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PINCHING THEOREMS FOR TOTALLY REAL MINIMAL SUBMANIFOLDS OF $CP^n(c)$

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Abstract. Let h be the second fundamental form of a compact minimal totally real submanifold M of a complex space form $CP^n(c)$ of holomorphic curvature c. For any $u \in TM$, set $\delta(u) = \|h(u, u)\|^2$. We prove that if $\delta(u) \leq c/12$ for any unit vector $u \in TM$, then either $\delta(u) \equiv 0$ (i.e. M is totally geodesic) or $\delta(u) \equiv c/12$. All compact minimal totally real submanifolds of $CP^n(c)$ satisfying $\delta(u) \equiv c/12$ are determined.

1. Introduction. Let M be an m-dimensional compact Riemannian manifold isometrically immersed in $CP^n(c)$, where $CP^n(c)$ is the complex projective space of constant holomorphic sectional curvature $c(>0)$ and of complex dimension n (all manifolds, mappings, functions and so on are assumed to be C^∞). Let h be the second fundamental form of the immersion. h is a symmetric bilinear mapping $TM_x \times TM_x \to TM_x^\perp$ for $x \in M$, where TM_x is the tangent space of M at x, and $TM_x^\perp$ is the normal space to M at x. Let $\Pi : UM \to M$ and UM_x be the unit tangent bundle of M and its fiber over $x \in M$, respectively. We set $\delta(u) = \|h(u, u)\|^2$ for any u in UM. $\delta(u)$ may be considered as a measure of the degree to which an immersion fails to be totally geodesic.

In a recent paper, Ros [10] proved that if M is a Kaehler submanifold of $CP^n(c)$ and if $\delta(u) < c/4$ for any $u \in UM$, then M is totally geodesic in $CP^n(c)$. In another paper [11], Ros gave a complete list of Kaehler submanifolds of $CP^n(c)$ satisfying the condition $\max_{u \in UM} \delta(u) = c/4$. In this paper, our purpose is to obtain the analogous results for another important class of submanifolds of $CP^n(c)$, namely, for totally real minimal submanifolds of $CP^n(c)$. Our main result is the following theorem.

THEOREM 1.1. *Let M be a compact totally real minimal submanifold of $CP^n(c)$. If $\delta(u) < c/12$ for any $u \in UM$, then M is totally geodesic in $CP^n(c)$.*

The above pinching for $\delta(u)$ is the best possible. Indeed, there exist submanifolds with $\max_{u \in UM} \delta(u) = c/12$, and Theorem 6.1 of Sec. 6 gives a complete list of such submanifolds. We will also show (Theorems 7.1–7.3 of Sec. 7) that in some cases the inequality $\delta(u) < c/12$ may be improved.

Our method is different from that of A. Ros. However we were influenced by his paper [10], as well as by paper [7] of N. Mok and T.-Q. Zhang. Results similar to that of

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Theorems 1.1 and 6.1 for minimal submanifolds of a sphere were proved recently in our paper [4]. There are also well known results of the type described in Theorems 1.1 and 6.1 which use $S(x)$ instead of $\delta(u)$, where $S(x)$ is the square of the length of the second fundamental form h at $x\in M$, [2], [5], [6].

2. Variational inequality. Let M be a compact m-dimensional Riemannian manifold isometrically immersed in an $(m+p)$-dimensional Riemannian manifold N. Let h be the second fundamental form of the immersed manifold M, and $\delta(u)=\|h(u,u)\|^2$ for $u\in UM$. Let $x\in M$. Suppose that $u\in UM_x$ satisfies $\delta(u)=\max_{v\in UM_x}\delta(v)$. We shall call u a *maximal direction* at x. Let $e_1,\cdots,e_{m+p}$ be an adapted frame at x. That means that $e_1,\cdots,e_m\in TM_x$ and therefore $e_{m+1},\cdots,e_{m+p}\in TM_x^{\perp}$. Assume that e_1 is a maximal direction at x. From now on let the indices $i,j,k,\cdots$ run from $1,\cdots,m$. Set $h_{ij}=h(e_i,e_j)\in TM_x^{\perp}$. Since e_1 is a maximal direction, we have at the point x for any $t,x^2,\cdots,x^m\in \boldsymbol{R}$

$$\left\|h\left(e_1+t\sum_{i=2}^{m}x^ie_i,\, e_1+t\sum_{i=2}^{m}x^ie_i\right)\right\|^2\le\left[1+t^2\sum_{i=2}^{m}(x^i)^2\right]^2\cdot\|h_{11}\|^2 . \tag{2.1}$$

