Tôhoku Math. J.
40 (1988), 633-643.

TRANSLATION AND CANCELLATION OF SOCLE SERIES PATTERNS*

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(Received June 16, 1987)

Let G be a simply connected semisimple linear algebraic group over an algebraically closed field K of characteristic $p > 0$, B a Borel subgroup of G, and T a maximal torus of B. Let G_n be the n-th Frobenius kernel of G, and G_nB the closed subgroup scheme of G generated by G_n and B. Then we have induction functors $\hat{Z}_n = \mathrm{Ind}_B^{G_nB}$, $\mathrm{Ind}_{G_nB}^G$ and Ind_B^G. The first one of these functors is exact, but the others are only left exact, so we can further construct their right derived functors $H_n^i = R^i\,\mathrm{Ind}_{G_nB}^G$ and $H^i = R^i\,\mathrm{Ind}_B^G$ for all $i \geqq 0$. Thanks to the transitivity of inductions we obtain that $H^0 = H_n^0 \circ \hat{Z}_n$, or more generally, $H^i = H_n^i \circ \hat{Z}_n$.

Let $X(T)$ be the character group of T. Let $\lambda \in X(T)$, canonically regarded as a 1-dimensional B-module. In this paper we shall reveal a connection between the G_n-socle series of $\hat{Z}_n(\lambda)$ and the G-socle series of $H^0(\lambda)$. Our main result (cf. (2.1)) is a generalization of a similar result of Andersen (cf. [3, (4.4)]). As an application of the main result, we shall also discuss the G-socle series of $H^0(\lambda)$ for non-generic λ in the B_2 case.

This research was carried out while the first author was a visiting scholar at East China Normal University. He would like to express his appreciation to his advisor Professor Cao Xi-hua. Moreover, we are greatly indebted to the referee for giving us some useful suggestions and telling us the fact that the same result with different proof was obtained independently by John B. Sullivan in his preprint "the Euler character and cancellation theorem for Weyl modules", Feb. 1987.

1. Preliminaries. Let R be the root system of G with respect T, and W its Weyl group. Choose the set of positive roots, denoted by R_+, such that B corresponds to $-R_+$, and denote the set of simple roots by S. Let E be the Euclidean space, spanned by R with W-invariant inner product $\langle\ ,\ \rangle$. Then we can identify $X(T)$ with the abstract weight lattice of R. Let $X(T)_+$ be the set of dominant weights with respect to the

* Supported by the Sience Fund of the Chinese Academy of Sciences, No. 842016.

choice of R_+. Set

$$X_n(T) = \{\lambda \in X(T)_+ \mid \langle \lambda, \alpha^\vee \rangle < p^n, \text{ for all } \alpha \in S\} ,$$

where $\alpha^\vee = 2\alpha/\langle\alpha, \alpha\rangle$ is the coroot of α. Then each $\lambda \in X(T)$ can be uniquely written as $\lambda = \lambda^0 + p^n\lambda^1$ with $\lambda^0 \in X_n(T)$, $\lambda^1 \in X(T)$. It is well-known that $X(T)_+$ (resp. $X(T)$, $X_n(T)$) parametrizes the isomorphism classes of simple G-modules (resp. G_nB-modules, G_n-modules), and we shall denote by $L(\lambda)$ (resp. $\hat{L}_n(\lambda)$, $L_n(\lambda)$) the simple G-module (resp. G_nB-module, G_n-module) with highest weight λ.

Also, we set

$$C_0 = \{x \in E \mid 0 < \langle x + \rho, \alpha^\vee \rangle < p, \text{ for all } \alpha \in R_+\} ,$$

where ρ is half the sum of positive roots, and let $\bar{C}_0$ be the Euclidean closure of C_0.

The following facts are well-known:

(1.1) PROPOSITION. (1) *If* $\lambda \in X(T)_+$, *then*

$$L(\lambda)|_{G_nB} \cong \hat{L}_n(\lambda^0) \otimes (L(\lambda^1)|_B)^{(n)} ,$$

$$L(\lambda)|_{G_n} \cong L_n(\lambda^0) \oplus \cdots \oplus L_n(\lambda^0) \quad (\dim L(\lambda^1) \textit{ copies}) ;$$

(2) *If* $\lambda \in X(T)$, *then* $\hat{L}_n(\lambda)|_{G_n} \cong L_n(\lambda^0)$;

(3) *If* $\lambda, \nu \in X(T)$, *then*

$$\hat{L}_n(\lambda + p^n\nu) = \hat{L}_n(\lambda) \otimes p^n\nu ,$$

$$\hat{Z}_n(\lambda + p^n\nu) = \hat{Z}_n(\lambda) \otimes p^n\nu .$$

Here by (n) we mean the n-th Frobenius twist . Note that if E is a B-module, then $E^{(n)}$ is a G_nB-module.

