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# COMPLETE NEGATIVELY PINCHED KÄHLER SURFACES OF FINITE VOLUME

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**0. Introduction.** Recently many differential geometers are interested in complete Riemannian manifolds of negative or nonpositive curvature. On the other hand, complete Kähler manifolds of negative curvature are interesting objects in complex analysis because of their function-theoretic properties. But few results are known about them. Moreover, in general it is hard to construct examples of negatively curved Kähler manifolds of finite volume. A typical example is an arithmetic quotient of a bounded symmetric domain of rank one.

In this paper, we shall investigate the 2-dimensional case. The purpose of this paper is to study how the differential geometric properties reflect the complex structures in the case of complete negatively pinched Kähler surfaces of finite volume. More precisely, we study complete Kähler surfaces $S$ such that

1. $\mathrm{vol}(S) < \infty$,
2. $-1 \leqq c^{-}(S) \leqq c^{+}(S) < 0$,

where $c^{+}(S)$, $c^{-}(S)$ denote the supremum and the infimum of the sectional curvatures of $S$, respectively. By [14, p. 363, Main Theorem], such a complete Kähler surface $S$ is a quasi-projective surface and can be compactified as a normal projective surface $\hat{S}$ by addition of one point to each end. We call $\hat{S}$ the Siu-Yau compactification of $S$. In this paper, we shall show that $\hat{S}$ has only almost simple elliptic singularities. In other words, we can determine the complex structure at infinity of $S$. Our proof depends on the classification of normal surface singularities with solvable local fundamental groups ([16]).

This paper is organized as follows. In Section 1, we determine the minimal resolution of the normal isolated singularities of $\hat{S}$ by using Wagreich's classification of normal isolated surface singularities with solvable local fundamental groups ([16]). In Section 2, we study the asymptotic behavior of the Kähler metric of $S$ toward infinity by using the existence of a complete Kähler-Einstein metric with negative scalar curvature ([6]). In Section 3, we prove a finiteness theorem of deformation types if we pinch the curvature and volume of the surfaces. In Section 4, we remark a duality between $L^2$-cohomology and intersection homology.

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**1. Complex structure at infinity.** Let $S$ be a complete Kähler surface of finite volume such that $-1 \leqq c^-(S) \leqq c^+(S) < 0$. Let $\hat{S}$ be the Siu-Yau compactification of $S$. Let $p$ be a point of $\hat{S} - S$. $p$ is a normal isolated singular point of $\hat{S}$ ($p$ is not a smooth point of $\hat{S}$ as we see below). Let $V$ be a Stein neighborhood of $p$ which is biholomorphic to a closed subvariety of the open unit ball $B^n$ in $\boldsymbol{C}^n$. We shall identify $V$ with the subvariety. We may assume without loss of generality that $p$ is the origin of $\boldsymbol{C}^n$. Let $S_\varepsilon^{2n-1}$ (resp. $B_\varepsilon^n$) denote the sphere of radius $\varepsilon$ around $p$. Then Milnor [9] has shown that the "neighborhood boundary"

$$K^3 = V \cap S_\varepsilon^{2n-1}, \quad 0 < \varepsilon \ll 1$$

is a $C^\infty$ 3-manifold, independent of $\varepsilon$ and $V \cap B_\varepsilon^n$ is the cone over $K$.

In [16], Wagreich classified normal surface singularities such that the fundamental group of the neighbourhood boundary is solvable.

On the other hand, the following holds.

PROPOSITION 1.1 ([13, p. 177, Proposition] and [3, p. 510, Theorem 3.1]). *Let $M$ be a complete Riemannian manifold of finite volume such that its sectional curvature is pinched between two negative constants.*

1. *If $n = \dim M \geqq 3$, the ends correspond bijectively to the conjugacy classes of the maximal nilpotent subgroups of rank $n-1$ of $\pi_1(M)$.*
2. *The ends have disjoint neighbourhood $U$ diffeomorphic to $N \times (0, \infty)$, where $N$ is a compact codimension one submanifold of $M$.*
3. *$N$ is diffeomorphic to $\Gamma \diagdown \boldsymbol{R}^{n-1}$, where $\Gamma$ is a maximal almost nilpotent subgroup of $\pi_1(M)$ of rank $n-1$.*

By Proposition 1.1, $\pi_1(K)$ is an almost nilpotent group of rank three. In particular, $p$ is a singular point of $\hat{S}$. Now we shall determine the complex analytic structure of the ends.