Expanding in terms of t, we obtain

$$4t\sum_{i=2}^{m}x^i\langle h_{11},h_{1i}\rangle+O(t^2)\le 0\,,$$

where $\langle\ ,\ \rangle$ denotes the scalar product in M. It follows that

$$\langle h_{11},h_{1i}\rangle=0\,,\qquad i=2,\cdots,m\,. \tag{2.2}$$

We now choose an adapted frame at $x\in M$ such that in addition to (2.2), we have (cf. [4], p. 782),

$$\langle h_{11},h_{ij}\rangle=0\,,\qquad i\neq j\,. \tag{2.3}$$

Once more expanding (2.1) in terms of t, we obtain

$$2t^2\left[\sum_{i=2}^{m}(\|h_{11}\|^2-\langle h_{11},h_{ii}\rangle-2\|h_{1i}\|^2)(x^i)^2-2\sum_{\substack{i,j=2\\ i\neq j}}^{m}\langle h_{1i},h_{1j}\rangle x^ix^j\right]+O(t^3)\ge 0\,. \tag{2.4}$$

Since (2.4) must hold for any real x^i, we obtain the following *variational inequality*:

$$\|h_{11}\|^2-\langle h_{11},h_{ii}\rangle-2\|h_{1i}\|^2\ge 0\,,\qquad i=2,\cdots,m\,. \tag{2.5}$$

3. Generalized Bochner's Lemma. Let M be a Riemannian manifold and L be a covariant tensor field on M of the type $(0,k)$. At any $x\in M$, L can be considered as a multilinear mapping $L\colon\ TM_x\times\cdots\times TM_x\to\boldsymbol{R}$. Suppose that $u\in UM_x$ satisfies $L(u,\cdots,u)=\max_{v\in UM_x}L(v,\cdots,v)$. We shall call u a *maximal direction* at x with respect to L. For any $x\in M$, we set $f_L(x)=L(u,\cdots,u)$, where u is a maximal direction at x with respect to L. The next proposition is an obvious generalization of [7], Proposition 3.1.

PROPOSITION 3.1 (generalized Bochner's Lemma). *Let M be a compact Riemannian manifold and L be a covariant tensor field on M of the type $(0, k)$. If $(\Delta L)(u, \cdots, u) \geq 0$ for any maximal direction u with respect to L, where Δ denotes the Laplace operator, then f_L=const on M and $(\Delta L)(u, \cdots, u)=0$ for any maximal direction u.*

PROOF. It is easy to see that f_L is a continuous function on M. We shall show that f_L is subharmonic in the generalized sense. Fix $x \in M$ and let u be a maximal direction at x. In an open neighbourhood U_x of x within the cut-locus of x we shall denote by $v(y)$ the tangent vector to M obtained by parallel transport of $u=v(x)$ along the unique geodesic joining x to y within the cut-locus of x. Define $g_x(y)=L(v(y), \cdots, v(y))$. Then

$$(\Delta g_x)(x)=\Delta[L(v(y), \cdots, v(y))]_{y=x}=(\Delta L)(u, \cdots, u) \geq 0 .$$