(1.2) PROPOSITION. (1) *If* $\lambda \in X(T)$, *then*

$$H^0(\lambda) \neq 0 \Leftrightarrow \lambda \in X(T)_+ \Rightarrow H^i(\lambda) = 0 \quad \textit{for all} \quad i > 0 ;$$

(2) *If* $\lambda \in X(T)_+ \cap \bar{C}_0$, *then* $H^0(\lambda) = L(\lambda)$;

(3) *If* $\lambda \in X(T)_+$, *then* Soc $H^0(\lambda)$ (*the largest semisimple submodule of* $H^0(\lambda)$) *is* $L(\lambda)$ *and the other composition factors of* $H^0(\lambda)$ *have highest weights strictly smaller than* λ.

(1.3) PROPOSITION. *If E is a B-module, then* $H^i_n(E^{(n)}) = H^i(E)^{(n)}$. *Therefore, for* $\lambda \in X(T)$, *we have*

$$H^i_n(\hat{L}_n(\lambda)) = L(\lambda^0) \otimes H^i(\lambda^1)^{(n)} .$$

In particular, if $\lambda^1 \in X(T)_+ \cap \bar{C}_0$, *then* $H^0_n(\hat{L}_n(\lambda)) = L(\lambda)$ *and* $H^i(\hat{L}_n(\lambda)) = 0$ *for all* $i > 0$; *if* $\lambda \notin X(T)_+$, *then* $H^0(\hat{L}_n(\lambda)) = 0$.

2. Main theorem. If H is an affine group scheme and M is an H-module, we can inductively define the socle series of M: Let $\mathrm{Soc}_H^1 M = \mathrm{Soc}_H M$, the largest semisimple submodule of M, and, if $i > 1$, let $\mathrm{Soc}_H^i M$ be the full inverse image of $\mathrm{Soc}_H(M/\mathrm{Soc}_H^{i-1} M)$ in M. If M is finite dimensional, then $\mathrm{Soc}_H^i M = \mathrm{Soc}_H^{i+1} M = \cdots$ for large i. From the definition we see immediately that $\mathrm{Soc}_H^i M/\mathrm{Soc}_H^{i-1} M$ is a semisimple H-module. Conversely, it is not difficult to show that if

$$0 = M^0 \subset M^1 \subset M^2 \subset \cdots \subset M^r = M$$

is an H-submodule filtration of M with semisimple quotients, then $M^i \subset \mathrm{Soc}_H^i M$. A composition factor of $\mathrm{Soc}_H^i M/\mathrm{Soc}_H^{i-1} M$ is called a composition factor of M with socle level i.

In particular, if $\lambda \in X(T)$, we set $\hat{S}_n^i(\lambda) = \mathrm{Soc}_{G_n}^i \hat{Z}_n(\lambda)$. Note that we take the G_n-(not the G_nB-)socle series. Similarly, if $\lambda \in X(T)_+$, we shall set $S^i(\lambda) = \mathrm{Soc}_G^i H^0(\lambda)$. The fact that $G_n \lhd G_nB$ ensures that $\hat{S}_n^i(\lambda)$ is a G_nB-submodule of $\hat{Z}_n(\lambda)$. The main theorem we shall prove is the following:

(2.1) THEOREM. *Let $\lambda \in X(T)_+$ with the following property:*

(∗) *If $\hat{L}_n(\mu)$ with $\mu \in X(T)_+$ is a composition factor of $\hat{Z}_n(\lambda)$, then $\mu^1 \in \bar{C}_0$.*

Then $H_n^0(\hat{S}_n^i(\lambda)) = S^i(\lambda)$ for all i.

Note that the condition (∗) on λ is much weaker than that in [3, (4.4)]. In particular, our result is applicable for "non-generic" λ.

We have to prove some lemmas first.

(2.2) LEMMA. *Let $\lambda \in X(T)_+$, V_1 and V_2 be non-zero submodules of $H^0(\lambda)$. Then*

$$\mathrm{Hom}_G(V_1, V_2) = \begin{cases} K, & \text{if } V_1 \subset V_2\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

PROOF. Clearly we only need to prove $\mathrm{Hom}_G(V_1, H^0(\lambda)) = K$. From the reciprocity of inductions (cf. [5, (12.1.3)] or [8, (3.5)]) we obtain that

$$\mathrm{Hom}_G(V_1, H^0(\lambda)) \cong \mathrm{Hom}_B(V_1, \lambda)\,.$$

Then the lemma follows from the fact that λ is the highest weight in V_1 with multiplicity 1. q.e.d.

