$, which shows (9). (10): We follow [4, Theorem 1]. Take $\varepsilon > 0$. Choose $h_1 \in C_c(D)$ so that $\|h - h_1\|_{p,\delta} < \varepsilon$. Put $h_2 = h - h_1$. Then $u = H_\delta^*[h_1] + H_\delta^*[h_2] =: u_1 + u_2$ and $M_q(u; t) \leqq M_q(u_1; t) + M_q(u_2; t)$. Since $M_\infty(u_1; t) \leqq \|h_1\|_\infty$ and since $M_\infty(u_2; t) < Ct^{-(n+1+\delta)/p}\varepsilon$ by (9), we get (10) in the case $q = +\infty$. Suppose $p < q < +\infty$. (9) implies that $M_q(u_1; t) \leqq C\|h_1\|_{q,\delta}t^{-((n+1+\delta)/q)-(n/q))}$, since $h_1 \in L^{q,\delta}(D)$, and $M_q(u_2; t) < Ct^{-((n+1+\delta)/p)-(n/q))}\varepsilon$, so (10) follows. (11): Define a measure $d\nu$ on $(0, +\infty)$ by $d\nu(t) = t^{n+\delta}dt$ and let $(Th)(t) = M_q(u; t)t^{-n/q}$, $t \in (0, +\infty)$, where $u = H_\delta^*[h]$ for $h \in L^{p,\delta}(D)$. Since $u(z) = H_\delta[h \circ \Psi^{-1}](\Psi(z))$, u is continuous on D. The conclusion of Lemma 3 holds for u, hence $M_\infty(u; t)$ is a continuous function of t. $M_q(u; t)$ is obviously measurable, if $p \leqq q < +\infty$. The inequality (11) can be seen as in Lemma 1.

LEMMA 6. *Denote by D_k the domain $\{(z_1, z') \in \boldsymbol{C}\times \boldsymbol{C}^{k-1} \mid \mathrm{Im}\, z_1 - |z'|^2 > 0\}$, $1 \leqq k \leqq n$. If $f \in A^{p,\delta}(D)$, $0 < p < +\infty$, then*

$$\left(\int_{D_k} |f(z', 0'')|^p \rho(z', z')^{n+\delta-k}dz'\right)^{1/p} \leqq A(n, k, p, \delta)\|f\|_{p,\delta}\ . \tag{12}$$

PROOF. For $g \in A^{p,\delta}(B)$, we define $(\Psi_\delta^* g)(z) = 2^{(2n+\delta)/p}(g \circ \Psi)(z)(z_1 + i)^{-2(n+1+\delta)/p}$, $z \in D$. It is easily seen that $\|g\|_{p,\delta} = \|\Psi_\delta^* g\|_{p,\delta}$ and that Ψ_δ^* is an isometry of $A^{p,\delta}(B)$ onto $A^{p,\delta}(D)$. Let Ψ_k be the Cayley transform of D_k onto B_k. Then, for $z = (z', 0'') \in D \cap L_k$, we can write $\Psi(z) = (\Psi_k(z'), 0'')$. The Jacobian of Ψ_k is $2^{2k}|z_1 + i|^{-2k-2}$. For $f \in A^{p,\delta}(D)$, take $g \in A^{p,\delta}(B)$ so that $f = \Psi_\delta^* g$. Applying Lemma 2 to g, we obtain (12).

THEOREM 3. *Let $f \in A^{p,\delta}(D)$, $0 < p < +\infty$. Put $\sigma = p^{-1}(n + 1 + \delta) - q^{-1}k$ for $p \leqq q \leqq +\infty$, $1 \leqq k \leqq n$. Then, for $p \leqq \lambda < +\infty$, the following hold:*

$$M_q(f, k; t) \leqq A(n, k, p, q, \delta)\|f\|_{p,\delta}t^{-\sigma}\ , \quad t > 0\ . \tag{13}$$

$$M_q(f, k; t) = o(t^{-\sigma}) \quad as \quad t \to 0^+\ . \tag{14}$$

$$\left(\int_0^{+\infty} M_q(f, k; t)^\lambda t^{\lambda\sigma-1}dt\right)^{1/\lambda} \leqq A(n, k, p, q, \delta, \lambda)\|f\|_{p,\delta}\ . \tag{15}$$