For the Laplacian of continuous functions, we have the generalized definition

$$(\Delta f_L)(x)=c \lim_{r \to 0} r^{-2}\left(\int_{B(x,r)} f_L \Big/ \int_{B(x,r)} 1 - f_L(x)\right),$$

where c is a positive constant and $B(x, r)$ denotes the geodesic ball of radius r with the center at x. With this definition f_L is subharmonic on M if and only if $(\Delta f_L)(x) \geq 0$ at each point $x \in M$. Since $g_x(x)=f_L(x)$ and $g_x \leq f_L$ on U_x, $(\Delta f_L)(x) \geq (\Delta g_x)(x) \geq 0$. Thus, $f_L(x)$ is subharmonic and hence constant on M. It now follows that $g_x(x)=L(u, \cdots, u)$ is the maximum value of g_x on U_x. Hence $(\Delta g_x)(x)=(\Delta L)(u, \cdots, u) \leq 0$. Comparing with $(\Delta L)(u, \cdots, u) \geq 0$, we obtain that $(\Delta L)(u, \cdots, u)=0$.

4. A formula for a Laplacian. Let M be a compact m-dimensional Riemannian manifold isometrically immersed in an $(m+p)$-dimensional locally symmetric Riemannian manifold N, where $p \geq 2$. For any point $x \in M$, let $e_1, \cdots, e_{m+p}$ be an adapted frame at x such that e_1 is a maximal direction at x, and $\langle h_{11}, h_{ij} \rangle = 0$ for $i \neq j$. Let us define a tensor field $L=(L_{ijkl})$ of the type $(0, 4)$ on M by the formula

$$L_{ijkl}=\langle h_{ij}, h_{kl} \rangle . \tag{4.1}$$

It is clear that $\delta(u)=L(u, u, u, u)$ for any $u \in UM$. Let the indices a, b, c, d run from $1, \cdots, m+p$, and the indices $\alpha, \beta, \gamma, \delta$ run from $m+1, \cdots, m+p$. Denote by $R=(R_{abcd})$ the curvature tensor of N. We shall also write $(\Delta L)_{ijkl}=(\Delta L)(e_i, e_j, e_k, e_l)$ and $h_{ij}=\sum_\alpha h_{ij}^\alpha e_\alpha$.

LEMMA 4.1.

$$\frac{1}{2}(\Delta L)_{1111}=4\sum_{\alpha,\beta,i} h^{\alpha}_{11}h^{\beta}_{1i}R_{\alpha\beta i1}+\sum_{\alpha,\beta,i} h^{\alpha}_{11}h^{\beta}_{11}R_{\alpha ii\beta}+2\|h_{11}\|^2\sum_i R_{1i1i}$$
$$-2\sum_i\langle h_{11},h_{ii}\rangle R_{1i1i}+n\sum_{\alpha,\beta} h^{\alpha}_{11}H^{\beta}R_{\alpha 1\beta 1}+n\|h_{11}\|^2\langle h_{11},H\rangle$$
$$-2\|h_{11}\|^2\sum_i\|h_{1i}\|^2+2\sum_i\langle h_{11},h_{ii}\rangle\|h_{1i}\|^2-\sum_i\langle h_{11},h_{ii}\rangle^2+\sum_i\|\nabla_i h_{11}\|^2. \tag{4.2}$$

where $H=\sum_\alpha H^\alpha e_\alpha$ denotes the mean curvature vector $H=1/m\sum_i h_{ii}$.

PROOF. $(1/2)(\Delta L)_{1111}=\langle h_{11},(\Delta h)_{11}\rangle+\sum_i\|\nabla_i h_{11}\|^2$. The lemma follows readily from J. Simon's formula for Δh, [3], [12].

5. Totally real minimal submanifolds of $CP^n(c)$. Let now M be a compact m-dimensional minimal totally real submanifold of $CP^n(c)$. Since M is minimal, the mean curvature vector $H\equiv 0$ on M. M is called *totally real* if for any $x\in M$, $J(TM_x)\subset TM_x^\perp$, where J is the almost complex structure of $CP^n(c)$. In what follows we will deal with adapted frames of the form

$$\{e_1,\cdots,e_m,e_{1^*},\cdots,e_{m^*},e_{2m+1},\cdots,e_{2m+q},e_{(2m+1)^*},\cdots,e_{(2m+q)^*}\},$$

where $e_{1^*}=Je_1,\cdots,e_{m^*}=Je_m$, $e_{(2m+1)^*}=Je_{2m+1},\cdots,e_{(2m+q)^*}=Je_{2m+q}$. Here $n=m+q$. Note that $e_1,\cdots,e_m\in TM_x$ and $e_{1^*},\cdots,e_{m^*},e_{2m+1},\cdots,e_{2m+q},e_{(2m+1)^*},\cdots,e_{(2m+q)^*}\in TM_x^\perp$. We will now prove our first main result.

