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ORTHOGONAL STANCE OF A MINIMAL SURFACE AGAINST ITS BOUNDING SURFACES

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0. Introduction. Suppose a minimal surface is spanning a certain nice surface besides the usual boundary curves. Then there arises a question: Under some additional condition, e.g., that the minimal surface considered is stable, does it touch the boundary surfaces orthogonally along the intersecting traces? Many soap film experiments such as those Gergonne observed, seem to have convinced us of the affirmative answer. On the other hand, mathematical descriptions suggesting these circumstances can also be found somewhere; indeed a mean orthogonal intersection in the weak sense is shown in Courant [1], pp. 207–208.

The present study is concerned with one of the simplest cases in this circle of ideas, namely, with an oriented minimal surface S of disk type, which spans partly a given Jordan arc γ, and whose remaining boundary arc complementary to γ lies on a sufficiently smooth surface T prescribed. Under these circumstances the minimal surface in question turns out to meet the base surface T orthogonally at almost all points of the intersection arc, which is our main assertion to be proved in the ultimate.

1. Preliminaries. 1.1. First of all we shall have to specify the base surface as well as the handle attached to it. Let G denote a simply connected region comprising the closed upper semi-disk $B=\{(u, v) \mid u^2+v^2 \leq 1, v \geq 0\}$ in the (u, v)-plane. Consider a surface T parametrized by the C^1-mapping $T=T(u, v)$ of the (u, v)-plane into $\boldsymbol{R}^3$ with full rank 2 everywhere on G. Extremities of the handle γ to be settled on T are denoted by $\mathrm{P}_{\pm 1}$, which may have the respective coordinates $T(\pm 1, 0)$ without loss of generality.

In the following, the notation $|T|$, $|\gamma|$ shall mean the loci of the surface T and of the arc γ respectively, i.e., the bare point set free from any parametrizations as submanifolds.

1.2. For the sake of technical convenience we will require further nice properties on γ: Jordan simplicity, rectifiability and disjointness with $|T|$ except at both terminal points. Hence it admits representation as a continuous VB-function (Abbreviation for *function of bounded variation*) $\gamma=\gamma(\theta)$ on the semi-circle $\beta=\{e^{\sqrt{-1}\theta} \mid 0 \leq \theta \leq \pi\}$ in such a way that the point $\gamma(\theta)$ moves from P_{+1} to P_{-1} as θ increases from 0 until π and that $\theta_1 \neq \theta_2$ implies $\gamma(\theta_1) \neq \gamma(\theta_2)$.

PROPOSITION 1. *There exists a simple rectifiable arc γ' lying on $|T|$ which connects the two points* P_{-1} *and* P_{+1}.

PROOF. One may choose as γ', e.g., the image of the diameter $I=[-1, 1]$ of B under the mapping T. In fact, $\gamma'=T(u, 0)$ parametrized on the u-interval $[-1, 1]$ is a simple arc joining a pair of terminal points on account of the regularity of T and of the requirement $T(\pm 1, 0)=P_{\pm 1}$, which is of length

$$L[\gamma']=\int_{-1}^{1} |\partial T/\partial u|du \leq 2\max_{u\in I} |\partial T/\partial u| < +\infty .$$

q.e.d.

From now on every such γ' as introduced above may always be supposed to have its parameter interval $I=[-1, 1]$ on the u-axis.

1.3. Let $\hat{\gamma}$ be defined as the sum of these two paths γ and γ' in the following sense: $\hat{\gamma}=\hat{\gamma}(w)$ shall be parametrized on ∂B, so that the point $\hat{\gamma}(w)$ first proceeds from P_{+1} to P_{-1} along $|\gamma|$, and then comes back from P_{-1} to P_{+1} along $|\gamma'|$, making a round along the rectifiable Jordan curve in $\boldsymbol{R}^3$ with locus $|\gamma| \cup |\gamma'|$, when the complex parameter $w=u+\sqrt{-1}v$, starting at $w=1$, goes round ∂B counter-clockwise.

DEFINITION 1. The family $\hat{\Gamma}$ consists of all such rectifiable Jordan curve $\hat{\gamma}$ as mentioned above, which is of course non-void owing to Proposition 1.