LEMMA 1.1. *Let $\mu: (\tilde{V}, \tilde{p}) \to (V, p)$ be a finite covering such that it is unramified over $V - \{p\}$ and the fundamental group of the neighborhood boundary of $(\tilde{V}, \tilde{p})$ is nilpotent. Let $\pi: V' \to \tilde{V}$ be the minimal resolution of $\tilde{V}$ and let $E$ be the exceptional divisor. Then one of the following holds.*

1. *$E$ is a nonsingular elliptic curve.*

2. *$E$ is a cycle of rational curves, i.e., every irreducible component of $E$ is a nonsingular rational curve and the dual graph of $E$ is a circle.*

3. *$E$. is a rational curve with one node.*

Befare proving Lemma 1.1, we shall review the definitions of simple elliptic and cusp singularities.

A normal surface singularity $(X, x)$ is called a simple elliptic singularity, if the exceptional set of the minimal resolution consists of a single nonsingular elliptic curve $A$. In this case $X - \{x\}$ is a quotient of the unit ball $\boldsymbol{B}^2$ in $\boldsymbol{C}^2$ as a germ [6, p. 49, Lemma 4]). See [11] for details.

Let $k$ be a totally real field of degree $n$ over the rationals and $M$ an additive subgroup of $k$ which is a free abelian group of rank $n$. Let $U_M^+$ be the group of those units $\varepsilon$ of $k$ which are totally positive and satisfy $\varepsilon M = M$. For a given pair $(M, E)$ with $E \subset U_M^+$ (where $E$ has rank $n-1$) one defines the group $G(M, E)$ consisting of the elements of the form:

$$\begin{pmatrix} \varepsilon & \mu \\ 0 & 1 \end{pmatrix},$$

where $\varepsilon \in E$, $\mu \in M$.

Let $\boldsymbol{H}$ be the upper half plane. The group $G(M, E)$ acts freely and properly discontinuously on $\boldsymbol{H}^n$ by $z_j \mapsto \varepsilon^{(j)} z_j + \mu^{(j)}$, where $x \mapsto x^{(j)}$, $1 \leqq j \leqq n$, denote the $n$ different embeddings of $k$ into the reals. Then $\boldsymbol{H}^n/G(M, E)$ defines a complex manifold which acquires a normal singularity when a point $\infty$ is added with neighborhoods $|\mathrm{Im}(z_1)\mathrm{Im}(z_2) \cdots \mathrm{Im}(z_n)| >$ const. The singularity at $\infty$ will be called a cusp singularity of type $(M, E)$.

Cusp singularities of dimension two are characterized as follows. Let $(X, x)$ be a normal surface singularity and let $\pi: \tilde{X} \to X$ be the minimal resolution of $(X, x)$. Let $A = \pi^{-1}(x)$ be the exceptional set. Then $(X, x)$ is a cusp singularity if and only if $A$ is an irreducible rational curve with a node singularity or $A$ is a "cycle" of rational curves (see [4], [5], [8]).

Proof of Lemma 1.1. First by assumption we may assume that $\pi_1(K)$ is nilpotent, hence solvable. In this case by the classification of [16], we can check easily that if $\pi_1(K)$ is nilpotent of rank three, then $E$ must be a nonsingular elliptic curve or a cycle of rational curves. In the first case, $(V, p)$ is a germ of a simple elliptic singularity and in the second case $(V, p)$ is a cusp singularity. q.e.d.

Let us consider the case in which $\pi_1(K)$ is not nilpotent. In this case, there exists a nilpotent subgroup $\Gamma$ of finite index in $\pi_1(K)$ by Proposition 1.1. Let $\mu: V_\Gamma^* \to V - \{p\}$ be the unramified covering corresponding to $\Gamma$. Then we can compactify one end of $V_\Gamma^*$ as a normal complex space

$V_\Gamma$ by adding one point. $V_\Gamma$ has a unique singular point $p_\Gamma$ whose local fundamental group is $\Gamma$. Hence $(V_\Gamma, p_\Gamma)$ is a germ of the simple elliptic singularity or a cusp singularity.

Hence we obtain the following lemma.