(2.3) COROLLARY. *Let $\lambda \in X(T)_+$ and $Ev_n \colon H^0(\lambda) \to \hat{Z}_n(\lambda)$ be the evaluation. Let V be a submodule of $H^0(\lambda)$. Then V is a submodule of $H_n^0(Ev_n(V))$.*

PROOF. The result is trivial if $V = 0$. Otherwise λ is a weight of V, and the restriction of Ev_n to the weight space of weight λ is non-zero. Hence $Ev_n|_V$ is non-zero. From the reciprocity

$$\mathrm{Hom}_{G_nB}(V, Ev_n(V)) \cong \mathrm{Hom}_G(V, H_n^0(Ev_n(V)))$$

we obtain a non-zero G-homomorphism $V \to H_n^0(Ev_n(V))$. Since $H_n^0(Ev_n(V))$ is also a submodule of $H^0(\lambda)$, V is a submodule of $H_n^0(Ev_n(V))$, by (2.2). q.e.d.

(2.4) LEMMA. *Let E be a G_nB-module, and it is semisimple as a G_n-module. Then*

$$E \cong \coprod_{\mu \in X_n(T)} \hat{L}_n(\mu) \otimes E(\mu)^{(n)} ,$$

where $E(\mu)$ is a B-module, for each $\mu \in X_n(T)$.

PROOF. cf. [3, (4.1)]. q.e.d.

PROOF OF (2.1). Step I: we shall show $H_n^0(\hat{S}_n^i(\lambda)) \subset S^i(\lambda)$. It is enough to show that $H_n^0(\hat{S}_0^i(\lambda))/H_n^0(\hat{S}_n^{i-1}(\lambda))$ is semisimple as a G-module. Let $E_i = \hat{S}_n^i(\lambda)/\hat{S}_n^{i-1}(\lambda)$. Applying the functor H_n^0 to the short exact sequence

$$0 \to \hat{S}_n^{i-1}(\lambda) \to \hat{S}_n^i(\lambda) \to E_i \to 0 ,$$

we obtain the following exact sequence:

$$0 \to H_n^0(\hat{S}_n^{i-1}(\lambda)) \to H_n^0(\hat{S}_n^i(\lambda)) \to H_n^0(E_i) .$$

Therefore, it is enough to show that $H_n^0(E_i)$ is semisimple as a G-module. By (2.4), we can decompose E_i as follows:

$$E_i \cong \coprod_{\mu \in X_n(T)} L_n(\mu) \otimes E_i(\mu)^{(n)} ,$$

where $E_i(\mu)$'s are B-modules. Then (1.3) together with the tensor identity (cf. [5, (12.1.6)] or [8, (3.6)]) gives

$$H_n^0(E_i) \cong \coprod_{\mu \in X_n(T)} L(\mu) \otimes H^0(E_i(\mu))^{(n)} .$$

Because of the condition (*) all dominant weights in $E_i(\mu)$ are in $\bar{C}_0$, so an easy induction on dimension shows that any composition factor of $H^0(E_i(\mu))$ has the form $L(\nu)$ with $\nu \in X(T)_+ \cap \bar{C}_0$. Thanks to the linkage principle (cf. [2]), there is no non-trivial extension between these $L(\nu)$'s. Hence $H^0(E_i(\mu))$ is semisimple as a G-module, and so is $H_n^0(E_i)$, by Steinberg's tensor product theorem (cf. [5, (6.2.2)] or [9, Th. 41]).

Step II: we shall prove $S^i(\lambda) = H_n^0(\hat{S}_n^i(\lambda))$ by induction on i. The case $i = 1$ is trivial, for $\hat{S}_n^1(\lambda) = \hat{L}_n(\lambda)$ and $S^1(\lambda) = L(\lambda)$. Now we assume that $H_n^0(\hat{S}_n^i(\lambda)) = S^i(\lambda)$. From the naturality of evaluations we see that $Ev_n|_{S^i(\lambda)}$ is exactly the evaluation from $H_n^0(\hat{S}_n^i(\lambda))$ into $\hat{S}_n^i(\lambda)$, so $Ev_n(S^i(\lambda)) \subset \hat{S}_n^i(\lambda)$, hence $(Ev_n(S^{i+1}(\lambda)) + \hat{S}_n^i(\lambda))/\hat{S}_n^i(\lambda)$ is a quotient module of $Ev_n(S^{i+1}(\lambda))/Ev_n(S^i(\lambda))$. On the other hand, the composite G_nB-homomorphism

$$S^{i+1}(\lambda) \twoheadrightarrow Ev_n(S^{i+1}(\lambda)) \twoheadrightarrow Ev_n(S^{i+1}(\lambda))/Ev_n(S^i(\lambda))$$

sends $S^i(\lambda)$ to zero, so it factors through $S^{i+1}(\lambda)/S^i(\lambda)$, which is a semisimple G-module and thus a semisimple G_n-module, by (1.1). It follows that $Ev_n(S^{i+1}(\lambda))/Ev_n(S^i(\lambda))$ is semisimple as a G_n-module, and so is its quotient module $(Ev_n(S^{i+1}(\lambda)) + \hat{S}^i_n(\lambda))/\hat{S}^i_n(\lambda)$. Therefore,