PROOF. We define $R_{k,n}$ by $(R_{k,n}f)(z') = f(z', 0'')$, $z' \in D_k$, for $f \in A^{p,\delta}(D)$, $1 \leqq k \leqq n-1$. Lemma 6 means that $R_{k,n}f \in A^{p,n+\delta-k}(D_k)$ with $\|R_{k,n}f\|_{p,n+\delta-k} \leqq A(n, k, p, \delta)\|f\|_{p,\delta}$. Hence it is sufficient to treat the case $k = n$. Now we have $|f|^{p/2} \leqq H_\delta^*[|f|^{p/2}] =: u$ with $(\| |f|^{p/2}\|_{2,\delta})^2 = (\|f\|_{p,\delta})^p$, by (3). From $M_q(f; t) \leqq M_{(2q/p)}(u; t)^{2/p}$, $t > 0$, (13) follows; also, (14) and (15) follow, in the case $p < q \leqq +\infty$. Next, rewriting the definition of $\|f\|_{p,\delta}$, we obtain

$$\left(2^\delta \int_0^{+\infty} M_p(f; t)^p t^\delta dt\right)^{1/p} = \|f\|_{p,\delta} .$$

This shows (15) in the case $p = q = \lambda$. The case $p = q < \lambda$ follows from (13). Finally, (14) can be proved for $p = q$, as follows: Letting $v = |f|^{p/2}$, we have $M_p(f; t)^p = M_2(v; t)^2$, a decreasing function of t by Lemma 4. It follows that, for $t > 0$,

$$\int_0^t M_p(f; s)^p s^\delta ds \geqq C M_p(f; t)^p t^{1+\delta} ;$$

this tends to 0 as $t \to 0^+$.

4. **The Mackey topology of** $A^{p,\delta}(B)$, $0 < p < 1$. Let $f \in A^{p,\delta}(B)$, $0 < p < +\infty$, and $c \geqq 1$. Then Theorem 2 implies that

$$\left(\int_{B_k} |f(z', 0'')|^{cp}(1 - |z'|^2)^{c(n+1+\delta)-k-1} dz'\right)^{1/(cp)} \leqq C\|f\|_{p,\delta} ,$$

an extension of Lemma 2. In particular, we have $\|f\|_{cp,c(n+1+\delta)-n-1} \leqq C\|f\|_{p,\delta}$, so $A^{p,\delta}(B) \subset A^{cp,c(n+1+\delta)-n-1}(B)$. This shows that Condition (1) of the proof of [9, Theorem 3] is satisfied. Moreover, $A^{p,\delta}(B)$ is an F-space with $(A^{p,\delta}(B))^*$ separating points of $A^{p,\delta}(B)$, by [7, (19)]. Thus, in the following, it suffices to see that Condition (2) in the proof of [9, Theorem 3] is satisfied.

THEOREM 4. *The Mackey topology of* $A^{p,\delta}(B)$, $0 < p < 1$, *is induced by the topology of* $A^{1,\sigma}(B)$, $\sigma = p^{-1}(n + 1 + \delta) - n - 1$.

PROOF. Fix $\beta > \sigma$. Put $(J(w))(z) = J(z, w) := (1 - |w|^2)^{-\sigma} K_\beta(z, w)$, z, $w \in B$. Then $J(w) \in A^{p,\delta}(B)$. We can see that $M := \sup\{\|J(w)\|_{p,\delta} \mid w \in B\} < +\infty$. Indeed, we have

$$(\|J(w)\|_{p,\delta})^p = A_0^p (1 - |w|^2)^{p(\beta-\sigma)} \int_B \frac{(1 - |z|^2)^\delta}{|1 - \langle z, w\rangle|^{p(n+1+\beta)}} dz ,$$