DEFINITION 2. Let $\Phi=\{\phi\}$ denote the family of all continuous monotone non-decreasing map ϕ of ∂B onto itself, which leaves ± 1 unaltered.

PROPOSITION 2. *Every possible parametrization of a member of $\hat{\Gamma}$ on ∂B is a continuous* VB-*function on ∂B into $\boldsymbol{R}^3$, which can be obtained by the composition $\hat{\gamma}\circ\phi(w)$ of any fixed parametrization $\hat{\gamma}(w)$ of it with some element $\phi\in\Phi$, and vice versa.*

PROPOSITION 3. *Let an arbitrary $\hat{\gamma}$ of $\hat{\Gamma}$ be fixed. Then there exists some parameter change $\phi_0\in\Phi$ such that its own parametrization $\hat{\gamma}\circ\phi_0(w)$ is of class* Lip 1 *on ∂B.*

PROOF. For any rectifiable curve the parametrization $\mathfrak{x}(s)$ by arc-length s satisfies the Hölder condition with exponent 1 and with coefficient 1:

$$|\mathfrak{x}(s_1)-\mathfrak{x}(s_2)| \leq |s_1-s_2| .$$

2. Existence of a minimal surface which spans the given eared surface. 2.1. In this subsection we fix an element $\hat{\gamma}$ of $\hat{\Gamma}$ once and for all.

PROPOSITION 4. *In order to have all harmonic surfaces spanning a given rectifiable contour $\hat{\gamma}=\hat{\gamma}(w)\in\hat{\Gamma}$, it is necessary and sufficient to solve the Dirichlet problem with boundary value $\hat{\gamma}\circ\phi(w)$ $(w\in\partial B)$ for all admissible parameter change $\phi\in\Phi$.*

The boundary value $\hat{\gamma}\circ\phi(w)$ $(w\in\partial B)$ may be identified with the unique harmonic extension into $\operatorname{Int} B$ induced from it, which will cause no confusion in expressing the latter one by one and the same notation $\hat{\gamma}\circ\phi(w)$ $(w\in\operatorname{Int} B)$.

PROPOSITION 5. *There exists at least one elment ϕ_0 of Φ, such that the Dirichlet energy integral $D[\hat{\gamma}\circ\phi_0]$ for the harmonic vector $\hat{\gamma}\circ\phi_0(w)$ $(w\in\operatorname{Int} B)$ satisfies the inequality*

$$D[\hat{\gamma}\circ\phi_0(w)]\leq cL[\hat{\gamma}]$$

with an absolute constant c depending only on $\hat{\gamma}$.

PROOF. Let $\phi_0\in\Phi$ be such that $\hat{\gamma}\circ\phi_0(w)$ is of class Lip 1 on ∂B (Proposition 3). Then it suffices to show that the real-valued function

$$f(\theta)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos n\theta+b_n\sin n\theta)$$

with period 2π satisfying $|f(\theta_1)-f(\theta_2)|\leq|\theta_1-\theta_2|$ gives the coefficient estimates

$$n|a_n|\leq W, \qquad n|b_n|\leq W \qquad (n=1, 2, \cdots) \tag{1}$$

and

$$\sum_{n=1}^{\infty}(|a_n|+|b_n|)<+\infty\,, \tag{2}$$

where W is the total variation of $f(\theta)$ over $0\leq\theta\leq 2\pi$.

Once (1) and (2) will have been proved, then computation of the Dirichlet integral for the harmonic function in the unit disk with boundary value $f(\theta)$ will produce the inequality

$$\pi\sum_{n=1}^{\infty}n(|a_n|^2+|b_n|^2)\leq\pi W\sum_{n=1}^{\infty}(|a_n|+|b_n|)$$

as desired. But (2) is a particular case of Bernstein's theorem, while (1) is an elementary property on the VB-functions (see Zygmund [4], pp. 240–241 and p. 48). q.e.d.