$$Ev_n(S^{i+1}(\lambda)) \subset \hat{S}^{i+1}_n(\lambda) .$$

Then (2.3) together with the left exactness of inductions gives

$$S^{i+1}(\lambda) \subset H^0_n(Ev_n(S^{i+1}(\lambda))) \subset H^0_n(\hat{S}^{i+1}_n(\lambda)) ,$$

which, together with Step I, ensures that

$$S^{i+1}(\lambda) = H^0_n(\hat{S}^{i+1}_n(\lambda)) . \qquad \text{q.e.d.}$$

3. Socle series patterns. We can illustrate the socle series of $\hat{Z}_n(\lambda)$ or $H^0(\lambda)$ in the following way: if $\mu \in X(T)$ (resp. $\mu \in X(T)_+$) such that $[\hat{Z}_n(\lambda): \hat{L}_n(\mu)] = r > 0$ (resp. $[H^0(\lambda): L(\mu)] = r > 0$), we attach r integers to the point $\mu \in E$, each integer pointing out the socle level of a factor that is isomorphic with $\hat{L}_n(\mu)$ (resp. $L(\mu)$). λ is the only point in E to which a single integer 1 is attached. Such a set of points in E together with the integers attached to each point is called the socle series pattern of $\hat{Z}_n(\lambda)$ (resp. $H^0(\lambda)$).

Note that from (1.1) we see that if λ, $\nu \in X(T)$, then

$$\hat{S}^i_n(\lambda + p^n\nu) = \hat{S}^i_n(\lambda) \otimes p^n\nu ,$$

hence

$$(3.1) \qquad [\hat{S}^i_n(\lambda + p^n\nu)/\hat{S}^{i-1}_n(\lambda + p^n\nu): \hat{L}_n(\mu + p^n\nu)] = [\hat{S}^i_n(\lambda)/\hat{S}^{i-1}_n(\lambda): \hat{L}_n(\mu)] ,$$

for all $\mu \in X(T)$. This means that the socle series pattern of $\hat{Z}_n(\lambda + p^n\nu)$ is exactly a translation of that of $\hat{Z}_n(\lambda)$. In this sense we say that $\hat{Z}_n(\lambda)$ and $\hat{Z}_n(\lambda + p^n\nu)$ have the same socle series pattern, or the socle series patterns of $\hat{Z}_n(\lambda)$'s are invariant under $p^nX(T)$-translations.

Recall that we call a weight $\lambda \in X(T)_+$ n-generic if every composition factor of $\hat{Z}_n(\lambda)$ has the form $\hat{L}_n(\mu)$ with $\mu^1 \in \bar{C}_0 \cap X(T)_+$. From our main results (2.1) and (1.3) we know that if λ is n-generic, then

$$(3.2) \qquad [S^i(\lambda)/S^{i-1}(\lambda): L(\mu)] = [\hat{S}^i_n(\lambda)/\hat{S}^{i-1}_n(\lambda): \hat{L}_n(\mu)] ,$$

for all $\mu \in X(T)$ (if $\mu \notin X(T)_+$ we understand that the number on the left hand side is 0). Therefore, the socle series pattern of $H^0(\lambda)$ is the same as that of $\hat{Z}_n(\lambda)$.

If both λ and $\lambda + p^n\nu$ (λ, $\nu \in X(T)_+$) are n-generic, then the socle series pattern of $H^0(\lambda + p^n\nu)$ is a translation of that of $H^0(\lambda)$. However, generally speaking, there is no $p^n X(T)$-translation invariance for the socle series patterns of $H^0(\lambda)$'s. The best thing we can say in this direction is the following weaker result:

(3.3) THEOREM (Translation and cancellation principle for socle series patterns). *Let λ be a dominant weight satisfying $(*)$ and $\nu \in X(T)_+$ such that $\lambda + p^n\nu$ is n-generic. Then*

$$[S^i(\lambda)/S^{i-1}(\lambda): L(\mu)] \leqq [S^i(\lambda + p^n\nu)/S^{i-1}(\lambda + p^n\nu): L(\mu + p^n\nu)]$$

for all $\mu \in X(T)_+$.