PROOF OF THEOREM 1.1. Let the indices A, B run from $1,\cdots,m,2m+1,\cdots,2m+q$, and let $e_{A^*}=Je_A$. By [14], p. 136, all components R_{abcd} of the curvature tensor of $CP^n(c)$ are equal to zero with the exception of the following components and the components obtained with the help of obvious symmetries:

$$\begin{aligned}&R_{ABAB}=R_{AB^*AB^*}=R_{ABA^*B^*}=R_{A^*BB^*A}=c/4, \quad (A\neq B),\\ &R_{AA^*BB^*}=c/2, \quad (A\neq B),\\ &R_{AA^*AA^*}=c.\end{aligned} \tag{5.1}$$

Substituting (5.1) into (4.2) and using the fact that $h^{i^*}_{ik}=h^{k^*}_{ii}$ (see, for example, [13]), we obtain

$$\frac{1}{2}(\Delta L)_{1111}=\sum_{i=1}(\|h_{11}\|^2-\langle h_{11},h_{ii}\rangle)(\|h_{11}\|^2-\langle h_{11},h_{ii}\rangle-2\|h_{1i}\|^2)$$
$$+2\sum_i(\|h_{11}\|^4-\langle h_{11},h_{ii}\rangle^2)+3m\|h_{11}\|^2(c/12-\|h_{11}\|^2)$$
$$+c/4\sum_i(h^{i^*}_{11})^2+\sum_i\|\nabla_i h_{11}\|^2. \tag{5.2}$$

Since $\delta(u)<c/12$ for any $u\in UM$, we have that $\|h_{11}\|^2<c/12$. This fact, the Cauchy-

Schwarz inequality, and the variational inequality (2.5) show that each summand in (5.2) is non-negative. By Proposition 3.1, $(\Delta L)_{1111}=0$. Hence $3m\|h_{11}\|^2(c/12-\|h_{11}\|^2)=0$. Therefore $\|h_{11}\|=0$, and M is totally geodesic.

6. The case: $\max_{u\in UM}\delta(u)=c/12$. In this case $\|h_{11}\|^2\equiv c/12$ on M. As in the proof of Theorem 1.1, we obtain $(\Delta L)_{1111}=0$. Since each summand in (5.2) is non-negative, we obtain for $i=1,\cdots,m$,

$$(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle)(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle-2\|h_{1i}\|^2)=0\,, \tag{6.1}$$

$$\|h_{11}\|^4-\langle h_{11}, h_{ii}\rangle^2=0\,, \tag{6.2}$$

$$h_{11}^{i*}=0\,, \tag{6.3}$$

$$\nabla_i h_{11}=0\,. \tag{6.4}$$

By (6.2), $\|h_{11}\|^4=\langle h_{11}, h_{ii}\rangle^2\leq\|h_{11}\|^2\,\|h_{ii}\|^2\leq\|h_{11}\|^4$. Therefore $h_{ii}=\pm h_{11}$ for each $i=1,\cdots,m$. Since $\sum_{i=1}^m h_{ii}=0$, we obtain that m is even, $m=2r$, and (after suitable renumbering of $e_1,\cdots,e_m$) we can write $h_{11}=h_{22}=\cdots=h_{rr}=-h_{r+1r+1}=\cdots=-h_{2r2r}$. Let the indices λ, μ, ν, ξ run from $1,\cdots,r$, and let $\bar\lambda=\lambda+r$. Then

$$h_{\lambda\lambda}=h_{11}\,,\qquad h_{\bar\lambda\bar\lambda}=-h_{11}\,. \tag{6.5}$$