LEMMA 1.2. *$(V_\Gamma, p_\Gamma)$ is a germ of a simple elliptic singularity or a cusp singularity.*

DEFINITION 1.1. A germ of a normal isolated surface singularity $(V, p)$ is said to be almost simple elliptic, if there exists a finite covering $(W, q) \to (V, p)$ which is unramified on $W - \{q\}$ such that $(W, q)$ is a germ of a simple elliptic singularity.

The following theorem will be proved in the next section.

THEOREM 1.1. *$\hat{S}$ has only almost simple elliptic singularities.*

**2. Asymptotic behavior of the Kähler metric.** In this section, we use the same notation as in Section 1.

LEMMA 2.1. *If we take $V$ sufficiently small, then there exists a Kähler-Einstein metric with constant negative scalar curvature on $V_\Gamma - \{p_\Gamma\}$ which is complete toward $p_\Gamma$.*

PROOF. We note that $\boldsymbol{B}^2$ and $\boldsymbol{H}^2$ have canonical Kähler-Einstein metrics which are invariant under their automorphism groups, respectively. Then the lemma follows from Lemma 1.1. q.e.d.

Let $\omega_E$ be the Kähler-Einstein metric constructed in Lemma 2.1. Let $\omega_\Gamma$ denote the pull-back of the original Kähler metric on $S$ by $\mu\colon V_\Gamma^* \to V - \{p\}$. We note that $-1 \leqq c^-(S) \leqq c^+(S) < 0$ holds. Then since $\omega_E$ is complete toward $p_\Gamma$, by Yau's Schwarz lemma (which is essentially the maximum principle), we see that there exists a positive constant $C_1 > 1$ such that

$$\omega_\Gamma \leqq C_1\omega_E \quad \text{on} \quad V_\Gamma^* \tag{1}$$

and

$$C_1^{-1}\omega_E^2 \leqq \omega_\Gamma^2 \leqq C_1\omega_E^2 \quad \text{on} \quad V_\Gamma^* \tag{2}$$

Hence there exists a positive constant $C_2 > 1$ such that

$$C_2^{-1}\omega_E < \omega_\Gamma < C_2\omega_E \quad \text{on} \quad V_\Gamma^* . \tag{3}$$

PROOF OF THEOREM 1.1. For $g \in \pi_1(V_\Gamma^*)(=[S^1, V_\Gamma^*])$, we define the length $|g|$ of $g$ by

$$|g| = \inf_{[\gamma]=g} \operatorname{length}(\gamma) ,$$

where length($\gamma$) is the length of the loop $\gamma$ which represents the homotopy class $g$ measured by $\omega_E$. Assume that $(V_\Gamma, p_\Gamma)$ is a germ of a cusp singularity. Then there exists $g$ such that $|g| > 0$ (cf. [4]). This contradicts (3). By Lemma 1.1 this completes the proof. q.e.d.

LEMMA 2.2. *Let $W$ be the blowing up of $V$ with center $p$. Then $W$ has only isolated quotient singularities.*

PROOF. This lemma is an immediate consequence of [18, p. 85, Theorem 3.9] and Theorem 1.1. q.e.d.

Let $\pi: \bar{S} \to \hat{S}$ be the blowing up with center $\hat{S} - S$. Then by Lemma 2.1, $\bar{S}$ has only normal isolated quotient singularities as singularities. In other words, $\bar{S}$ is a normal $V$-surface. Hence in particular $\bar{S}$ is $\boldsymbol{Q}$-factorial, i.e., every Weil divisor on $\bar{S}$ is a $\boldsymbol{Q}$-Cartier divisor. Let $D$ be the exceptional divisor of $\pi$.

THEOREM 2.1. *$K_{\bar{S}} + D$ is ample modulo $D$ and numerically trivial on $D$.*

PROOF. First we assume that $\hat{S}$ has only simply elliptic singularities. In this case $\bar{S}$ is smooth and $\omega$ is equivalent to the natural locally symmetric metric toward the ends. By computing the Lelong number of the current $-\mathrm{Ric}_\omega$ on $\bar{S}$ along $D$, we have

$$[-\mathrm{Ric}_\omega] = 2\pi[K_{\bar{S}} + D]\,. \tag{4}$$

Since there exists a positive constant $C_3 > 1$ such that

$$C_3^{-1}\omega < -\mathrm{Ric}_\omega < C_3\omega \quad \text{on} \quad S \tag{5}$$

by assumption, $K_{\bar{S}} + D$ is ample modulo $D$ and numerically trivial on $D$.