PROOF. Thanks to (2.1), we have the following exact sequence:

$$0 \to S^{i-1}(\lambda) \to S^i(\lambda) \to H^0_n(\hat{S}^i_n(\lambda)/\hat{S}^{i-1}_n(\lambda)) .$$

From this, and by (1.3), (3.1) and (3.2) we deduce that

$$\begin{aligned}[S^i(\lambda)/S^{i-1}(\lambda): L(\mu)] \leqq [H^0_n(\hat{S}^i_n(\lambda)/\hat{S}^{i-1}_n(\lambda)): L(\mu)] &\leqq [\hat{S}^i_n(\lambda)/\hat{S}^{i-1}_n(\lambda): \hat{L}_n(\mu)] \\ &= [\hat{S}^i_n(\lambda + p^n\nu)/\hat{S}^{i-1}_n(\lambda + p^n\nu): \hat{L}_n(\mu + p^n\nu)] \\ &= [S^i(\lambda + p^n\nu)/S^{i-1}(\lambda + p^n\nu): L(\mu + p^n\nu)] . \qquad \text{q.e.d.}\end{aligned}$$

Thanks to (3.3), in order to obtain the socle series pattern of $H^0(\lambda)$, λ as in (3.3), one can translate the socle series pattern of $H^0(\lambda + p^n\nu)$ for some n-genetic $\lambda + p^n\nu$ by $-p^n\nu$, then cancel all non-dominant points (and all integers attached to these points) and perhaps some dominant points or some integers attached to dominant points. Another interesting thing is that after the cancellation, the left integers in the pattern are still consecutive, starting from 1. In other words, if all composition factors with socle level i are cancelled in the above-mentioned procedure, then all composition factors with socle level $\geqq i$ are cancelled, too. This follows immediately from (2.1) and (3.3).

It is also well-known that if $\lambda \in X(T)_+$ is p-regular (i.e. $\langle \lambda + \rho, \alpha^\vee \rangle \not\equiv 0 \bmod p$ for all $\alpha \in R_+$), then the socle series pattern of $H^0(\lambda)$ only depends on the alcove to which λ belongs. More precisely, if λ and λ' are p-regular dominant weights in the same alcove, then

$$[S^i(\lambda)/S^{i-1}(\lambda): L(\mu)] = [S^i(\lambda')/S^{i-1}(\lambda'): L(\mu')]$$

for all $\mu \in X(T)_+$ linked to λ, where μ' is the unique weight that is contained in the alcove containing μ and that is linked to λ'. There is a similar story about $\hat{Z}_n(\lambda)$'s. Therefore, it is more convenient to picture the socle series patterns of $H^0(\lambda)$'s or $\hat{Z}_n(\lambda)$'s for p-regular λ's by attaching integers to alcoves other than points. In this way, the types of socle series patterns of $\hat{Z}_n(\lambda)$'s for p-regular λ's, or equivalently, those of $H^0(\lambda)$ for n-generic and p-regular λ's, are reduced to a small number. For

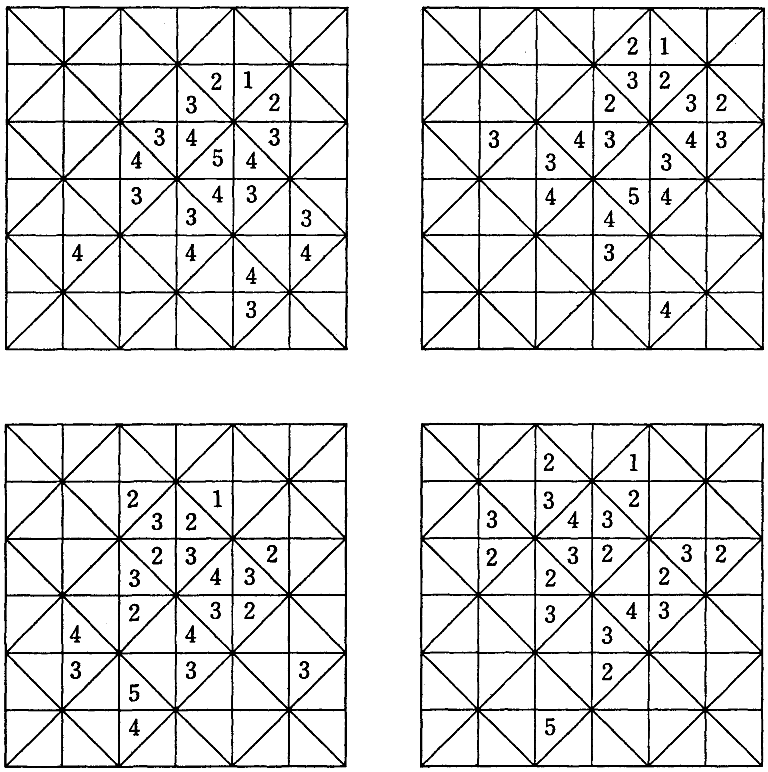

FIGURE 1. The 1-generic and p-regular socle series patterns in the B_2 case

example, in the B_2 case there are only four types of 1-generic and p-regular socle series patterns, which are shown in Figure 1 (cf. [6]).

4. Non-generic patterns in the B_2 case. Now we assume G to be of the B_2 type, and we limit ourselves to the case $n = 1$. In this case the cancellation is much easier to control, for each composition factor of $H^0(\lambda)$, where λ satisfies $(*)$ for $n = 1$, is of multiplicity 1, as shown in Figure 1. The only thing we have to do is to determine the composition factors of $H^0(\lambda)$ for non-generic λ.