DEFINITION 3 (The three-points-condition). Taking a point P_i on $|\gamma|$ distinct from both end-points $P_{\pm 1}$, we assume from now on that the condition $\gamma\circ\phi(\sqrt{-1})=P_i$ is always fulfilled for all $\phi\in\Phi$ and $\hat{\gamma}\in\hat{\Gamma}$ without particular mention.

PROPOSITION 6. *Let $\hat{\gamma}\in\hat{\Gamma}$ be fixed at will. Then the family $\{\hat{\gamma}\circ\phi(w)\mid\phi\in\Phi\}$ of harmonic vectors is normal on B.*

We owe this fruitful lemma to Courant ([1], pp. 103–104), to which the readers are referred.

PROPOSITION 7. *The infimum of $D[\hat{\gamma}\circ\phi(w)]$ over all $\phi\in\Phi$ is attained by some harmonic vector $\hat{\gamma}\circ\phi_0(w)$ on* Int *B, which is a minimal surface $M[\hat{\gamma}]$ spanning $\hat{\gamma}$ and it is* *noteworthy that* $D[M[\hat{\gamma}]] \leq cL[\hat{\gamma}]$.

The first part of this proposition is the well known existence theorem of a minimal surface, while the second an immediate consequence from the Dirichlet Principle and Proposition 5.

PROPOSITION 8. *The correspondence* $\partial B \mapsto \partial |M[\hat{\gamma}]|$ *is topological.*

These classical but significant results as well as the proofs are attributed to Douglas-Courant, which are included also in Courant's monograph above cited, or in Nitsche [2].

2.2. Now Proposition 7 allows us to confine ourselves to the subfamily $\hat{\Gamma}_0$ of contours, for which $L[\hat{\gamma}]$ is bounded above by some finite constant L_0, so far as we aim at the infimum of $D[\hat{\gamma} \circ \phi(w)]$ over all $\hat{\gamma} \in \hat{\Gamma}$ together with all $\phi \in \Phi$, and have only to deal with the minimal surfaces $\{M[\hat{\gamma}]\}$ under the restriction $\hat{\gamma} \in \hat{\Gamma}_0$.

PROPOSITION 9. *The family* $\{M[\hat{\gamma}]\}$ *with all* $\hat{\gamma} \in \hat{\Gamma}_0$ *is normal on* B.

PROOF. The assertion follows from Propositions 5 and 6.

An arbitrary minimizing sequence $\{M[\hat{\gamma}_n]\}_{n=1,2,\dots}$ for the functional $D[M[\hat{\gamma}]]$ on $\{M[\hat{\gamma}] \mid \hat{\gamma} \in \hat{\Gamma}_0\}$ contains a subsequence $\{S_n(w)\}_{n=1,2,\dots}$ converging uniformly on B to a continuous mapping $S^*(w)$ of B into $\boldsymbol{R}^3$, which is harmonic on $\operatorname{Int} B$.

It is clear that the harmonic map $S^*(w)$ of $\operatorname{Int} B$ into $\boldsymbol{R}^3$ spans partly the given arc $|\gamma|$, since it is continuous up to B and $|\gamma|$ is the topological image of β under the mapping S^*.

PROPOSITION 10. *The restriction of the mapping* $S^*(w)$ *to* I *parametrizes a rectifiable arc* γ^* *connecting the points* $\mathrm{P}_{\pm 1}$ *on* $|T|$, *which satisfies* $L[\gamma^*] \leq L_0$.

PROOF. The restricted mappings $M[\hat{\gamma}_n](w)$ $(n=1, 2, \cdots)$ to I can be regarded as parametrizations of rectifiable Jordan arcs, which are of uniformly bounded variations on I. So the uniform limit $S^*(w)$, again restricted to I, is a VB-function too by Helly's theorem, i.e., $S^*(I)$ is a rectifiable arc, lying on $|T|$ and connecting the points $\mathrm{P}_{\pm 1}$. Since the sequence $\{M[\hat{\gamma}_n](I)\}_{n=1,2,\dots}$ of arcs converges to the arc $S^*(I)$ in Fréchet's sense, we have the lower semi-continuity

$$L[\gamma^*] \leq \liminf_{n \to \infty} L[M[\hat{\gamma}_n](I)] \leq L_0 .$$

q.e.d.