It follows from (2.5) and (6.5) that $h_{1\lambda}=0$, $\lambda\neq1$. Since, by (6.5), each direction e_i is maximal, it follows that

$$h_{\lambda\mu}=h_{\bar\lambda\bar\mu}=0\,,\qquad \lambda\neq\mu\,. \tag{6.6}$$

By (6.1), $\|h_{1\bar\lambda}\|^2=\|h_{11}\|^2$. Therefore

$$\|h_{\lambda\bar\mu}\|^2=\|h_{11}\|^2\,. \tag{6.7}$$

Expansion (2.4) now takes the form

$$-4t^2\sum_{\substack{i,j=2\\ i\neq j}}^{m}\langle h_{1i}, h_{1j}\rangle x^ix^j+O(t^3)\geq0\,.$$

Hence $\langle h_{1i}, h_{1j}\rangle=0$, $i\neq j$; $i,j\neq1$. Since each direction e_i is maximal, we have

$$\begin{aligned}&\langle h_{\lambda\bar\mu}, h_{\lambda\bar\nu}\rangle=0\,,\qquad \bar\mu\neq\bar\nu\,,\\ &\langle h_{\lambda\bar\nu}, h_{\mu\bar\nu}\rangle=0\,,\qquad \lambda\neq\mu\,.\end{aligned} \tag{6.8}$$

Once more, expanding (2.1) in terms of t, we find that

$$t^3\sum_{i,j,k}\langle h_{1i}, h_{jk}\rangle x^ix^jx^k+O(t^4)\leq0\,.$$

Hence $\langle h_{1i}, h_{jk}\rangle+\langle h_{1j}, h_{ki}\rangle+\langle h_{1k}, h_{ij}\rangle=0$, $i,j,k\neq1$. By (6.5)–(6.8) and since each vector e_i is a maximal direction, we obtain

$$\langle h_{\lambda\bar{\nu}}, h_{\mu\bar{\xi}}\rangle + \langle h_{\lambda\bar{\xi}}, h_{\mu\bar{\nu}}\rangle = 0\,, \qquad \lambda \neq \mu \quad \text{or} \quad \bar{\nu} \neq \bar{\xi}\,. \tag{6.9}$$

Using (2.3) and (6.5)–(6.9), we obtain by direct computation that $\delta(u)=c/12$ for any $u\in UM$. B. O'Neill [9] calls an immersion λ-isotropic if $\|h(u,u)\|=\lambda$ for any $u\in UM$. Therefore, the *immersion under consideration is* $\sqrt{c/12}$*-isotropic*. By (6.4), $\nabla_i h_{jj}=0$. By polarization, $\nabla_i h_{jk}=0$ for all i, j, k. Therefore, the *second fundamental form of the immersion is parallel*. From (6.3), it follows that $h^{i*}_{jj}=0$. By polarization,

$$h^{i*}_{jk}=0\,, \qquad i, j, k=1, \cdots, m\,. \tag{6.10}$$

For $x\in M$, put $N^1M_x=\{h(X, Y)\,|\,X, Y\in TM_x\}_{\boldsymbol{R}}$, where $\{*\}_{\boldsymbol{R}}$ denotes the real vector space spanned by $*$. N^1M_x is called the first normal space at x. Let $(N^1M_x)^{\perp}$ be the orthogonal complement of (N^1M_x) in $TM_x^{\perp}$. By (6.10),

$$J(TM_x)\subset(N^1M_x)^{\perp}\,. \tag{6.11}$$

H. Naitoh, [8], calls a submanifold satisfying condition (6.11) a submanifold of the type $P(\boldsymbol{R})$. Thus, the immersion under consideration is $\sqrt{c/12}$*-isotropic with parallel second fundamental form and of the type* $P(\boldsymbol{R})$.