For a G-module M, let ch M be the formal character of M. By Kempf's theorem we have $\operatorname{ch} H^i(\lambda) = 0$ for $\lambda \in X(T)_+$ and $i > 0$. So

$$\operatorname{ch} H^0(\lambda) = \sum_i (-1)^i \operatorname{ch} H^i(\lambda)$$

for $\lambda \in X(T)_+$. The left hand side of the above equality is the so-called "Euler character", which has a well-known "additivity". From it we get

$$\begin{aligned}\operatorname{ch} H^0(\lambda) &= \sum_i (-1)^i \operatorname{ch} H^i(\lambda) = \sum_i (-1)^i \operatorname{ch} H_1^i(\hat{Z}_1(\lambda)) \\ &= \sum_\mu [\hat{Z}_1(\lambda) : \hat{L}_1(\mu)] \sum_i (-1)^i \operatorname{ch} H_1^i(\hat{L}_1(\mu)) .\end{aligned}$$

If we write $\mu = \mu^0 + p\mu^1$ with $\mu^0 \in X_1(T)$, $\mu^1 \in X(T)$, then by (1.3), $H_1^i(\hat{L}_1(\mu)) = L(\mu^0) \otimes H^i(\mu^1)^{(1)}$. Hence we get

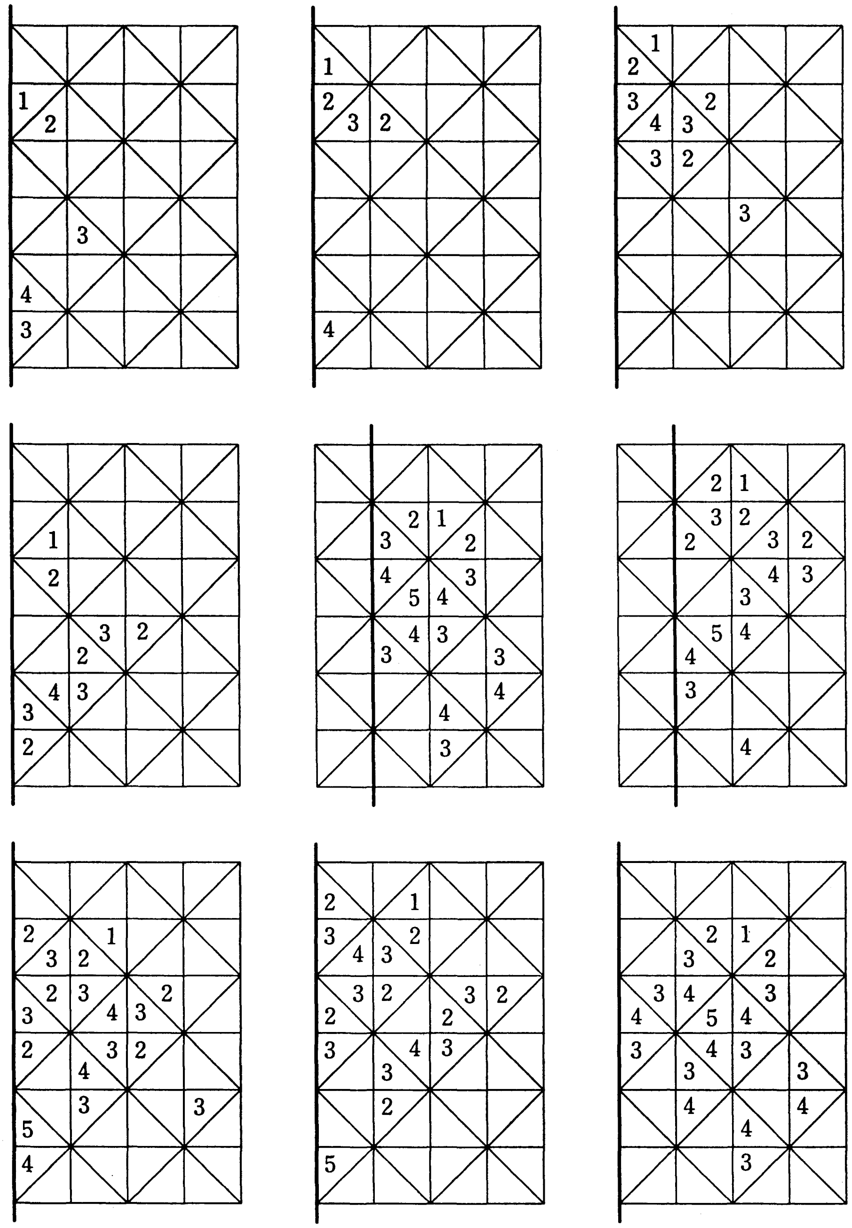

FIGURE 2. Non-generic patterns in the B_2 case
(Thick lines indicate the walls of the dominant chamber)

$$\begin{aligned}\operatorname{ch} H^0(\lambda) &= \sum_{\mu} [\hat{Z}_1(\lambda)\colon \hat{L}_1(\mu)] \sum_{i} (-1)^i \operatorname{ch} L(\mu^0) \operatorname{ch} H^i(\mu^1)^{(1)} \\ &= \sum_{\mu \in X(T)_+} \Big(\sum_{w \in W} (-1)^{l(w)} [\hat{Z}_1(\lambda)\colon \hat{L}_1(\mu^0 + pw.\,\mu^1)] \Big) \operatorname{ch} L(\mu^0) \operatorname{ch} H^0(\mu^1)^{(1)} \,.\end{aligned}$$