PROPOSITION 11. γ^* *is a Jordan arc.*

PROOF. Suppose, contrary to the assertion, that I contains a pair of points $u < u'$ satisfying $S^*(u) = S^*(u')$. Then for a given $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \boldsymbol{Z}^+$, such that $n \geq N$ implies $|S_n(u) - S_n(u')| < \varepsilon$, where S_n is an abbreviation for $M[\gamma_n]$. The interior of the quadrilateral with the three rectilinear sides $\overline{-1, u}$, $\overline{u, u'}$, $\overline{u', 1}$ and a circular side $\widehat{1, -1}$ is mapped by S_n conformally onto the quadrilateral with vertices P_{-1}, $S_n(u)$, $S(u')$ and P_{+1} lying on $|S_n|$. But $\varepsilon > 0$ can be chosed as small as one pleases, hence the moduli of these two quadrilaterals can never be equal; in fact the one remains constant and the other becomes unbounded as $\varepsilon \to 0$, contrary to the conformal invariance of the modulus, which is absurd.

Thus we have proved (cf. Courant [1], pp. 105–107).

THEOREM 1. *There exists a minimal surface S^* of least area among all minimal surfaces bounded by both $|\gamma|$ and by any one of rectifiable Jordan arcs joining the points $P_{\pm 1}$ on $|T|$.*

3. Orthogonal intersection with the basis. 3.1. Now that we have at least one surface of least area as a solution to the variational problem, we may assume in advance that the upper bound L_0 of lengths in admissible rectifiable arcs is not smaller than $2L[\gamma^*]$. So we are prepared to construct many admissible contours $\{\hat{\gamma}\} \subset \hat{\Gamma}_0$ satisfying $L[\hat{\gamma}] \leq L_0$ with such L_0 in the neighbourhood of $|\gamma^*|$ as well as the admissible harmonic surfaces bounded by them.

3.2. Let $g(u) = g(u; m)$ denote the straight line $V = m(U-u)$ in the (U, V)-plane passing through the point $(u, 0)$ on I with slope m. Keeping $m \in \boldsymbol{R} \cup \{\pm\infty\}$ fixed and letting u vary on the interval I, we have a 1-parameter family $\{T[g(u)] \mid u \in I\}$ of curves on $|T|$. On the other hand let $\lambda(u)$ denote an arbitrary C^1-mapping of $I = [-1, 1]$ into $\boldsymbol{R}$, such that $\lambda(-1) = \lambda(1) = 0$. Now we mark the point $P(u)$ on the curve $T[g(u)]$, so that the vector $\eta = \eta(u, \varepsilon) = \overrightarrow{\gamma^*(u), P(u)}$ with a real parameter ε may satisfy the following conditions:

1° the absolute value of $\eta(u, \varepsilon)$ is equal to $|\varepsilon\lambda(u)|$;

2° the orientation of $\eta(u, \varepsilon)$ changes at the point $u \in I$, where $\lambda(u)$ changes its sign.

The locus of these extremities $P(u)$ for all u ranging over the interval $[-1, 1]$ shall be denoted by $C_{\varepsilon\lambda}(u)$.

PROPOSITION 12. *The point set $C_{\varepsilon\lambda}(u)$ is parametrizable on I as a rectifiable Jordan arc, which connects P_{-1} with P_{+1}.*

PROOF. Since $C_{\varepsilon\lambda}(u)$ is a single-valued continuous mapping of the interval I into $|T|$ satisfying $C_{\varepsilon\lambda}(\pm 1) = P_{\pm 1}$, and further $u_1 \neq u_2$ implies $C_{\varepsilon\lambda}(u_1) \neq C_{\varepsilon\lambda}(u_2)$ by definition, it is a Jordan arc connecting P_{-1} with P_{+1} on $|T|$.