Now the proof for the general case follows from the logarithmic ramification formula and Theorem 1.1. q.e.d.

**3. A finiteness theorem.** In this section, we shall give an application of Theorem 2.1. The following theorem was motivated by [3, p. 498, Theorem I].

THEOREM 3.1. *Let $C$ be a positive number. Then there exists only a finite number of deformation types in the set of complete Kähler surfaces $S$ such that $-1 \leqq c^-(S) \leqq c^+(S) < 0$ and $\mathrm{vol}(S) \leqq C$.*

REMARK 3.1. In the case of locally symmetric spaces, Theorem 3.1 is a special case of Wang's finiteness theorem ([17]).

PROOF OF THEOREM 3.1. We note that since $\bar{S}$ is a $V$-surface, all the results in transcendental algebraic geometry (Hodge decomposition, Riemann-Roch theorem, Hodge index theorem, Ramanujum vanishing theorem) still hold for $\bar{S}$ (with minor modifications). Let $L$ denote the $\boldsymbol{Q}$-line bundle $K_{\bar{S}}+D$.

First we shall consider the special case where $K_{\bar{S}}$ is a Cartier divisor. In this case, since $L$ is numerically trivial on $D$ by Theorem 2.1, $D$ is a Cartier divisor. We claim that the complete linear system $|5L|$ is base point free and embeds $S$ and contracts $D$ to finite points. We note that since $L$ is numerically trivial on $D$, $\mathrm{Bs}|mL|\cap D$ consists of irreducible components of $D$ for every $m>0$. Now we can prove the claim by entirely the same argument as in the proof of [12, p. 110, Theorem 5.8]. On the other hand, by the proof of Theorem 2.1, we see that there exists a positive constant $K$ such that

$$L^2 \leqq K\,\mathrm{vol}(S)\,. \tag{6}$$

Hence $|5L|$ gives an embedding of $\hat{S}$ onto a projective surface of degree $\leqq 25K\,\mathrm{vol}(S)$ in some $\boldsymbol{P}^N$. By using a generic projection from $\boldsymbol{P}^N$ to $\boldsymbol{P}^5$, we can embed $\hat{S}$ as a projective surface of degree $\leqq 25K\,\mathrm{vol}(S)$ in $\boldsymbol{P}^5$. Now the theorem follows from the finiteness property of the Hilbert scheme.

If $K_{\bar{S}}$ is not a Cartier divisor, then we take a canonical covering $f\colon \bar{S}_{\mathrm{can}}\to\bar{S}$ (cf. [7, p. 608]). We note that the canonical covering is a cyclic Galois covering. Let $D_{\mathrm{can}}$ denote the divisor $(f^*D)_{\mathrm{red}}$. By the logarithmic ramification formula, we see that

$$f^*(K_{\bar{S}}+D) = K_{\bar{S}_{\mathrm{can}}} + D_{\mathrm{can}}\,. \tag{7}$$

Then it is standard to see that $\deg f\cdot L$ is a Cartier divisor on $\bar{S}$ and $|5\deg f\cdot L|$ is base point free and embeds $S$ and contracts $D$, since $f\colon \bar{S}_{\mathrm{can}}\to\bar{S}$ is a cyclic Galois covering. The rest of the proof is the same as in the special case. q.e.d.

**4. A remark on the duality between $L^2$-cohomology and intersection homology.** Let $S$ be a complete Kähler manifold of finite volume such that $-1\leqq c^-(S)\leqq c^+(S)<0$ and let $\hat{S}$ be the Siu-Yau compactification of $S$. In Section 2, we saw that the Kähler metric of $S$ is quasi-isometric to the natural locally symmetric one near the simple elliptic singularities up to a finite covering toward every end of $\hat{S}$. Comparing this with Saper's metric in [15], we have the following theorem.

THEOREM 4.1.

$$H^i_{(2)}(S)\simeq IH_{4-i}(\hat{S})\,,\quad 0\leqq i\leqq 4\,,$$

*where $H^*_{(2)}$ and $IH_*$ denote the $L^2$-cohomology group and the (middle) intersection homology group, respectively.*

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