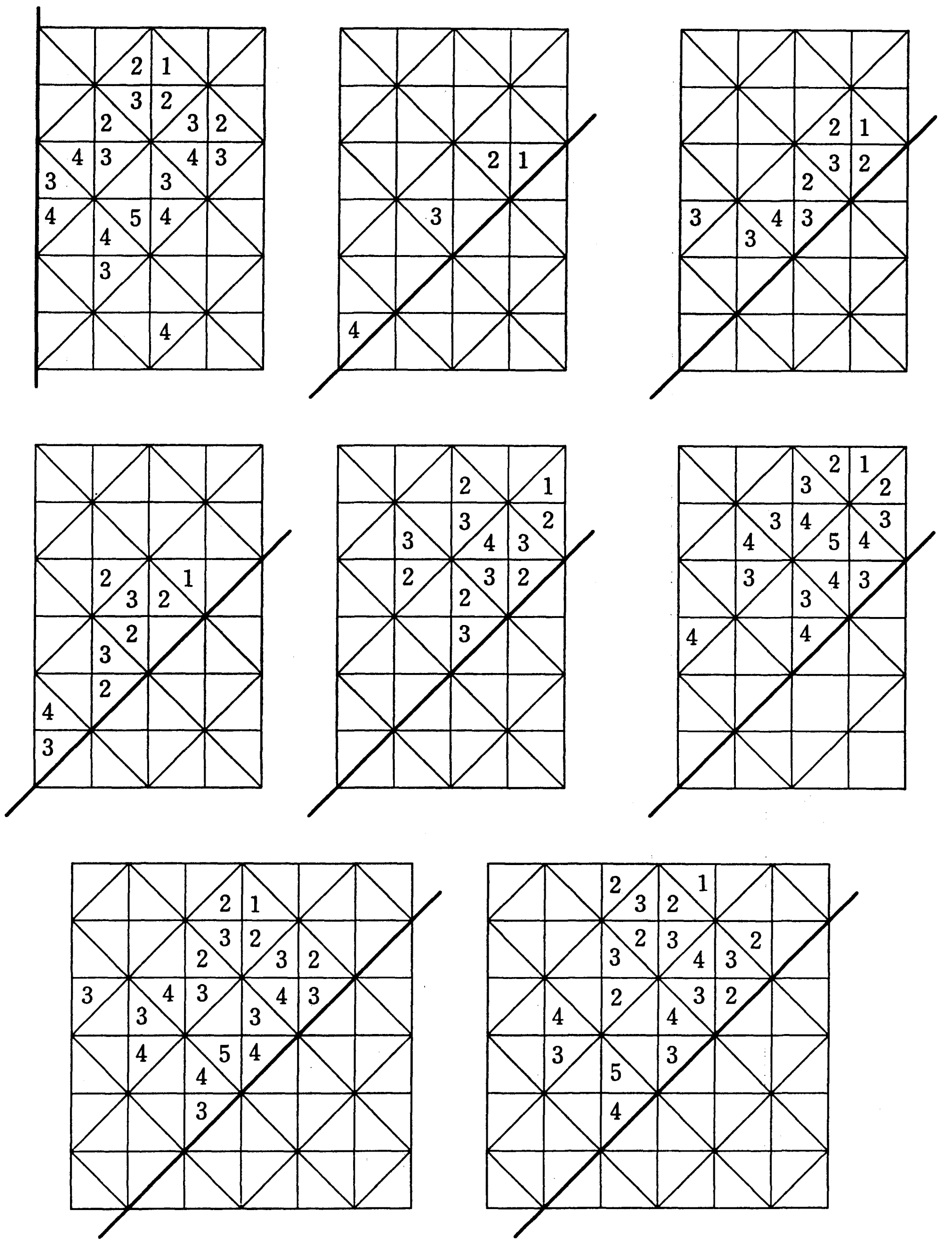

FIGURE 2. (continued)

Note that λ satisfies $(*)$ for $n = 1$. So the non-zero summands in the above formula are those with $\mu^1 \in \bar{C}_0$. Hence $H^0(\mu^1) = L(\mu^1)$. We have

