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ADDENDUM TO: TORSION AND DEFORMATION OF CONTACT METRIC STRUCTURES ON 3-MANIFOLDS

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Chern and Hamilton [1] introduced the torsion invariant $|\tau|$ on a compact contact Riemannian 3-manifold (M, g), where $\tau = L_{X_0} g$, and they conjectured that for fixed contact form ω, with X_0 inducing a Seifert foliation, there exists a complex structure $\varphi | B$ on $B = \ker \omega$ with the property that the Dirichlet energy

$$(1) \qquad E(g) = \frac{1}{2} \int_M |\tau|^2 \operatorname{vol}(M, g)$$

is critical over all CR structures. A CR structure on a manifold is a contact structure together with a complex structure on B. Since $\dim M = 3$, B has dimension 2, so a complex structure on B is equivalent to a conformal structure, i.e., knowing how to rotate through 90°. Hence, a Riemannian metric on a contact 3-manifold gives rise to a CR structure.

Let M be a contact manifold with a fixed contact form ω. Denote the space of all associated Riemannian metrics to the contact form ω by $\mathscr{M}(\omega)$. Since $E(g)$ vanishes, if and only if τ vanishes, this implies that X_0 is a Killing vector field. Can E have a critical point which is not a zero of E? Let g be a point of $\mathscr{M}$, and $\{g(t)\}$ be a curve in $\mathscr{M}$ with $g(0) = g$. Tanno [5] showed that g is a critical point of E, i.e., $(dE/dt)(0) = 0$, if and only if

$$(2) \qquad \nabla_{X_0} \tau = 2\tau \cdot \varphi .$$

Thus, following [1], $E(g)$ is critical over all CR structures if and only if (2) is satisfied. This is different from the condition $\nabla_{X_0} \tau = 0$ incorrectly obtained in [1, Theorem 5.4]. The reason for this discrepancy is given by Tanno in [5, p. 15]. The statement of the Theorem in [3] should therefore be modified by replacing the phrase "and critical torsion" by the phrase "such that $\nabla_{X_0} \tau = 0$". (The condition $\nabla_{X_0} \tau = 0$ is equivalent to the statement that the sectional curvature of all planes at a given point perpendicular to B are equal (see [1] and [5]).)

In the sequel, a critical point of E will be called a *critical metric*. If g is a critical metric, then by [3, Proposition 1, formula (ii)], $\nabla_{X_0} \tau = -2\psi$, where $\psi(X, Y) =$

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$g((L_{X_0}\varphi)X, Y)$. Formula (4) of [3] with $n=1$ and $b=a^2-a$ then reduces to

$$\tilde{S}=S+2(1-a)g+2(a-1)(a+2)\omega\otimes\omega \tag{3}$$

which is somewhat surprising since the torsion terms cancel. (This formula is the same as the one obtained in [2, p. 654], where it is assumed that X_0 is a Killing vector field.) With several important modifications, the method of proof of the Theorem in [3] also gives rise to the following:

THEOREM. *Let M be a compact orientable 3-manifold with contact metric structure $(\varphi, X_0, \omega, g)$ where g is critical. If there exists a constant a, $0<a<1$, such that $c<2a$, and if*

$$|\sigma|^2<2\left(a^2-\frac{c^2}{4}\right)\left(\frac{r}{2}+\frac{c^2}{4}+1-2a-c\right), \tag{4}$$

then M admits a contact metric of positive Ricci curvature. If, in addition, M is simply connected, it is diffeomorphic with the standard 3-sphere S^3.

PROOF. We show that $\tilde{S}>0$ at each $x\in M$. To this end, we determine the matrix $\tilde{S}$ in (3) with respect to a suitable adapted basis $\{E, \varphi E, X_0\}$ of T_xM, and compute the respective subdeterminants along the main diagonal. Assume $\sigma\neq 0$. (If $\sigma=0$, the same argument applies.) Since σ is a linear form on B, $\ker\sigma\neq 0$. Choose E such that $\sigma(E)=0$ and $\sigma(\varphi E)=|\sigma|$. Then,

$$\tilde{S}=\begin{bmatrix} S(E, E)+2(1-a) & S(E, \varphi E) & 0 \\ S(\varphi E, E) & S(\varphi E, \varphi E)+2(1-a) & |\sigma| \\ 0 & |\sigma| & S(X_0, X_0)+2(a^2-1) \end{bmatrix}.$$