All minimal totally real λ-isotropic immersions into $\boldsymbol{C}P^n(c)$ of the type $P(\boldsymbol{R})$ with parallel second fundamental form were completely classified by H. Naitoh in [8]. According to this classification, if we take $\lambda=\sqrt{c/12}$, we obtain one of the following immersions:

$$\varphi_{1,p}\colon\ \boldsymbol{R}P^2(c/12)\to\boldsymbol{C}P^{4+p}(c)\,,$$

$$\varphi_{2,p}\colon\ S^2(c/12)\to\boldsymbol{C}P^{4+p}(c)\,,$$

$$\varphi_{3,p}\colon\ \boldsymbol{C}P^2(c/3)\to\boldsymbol{C}P^{7+p}(c)\,,$$

$$\varphi_{4,p}\colon\ QP^2(c/3)\to\boldsymbol{C}P^{13+p}(c)\,,$$

$$\varphi_{5,p}\colon\ \mathrm{Cay}\,P^2(c/3)\to\boldsymbol{C}P^{25+p}(c)\,,$$

where $p=0, 1, 2, \cdots$, $S^2(c/12)$ is a sphere of curvature $c/12$, $\boldsymbol{R}P^2(c/12)$ is a real projective plane of curvature $c/12$, $QP^2(c/3)$ is a quaternion projective plane of Q-sectional curvature $c/3$, Cay $P^2(c/3)$ is a Cayley projective plane of c-sectional curvature $c/3$, and where $\varphi_{i,p}$, $(i=1, \cdots, 5;\ p=0, 1, 2, \cdots)$, are defined as follows:

Let $\pi_m\colon S^m(c/4)\to\boldsymbol{R}P^m(c/4)$ be the covering map, $\mu_{n,p}\colon\boldsymbol{R}P^n(c/4)\to\boldsymbol{C}P^{n+p}(c)$ be the natural totally geodesic imbedding, and let

$$\psi_1\colon\ \boldsymbol{R}P^2(c/12)\to S^4(c/4)\,,$$

$$\psi_3\colon\ \boldsymbol{C}P^2(c/3)\to S^7(c/4)\,,$$

$$\psi_4\colon\ QP^2(c/3)\to S^{13}(c/4)\,,$$

$$\psi_5: \quad \text{Cay}\, P^2(c/3) \to S^{25}(c/4)$$

be the first standard imbeddings of projective spaces, [1], p. 141. Set $\psi_2 = \psi_1 \circ \pi_2$, $n_1 = n_2 = 4$, $n_3 = 7$, $n_4 = 13$, $n_5 = 25$. Now we are able to give a formula for $\varphi_{i,p}$:

$$\varphi_{i,p} = \mu_{n_{i,p}} \circ \pi_{n_i} \circ \psi_i, \qquad i = 1, \cdots, 5; \quad p = 0, 1, 2, \cdots. \tag{6.12}$$

Thus, we obtain the following theorem:

THEOREM 6.1. *Let M be a compact m-dimensional manifold minimally immersed in $CP^n(c)$. Assume that M is totally real in $CP^n(c)$ and that* $\max_{u \in UM} \delta(u) = c/12$. *Then* $\delta(u) \equiv c/12$ *on UM and the immersion of M into $CP^n(c)$ is one of the immersions $\varphi_{i,p}$ defined by* (6.12).

7. Several additional results. Assume that $\dim_{\boldsymbol{R}} M = \dim_{\boldsymbol{C}} CP(c)$, that is, we have an immersion of M^m into $CP^m(c)$. Then $\sum_i (h_{11}^{i*})^2 = \|h_{11}\|^2$. In this case formula (5.2) takes the form

$$\frac{1}{2}(\Delta L)_{1111} = \sum_{i \neq 1} (\|h_{11}\|^2 - \langle h_{11}, h_{ii}\rangle)(\|h_{11}\|^2 - \langle h_{11}, h_{ii}\rangle - 2\|h_{1i}\|^2)$$
$$+ 2\sum_i (\|h_{11}\|^4 - \langle h_{11}, h_{ii}\rangle^2) + 3m\|h_{11}\|^2(c(m+1)/12m - \|h_{11}\|^2) + \sum_i \|\nabla_i h_{11}\|^2.$$

If $\|h_{11}\|^2 < c(m+1)/12$, then $(\Delta L)_{1111} \geq 0$ and we obtain the following theorem:

THEOREM 7.1. *Let M be a compact m-dimensional totally real minimal submanifold of $CP^m(c)$. If $\delta(u) < c(m+1)/12m$ for any $u \in UM$, then M is totally geodesic in $CP^m(c)$.*

The result in Theorem 7.1 is the best possible, since for $m = 2$ there is an example of a minimal totally real immersion $M^2 \to CP^2(c)$ with $\delta(c) \equiv c/8$, [8], p. 438.