Next we show the rectifiability of the arc $C_{\varepsilon\lambda}(u)$: indeed its length is seen to have a rough estimate from above

$$L[C_{\varepsilon\lambda}] \leq L[\gamma^*] + |\varepsilon| \int_{-1}^{1} \left| \frac{d\lambda(u)}{du} \right| du$$

in the following way. Take a finite set of points

$$-1<u_0<u_1<\cdots<u_{\nu-1}<u_\nu<\cdots<u_{N-1}<u_N=1$$

on I. Let the curve $T[g(u_\nu)]$ intersect the arc $C_{\varepsilon\lambda}(u)$ at the points $P_\nu = P(u_\nu)$ ($\nu=0, 1, 2, \cdots, N$), which form the vertex of the inscribed polygon $\Pi: P_0P_1\cdots P_{N-1}P_N$.

Consider the parallel displacement of the vector $\eta(u_{\nu-1})$ by the vector $\overrightarrow{\gamma^*(u_{\nu-1}), \gamma^*(u_\nu)}$, whose terminal point shall be denoted by Q_ν. If the division of I is sufficiently fine for a given $\varepsilon\in \boldsymbol{R}$, we have

$$|\overrightarrow{P_{\nu-1}, P_\nu}|<|\overrightarrow{P_{\nu-1}, Q_\nu}|+|\overrightarrow{Q_\nu, P_\nu}|=|\overrightarrow{\gamma^*(u_{\nu-1}), \gamma^*(u_\nu)}|+|\eta(u_\nu)-\eta(u_{\nu-1})|$$
$$<|\overrightarrow{\gamma^*(u_{\nu-1}), \gamma^*(u_\nu)}|+|d\eta(u_{\nu-1})|+o(|u_\nu-u_{\nu-1}|)\,.$$

Hence

$$\sum_{\nu=1}^{N}|\overrightarrow{P_{\nu-1}, P_\nu}|=\sum_{\nu=1}^{N}|\overrightarrow{\gamma^*(u_{\nu-1}), \gamma^*(u_\nu)}|+\sum_{\nu=1}^{N}\left|\frac{d\eta(u)}{du}\right|_{u_{\nu-1}}|u_\nu-u_{\nu-1}|+|\varepsilon|$$
$$\leq L[\gamma^*]+\sum_{\nu=1}^{N}\left|\frac{d\eta(u)}{du}\right|_{u_{\nu-1}}|u_\nu-u_{\nu-1}|+|\varepsilon|\,,$$

the right-hand side of which tends to

$$L[\gamma^*]+\int_{-1}^{1}\left|\frac{d\eta(u)}{du}\right|du+|\varepsilon|$$

as $\operatorname{Max}|u_\nu-u_{\nu-1}|\to 0$. So we finally have

$$L[C_{\varepsilon\lambda}]\leq L[\gamma^*]+|\varepsilon|\int_{-1}^{1}\left|\frac{d\lambda(u)}{du}\right|du\,.$$

q.e.d.

It follows immediately that

PROPOSITION 13. *The sum of paths $\gamma+C_{\varepsilon\lambda}$ parametrized on ∂B belongs to Γ, if and only if $|\varepsilon|$ is sufficiently small.*

3.3. Consider the unique harmonic vector in $\operatorname{Int} B$ with boundary value

$$\begin{cases}\eta(u, \varepsilon)\,, & (u\in I)\,,\\ 0\,, & (u\in|\beta|)\,,\end{cases}$$

which may be denoted again by $\eta=\eta(w, \varepsilon)$ without any confusion. The harmonic surface $S^*(w)+\eta(w, \varepsilon)$ then not only spans a contour of $\hat{\Gamma}_0$ but also has a finite Dirichlet integal on $\operatorname{Int} B$ for sufficiently small $\varepsilon\in\boldsymbol{R}$ (Proposition 5, 13), which is accordingly admitted to concurrence for our variational problem of partially variable boundary. Hence the minimality of $D[S^*(w)]$ yields that

(3) $0=\frac{d}{d\varepsilon}D[S^*(w)+\eta(w,\varepsilon)]\Big|_{\varepsilon=0}=\int_B\int\left\{\frac{\partial S^*}{\partial u}\frac{\partial}{\partial u}\left(\frac{\partial\eta}{\partial\varepsilon}\right)_{\varepsilon=0}+\frac{\partial S^*}{\partial v}\frac{\partial}{\partial v}\left(\frac{\partial\eta}{\partial\varepsilon}\right)_{\varepsilon=0}\right\}du\wedge dv\,.$