where the integral is $\approx (1 - |w|^2)^{n+1+\delta-p(n+1+\beta)}$ as $|w| \to 1$. Put $V = \{f \in A^{p,\delta} \mid \|f\|_{p,\delta} \leqq M\}$ and $W = \{f \in A^{p,\delta} \mid \|f\|_{1,\sigma} \leqq 1\}$. We denote by $[V]$ and $\overline{[V]}$, respectively, the absolutely convex hull of V and its $A^{p,\delta}$-closure and show that $W \subset \overline{[V]}$. Take $f \in W$. Then $f \in A^{1,\beta}$, so we can see that $f = K_\beta[f]$ in the same way as in (1). Since $f_r \to f$ in $A^{p,\delta}$, as $r \to 1$, we need only to show that $f_r \in \overline{[V]}$, $0 \leqq r < 1$. Now

$$f_r(z) = \int_B J(rz, w)(1 - |w|^2)^\sigma f(w) dw \ , \quad z \in B \ .$$

Let $\varepsilon > 0$. Since $J(rz, w)$ is uniformly continuous on $\bar{B} \times \bar{B}$, we can choose closed subsets of B, B_j, $1 \leqq j \leqq m$, with the interior being mutually disjoint, so that $\cup B_j = B$ and $|J(rz, w) - J(rz, u)| < \varepsilon$ for $z \in B$, w, $u \in B_j$, $1 \leqq j \leqq m$. Taking arbitrary $w_j \in B_j$ and putting $d\mu(w) = (1 - |w|^2)^\sigma f(w) dw$, we define $S_\varepsilon(z) = \sum_{j=1}^m J(rz, w_j)\mu(B_j)$. Then $S_\varepsilon \in [V]$ and $|f_r(z) - S_\varepsilon(z)| < \varepsilon$, $z \in B$. This completes the proof.

NOTE. After submission of the manuscript, K. Izuchi showed that Theorem 2 can directly be derived from [7, Theorem 4] by computation, without any use of H_δ. In this connection, we note here that [7, Theorem 4] is, conversely, an easy consequence of Theorem 2 and others. This method seems to have an advantage of being applicable in the setting of the domain D. We shall state the result as follows:

THEOREM 5. *Suppose* $f \in H^p(D)$, $0 < p < +\infty$. *Let* $p \leqq q \leqq +\infty$ ($p < q$, *when* $k = n$ *in* (17) *and* (18)) *and put* $\alpha = p^{-1}n - q^{-1}k$, $1 \leqq k \leqq n$. *Then the following hold.*

(16) $$M_q(f, k; t) \leqq A(n, k, p, q)\|f\|_p t^{-\alpha} \ , \quad t > 0 \ .$$

(17) $$M_q(f, k; t) = o(t^{-\alpha}) \quad as \quad t \to 0^+ \ .$$

(18) *For* $p \leqq \lambda < +\infty$,

$$\left(\int_0^{+\infty} M_q(f, k; t)^\lambda t^{\lambda\alpha - 1} dt\right)^{1/\lambda} \leqq A(n, k, p, q, \lambda)\|f\|_p \ .$$

PROOF. First suppose $p < q \leqq +\infty$, and take $c > 1$ so that $cp < q$. Then [7, Theorem 2, (4)] implies that $H^p(D) \subset A^{cp, cn-n-1}(D)$ with $\|f\|_{cp, cn-n-1} \leqq C(n, c)\|f\|_p$ for $f \in H^p(D)$. Theorem 3, (13) shows that $M_q(f, k; t) \leqq A(n, k, p, q)\|f\|_p t^{-\alpha}$, $1 \leqq k \leqq n$. Next let $p = q$ and $1 \leqq k \leqq n - 1$, (16) being trivial in the case $k = n$. Then [7, Theorem 2, (4)] again implies that $R_{k,n}f \in A^{p, n-k-1}(D_k)$ with $\|R_{k,n}f\|_{p, n-k-1} \leqq C(n, k)\|f\|_p$ for $f \in H^p(D)$. Applying Theorem 3, (13) to $R_{k,n}f$ on D_k, we obtain $M_p(f, k; t) \leqq C(n, k, p)\|f\|_p t^{-\alpha}$. (17) and (18) can similarly be verified.

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