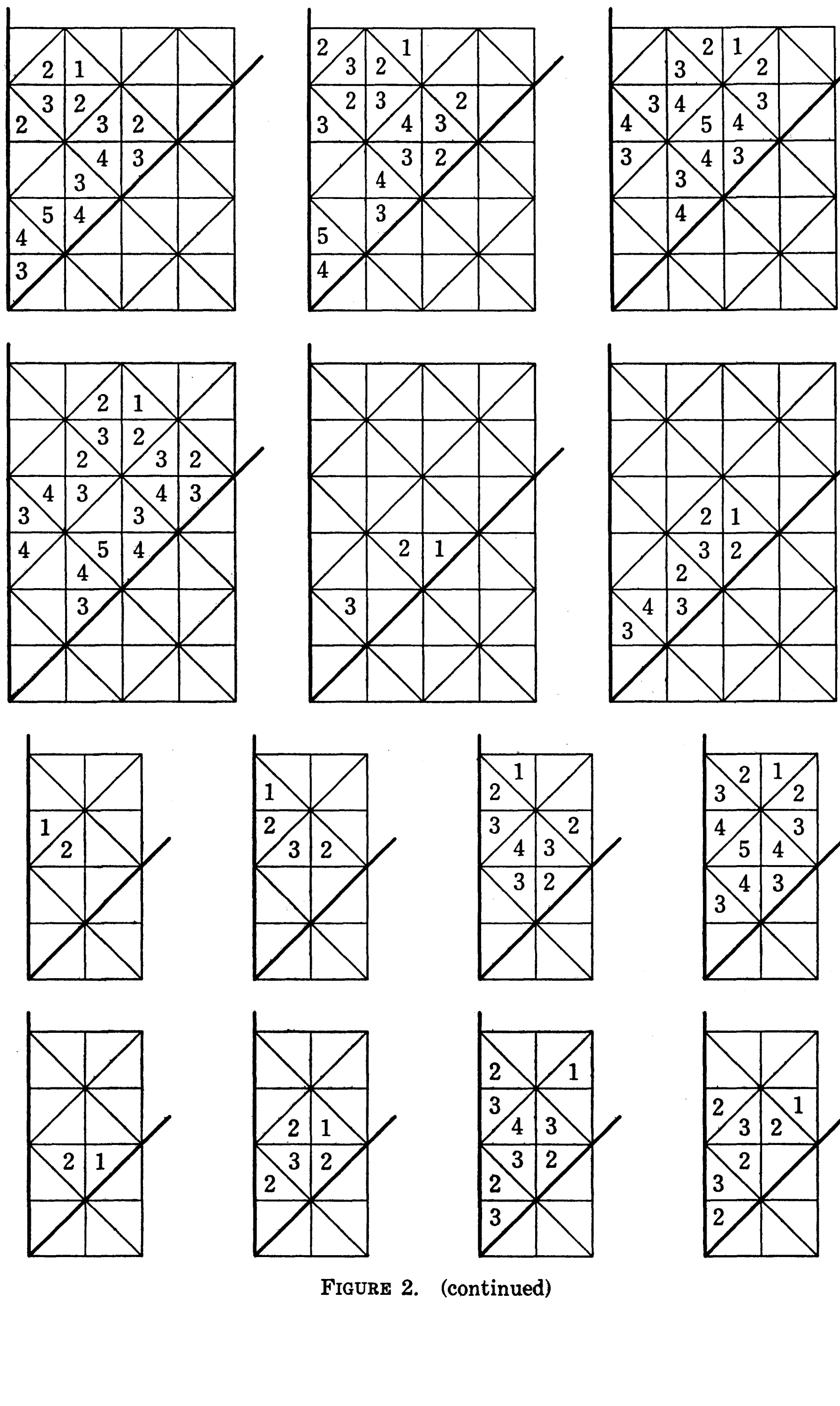

FIGURE 2. (continued)

$$\text{(4.1)} \qquad \operatorname{ch} H^0(\lambda) = \sum_{\mu \in X(T)_+} \Big(\sum_{w \in W} (-1)^{l(w)} [\hat{Z}_1(\lambda) : \hat{L}_1(\mu^0 + pw.\mu^1)] \Big) \operatorname{ch} L(\mu) .$$

The nice thing is that $[\hat{Z}_1(\lambda): \hat{L}_1(\mu)] = [\hat{Z}_1(\lambda + p\nu): \hat{L}_1(\mu + p\nu)]$ for any $\lambda, \mu, \nu \in X(T)$ (cf. (3.1)). Then using (4.1), together with Steinberg's tensor product theorem, one can determine the formal character of $H^0(\lambda)$, or equivalently, the composition factors of $H^0(\lambda)$. Hence we have determined the socle series pattern of $H^0(\lambda)$.

The non-generic patterns are shown in Figure 2. The patterns for the weights in $X_1(T)$ are well-known, so they are not shown.

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