3.4. Let Int B be mapped conformally onto the interior of the unit disk $\tilde{B}$ in the complex $re^{\sqrt{-1}\theta}$-plane ($r,\theta\in\boldsymbol{R}$), so that the peripheral points $e^{\sqrt{-1}k\pi/2}$ ($k=0,1,2$) may be kept invariant. On setting $\tilde{S}(r,\theta)=S^*(w)$, $\tilde{\eta}(r,\theta)=\eta(w)$, we readily have the admissibility of $\tilde{S}+\tilde{\eta}$ as well as the minimality of $D[\tilde{S}]$ in reference to the parameter domain $\tilde{B}$ on account of the conformal invariance of Dirichlet integrals.

An arbitrary component X of the vector $\tilde{S}$ is a real-valued function continuous on Clo $\tilde{B}$, harmonic on Int $\tilde{B}$, and absolutely continuous on $\partial\tilde{B}$. The partial derivative $\partial Y(r,\theta)/\partial\theta$ of its harmonic conjugate $Y(r,\theta)$ has a non-tangential limit on $\partial\tilde{B}$ for every θ belonging to a measurable subset E of $[0,2\pi]$ with mes $E=2\pi$ (cf. Zygmund [4], p. 253, Theorem 1.6). We summarize the result in

PROPOSITION 14. *The radial limit*

$$\lim_{r\to1}\frac{\partial S(r,\theta)}{\partial r}$$

exists almost everywehre on $\partial\tilde{B}$.

3.5. Fix an arbitrary θ_0 of E. Then for any r on the interval $(0,1)$ there is a $\rho\in(r,1)$ satisfying

$$\frac{\tilde{S}(1,\theta_0)-\tilde{S}(r,\theta_0)}{1-r}=\frac{\partial\tilde{S}(r,\theta_0)}{\partial r}\Big|_{r=\rho} \tag{4}$$

by the Mean Value Theorem. When r increases to 1, the right-hand side of (4) has a finite limit in view of Proposition 14, since then ρ grows to 1. Hence

PROPOSITION 15. *The inward-pointing normal* $\partial\tilde{S}(1,\theta)/\partial n$ on $\partial\tilde{B}$ *exists and is equal to* $\lim_{r\to1}\partial\tilde{S}(r,\theta)/\partial r$ *for almost every* θ *of* $[0,2\pi]$.

3.6. Here we will prove the following modification of Green's theorem.

PROPOSITION 16. *Let* $R=\{(x,y)\,|\,a\le x\le b,\ \alpha\le y\le\beta\}$ *be a closed rectangle in the* (x,y)*-plane and let* $P(x,y)$, $Q(x,y)$ *real-valued* C^2*-functions defined in the interior of* R *satisfying the three requirements*:

1° *for a.e. value* y_0 *on the* y*-interval* $[\alpha,\beta]$, $\lim_{x\to b-0}P(x,y_0)$ *exists finitely, while* $\lim_{x\to a+0}P(x,y_0)=0$;

2° *for a.e. value* x_0 *on the* x*-interval* $[a,b]$, $\lim_{y\to\beta-0}Q(x_0,y)$ *exists finitely, while* $\lim_{y\to\alpha+0}Q(x_0,y)=0$;

3° $$A[R]=\int_R\int\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\wedge dy<+\infty\,.$$

Under these assumptions it holds that

$$\int_{\partial R}(Pdy - Qdx) = \int_R\int\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)dx \wedge dy\,.$$

PROOF. Owing to the assumption 3° the y-interval $[\alpha, \beta]$ contains some measurable subset E_1, such that $\operatorname{mes} E_1 = \beta - \alpha$ and that for every value $y_1 \in E_1$ the single integral

$$\int_a^b \left(\frac{\partial P(x, y_1)}{\partial x} + \frac{\partial Q(x, y)}{\partial y}\bigg|_{y=y_1}\right)dx$$

is well defined. On the other hand for any subinterval $[a', b']$ of (a, b) we have

$$\int_{a'}^{b'} \frac{\partial P(x, y_1)}{\partial x}dx = P(b', y_1) - P(a', y_1)\,.$$