Let us now assume that $\dim_{\boldsymbol{R}} M$ is an odd number, that is, $m = 2r + 1$. By (5.2),

$$\frac{1}{2}(\Delta L)_{1111} \geq 2\sum_i (\|h_{11}\|^4 - \langle h_{11}, h_{ii}\rangle^2) + 3m\|h_{11}\|^2(c/12 - \|h_{11}\|^2) \tag{7.1}$$
$$= cm/4\|h_{11}\|^2 - (m+2)\|h_{11}\|^4 - 2\sum_{i=2}^{m} (b_i)^2,$$

where $b_i = \langle h_{11} h_{ii}\rangle$. Since e_1 is a maximal direction, we have

$$-\|h_{11}\|^2 \leq b_i \leq \|h_{11}\|^2, \qquad i = 2, \cdots, m. \tag{7.2}$$

Because of minimality of the immersion,

$$\sum_{i=2}^{m} b_i = -\|h_{11}\|^2. \tag{7.3}$$

It is easily seen that the convex function $f(b_2, \cdots, b_m)=\sum_{i=2}^{m}(b_i)^2$ of $(m-1)$ variables $b_2, \cdots, b_m$ subject to linear constraints (7.2), (7.3) attains its maximal value when (after suitable renumbering of $e_1, \cdots, e_m$)

$$b_2=\cdots=b_r=-b_{r+1}=\cdots=-b_{2r}=\|h_{11}\|^2\,; \qquad b_{2r+1}=0\,.$$

By (7.1), we obtain that

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111} &\geq cm/4\|h_{11}\|^2-(m+2)\|h_{11}\|^4-2(m-2)\|h_{11}\|^4\\ &=(3m-2)\|h_{11}\|^2(cm/4(3m-2)-\|h_{11}\|^2)\,.\end{aligned}$$

If $\|h_{11}\|^2<cm/4(3m-2)$, then $(\Delta L)_{1111}\geq 0$, and we obtain:

THEOREM 7.2. *Let M be a compact m-dimensional totally real minimal submanifold of $CP^n(c)$. Assume that m is odd. If $\delta(u)<mc/4(3m-2)$ for any $u\in UM$, then M is totally geodesic in $CP^n(c)$.*

Combining the method of proofs of Theorems 7.1 and 7.2, we obtain:

THEOREM 7.3. *Let M be a compact m-dimensional totally real minimal submanifold of $CP^m(c)$. Assume that m is odd. If $\delta(u)<c(m+1)/4(3m-2)$ for any $u\in UM$, then M is totally geodesic in $CP^m(c)$.*

8. Remark. Assume that M is a compact Kaehler submanifold of $CP^n(c)$. Then

$$\begin{aligned}\frac{1}{2}(\Delta L)_{1111}=&\sum_{i\neq 1,1^*}(\|h_{11}\|^2+\langle h_{11}, h_{ii}\rangle)(\|h_{11}\|^2-\langle h_{11}, h_{ii}\rangle-2\|h_{1i}\|^2)\\ &+(n+4)\|h_{11}\|^2(c/4-\|h_{11}\|^2)+\sum_i\|\nabla_i h_{11}\|^2\,.\end{aligned}$$

If $\|h_{11}\|^2<c/4$, then $(\Delta L)_{1111}\geq 0$. Therefore, if $\delta(u)<c/4$, then M is totally geodesic. Thus, we obtain a different proof of a result of A. Ros, [10], mentioned in Section 1.

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