By [3, p. 370], $S(X_0, X_0)=2(1-c^2/4)$. By [5, Lemma 7.1], the sectional curvatures $K(X_0, X)$ and $K(X_0, \varphi X)$ are related by

$$K(X_0, X)-K(X_0, \varphi X)=-(\nabla_{X_0}\tau)(X, X)=2\psi(X, X)$$

for any unit vector $X\in B$. Thus,

$$S(E, E)=S(\varphi E, \varphi E)+2\psi(E, E)\,.$$

By polarization,

$$S(E, \varphi E)=\psi(E, \varphi E)\,.$$

For,

$$S(E+\varphi E, E+\varphi E)-S(\varphi E-E, \varphi E-E)=2\psi(E+\varphi E, E+\varphi E)\,,$$

from which $4S(E, \varphi E)=2[\psi(E, E)+\psi(E, \varphi E)+\psi(\varphi E, E)+\psi(\varphi E, \varphi E)]=4\psi(E, \varphi E)$ since by [3, Proposition 1], trace $\psi=0$ and φ is symmetric with respect to ψ.

Now, the scalar curvature $r = \operatorname{trace} S$, so that

$$\begin{aligned} r &= S(E, E) + S(\varphi E, \varphi E) + S(X_0, X_0) \\ &= S(E, E) + S(E, E) - 2\psi(E, E) + 2\left(1 - \frac{c^2}{4}\right) \end{aligned}$$

from which

$$S(E, E) = \frac{r}{2} + \frac{c^2}{4} - 1 + \psi(E, E)$$

and

$$S(\varphi E, \varphi E) = \frac{r}{2} + \frac{c^2}{4} - 1 - \psi(E, E).$$

It follows that

$$\tilde{S} = \begin{bmatrix} \frac{r}{2} + \frac{c^2}{4} + 1 - 2a + \psi(E, E) & \psi(E, \varphi E) & 0 \\ \psi(\varphi E, E) & \frac{r}{2} + \frac{c^2}{4} + 1 - 2a - \psi(E, E) & |\sigma| \\ 0 & |\sigma| & 2\left(1 - \frac{c^2}{4}\right) + 2(a^2 - 1) \end{bmatrix}.$$

The inequalities $c < 2a$ and (4) ensure that $\tilde{S} > 0$ at $x \in M$. Indeed, since $c^2 = \psi(E, E)^2 + \psi(E, \varphi E)^2$, $\tilde{S}(E, E) > 0$, and

$$\begin{aligned} \tilde{S}(E, E)\tilde{S}(\varphi E, \varphi E) - \tilde{S}(E, \varphi E)^2 &= \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - \psi(E, E)^2 - \psi(E, \varphi E)^2 \\ &= \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - c^2 > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} &-|\sigma|\left\{|\sigma|\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right) + \psi(E, E)\right\} \\ &\quad + \left\{2\left(1 - \frac{c^2}{4}\right) + 2\left(a^2 - \frac{c^2}{4}\right)\right\}\left\{\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - c^2\right\} \\ &> -|\sigma|^2\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a + c\right) + 2\left(a^2 - \frac{c^2}{4}\right)\left\{\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a\right)^2 - c^2\right\} \\ &= \left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a + c\right)\left\{2\left(a^2 - \frac{c^2}{4}\right)\left(\frac{r}{2} + \frac{c^2}{4} + 1 - 2a - c\right) - |\sigma|^2\right\} > 0. \end{aligned}$$

This completes the proof. The last part is a consequence of a theorem of Hamilton [4].

COROLLARY. *Let M be a compact oriented 3-manifold with K-contact metric structure $(\varphi, X_0, \omega, g)$. If $r > -2$, then M admits a contact metric of positive Ricci curvature.*

REMARK. The quantity $r/2 + c^2/4 + 1$ in (4) is equal to $r^*/2$, where r^* is the generalized Tanaka-Webster scalar curvature defined in [5, p. 21].

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