Since it loses no generality to suppose here the assumption 1° is fulfilled on $y = y_1$, it follows that

$$\int_a^b \frac{\partial P(x, y_1)}{\partial x}dx = P(b, y_1) - P(a, y_1)\,.$$

Hence Fubini's theorem yields

$$\begin{aligned} A[R] &= \int_\alpha^\beta [P(b, y) - P(a, y)]dy + \int_\alpha^\beta \left(\int_a^b \frac{\partial Q(x, y)}{\partial y}dx\right)dy = \int_\alpha^\beta P(b, y)dy \\ &+ \int_a^b \left(\int_\alpha^\beta \frac{\partial Q(x, y)}{\partial y}dy\right)dx = \int_\alpha^\beta P(b, y)dy + \int_a^b [Q(x, \beta) - Q(x, \alpha)]dx \\ &= \int_\alpha^\beta P(b, y)dy + \int_a^b Q(x, \beta)dx = \int_{\partial R} [P(x, y)dy + Q(x, y)dx]\,. \end{aligned}$$

q.e.d.

Now let the open disk $\operatorname{Int} B$ in the $re^{\sqrt{-1}\theta}$-plane be put into correspondence in the one-to-one conformal manner with the interior of the rectangle R in the (x, y)-plane, so that the peripheral points $e^{2\pi\sqrt{-1}}$, $e^{\pi\sqrt{-1}/2}$ and $e^{\pi\sqrt{-1}}$ located on $\partial\tilde{B}$ may go to the corners (a, β), (a, α) and (b, α) on ∂R. Regarding the j-th components of the real vectors both $[\partial\tilde{\eta}/\partial\varepsilon]_{\varepsilon=0}$ and $\tilde{S}$ as functions in the variables x, y, we denote them, for brevity, by $X(x, y)$ and $Y(x, y)$ respectively ($j = 1, 2, 3$). Then $P = X(\partial Y/\partial x)$, $Q = X(\partial Y/\partial y)$ fulfill the requirements in Proposition 15 (see 3.5, 3.6), which guarantees that

$$\int_R\int\left(\frac{\partial X}{\partial x}\frac{\partial Y}{\partial x} + \frac{\partial X}{\partial y}\frac{\partial Y}{\partial y}\right)dx \wedge dy = \int_{\partial R} X\frac{\partial Y}{\partial n}ds\,. \tag{5}$$

Substitution of (3) into (5) implies

$$\int_{\gamma^*}\left(\frac{\partial\eta}{\partial\varepsilon}\right)_{\varepsilon=0}\frac{\partial S^*}{\partial n}ds=0\,. \tag{6}$$

3.7. Let $u\in I$ be fixed. If we denote by $t(u, m)$ the tangent vector to the curve $T[g(u; m)]$ at the point $(u, 0)$, then (6) is equivalent to

$$\int_{-1}^{1}\lambda(u)t(u; m)\frac{\partial S^*}{\partial n}du=0\,.$$

Therefore, in view of the arbitrariness of the real-valued C^1-function $\lambda(u)$ there exists a measurable subset E of I with mes $E=2$, such that

$$t(u; m)\perp\frac{\partial S^*(u, v)}{\partial v}\bigg|_{v=0}\qquad\text{for every}\quad u\in E\,. \tag{7}$$

Since the relation (7) ought to remain valid for every direction $m\in\boldsymbol{R}\cup\{\pm\infty\}$, we see finally that $\partial S^*(u, v)/\partial v\,|_{v=0}$ is a normal vector to the surface $|T(u, v)|$ at the point $\gamma^*(u)$, so far as u belongs to E. But the linear measure of the image $S^*(I\setminus E)$ is zero, since S^* is a minimal surface (Tsuji [3]; see Nitsche [2], pp. 288–289, too). We conclude

THEOREM 2. *The minimal surface S^* of least area spanning the given surface T with handle γ touches T orthogonally at almost all points of some rectifiable simple arc, which connects both terminal points of γ on T.*

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