Tôhoku Math. J.
42 (1990), 109–117

REPRESENTATION OF THE SOLUTION OPERATOR GENERATED BY FUNCTIONAL DIFFERENTIAL EQUATIONS

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(Received January 25, 1989, revised June 26, 1989)

1. Introduction. Hale gave a representation theorem for the solution operator generated by a neutral functional differential equation (NFDE) with finite delay, which represents the solution operator as the sum of a bounded linear operator with zero spectrum and a conditionally completely continuous operator (cf. [1]). This result proved to be very useful in studying the existence of periodic solutions of NFDE, and has been generalized to functional differential equations (FDE) with infinite delay (cf. [2]). In this paper, we generalize the latter result to NFDE with infinite delay, and give some applications to the existence of periodic solutions.

In the present paper, we denote the segment of a function $x(s)$ for $-\infty<s\leqslant t$ by x_t, and let X be a Banach space of some real functions $\phi: (-\infty, 0]\to R^n$ with the norm $\|\phi\|$ having the following properties:

(H_1) If $x: (-\infty, \sigma+A)\to R^n$, $A>0$, $\sigma\geqslant 0$, is continuous for $t\in[\sigma, \sigma+A)$ and $x_\sigma\in X$, then $x_t\in X$ and x_t is continuous for $t\in[\sigma, \sigma+A)$.

(H_2) There is a positive constant k_0 such that $|\phi(0)|\leqslant k_0\|\phi\|$, for $\phi\in X$, where $|\cdot|$ stands for a norm in R^n.

(H_3) There are positive constants K and M such that if x satisfies (H_1) then

$$\|x_t\|\leqslant K\sup_{u\in[\sigma,t]}|x(u)|+M\|x_\sigma\|\,,\qquad t\geqslant\sigma\,.$$

A continuous functional $D: [0, \infty)\times X\to R^n$ is said to be atomic (cf. [3]), if it can be represented as

$$D(t)\phi=A(t)\phi(0)-L(t)\phi\,,\qquad t\geqslant 0\,,\quad \phi\in X\,,$$

with a continuous nonsingular $n\times n$ matrix $A(t)$ and a bounded linear operator $L(t): X\to R^n$ which satisfy $\sup_{t\geqslant 0}|L(t)|\leqslant L$, $\sup_{t\geqslant 0}(|A(t)|+|A^{-1}(t)|)\leq A$ and $|L(t)\phi|\leqslant\gamma(\beta)\|\phi\|$ for $t\geqslant 0$, $\beta\geqslant 0$ and $\phi\in X$ with compact support contained in $(-\infty, 0]$, where L and A are positive constants and γ is a nonnegative continuous function on $[0, \infty)$ with $\gamma(0)=0$. Here and hereafter, $|L(t)|$ and $|A(t)|$ stand for the operator norms of $L(t)$ and $A(t)$, respectively.

For any atomic D and any $H\in C([\sigma, \infty), R^n)$, the equation

$$D(t)z_t=H(t)\,,\quad t\geqslant\sigma\geqslant 0\,,\quad z_\sigma=\phi\in X\,,\quad H(\sigma)=D(\sigma)\phi\,, \tag{1.1}$$

has a unique solution. Henceforth, we will denote the solution by $z(\sigma, \phi, H)$. If the zero solution of (1.1) with $H \equiv 0$ is uniformly asymptotically stable in the usual sense, then D is said to be stable. The following estimate for $\|z_t(\sigma, \phi, H)\|$ plays an important role in this paper:

LEMMA 1.1. (cf. [4]). *If D is table, then there exist positive constants a and b such that*

$$\|z_{\sigma+t}(\sigma, \phi, H)\| \leq be^{-at}\|\phi\| + b \sup_{u \in [\sigma, \sigma+t]} |H(u)| .$$

2. The representation theorem. In this paper, we consider the NFDE with infinite delay:

$$\frac{d}{dt}(D(t)x_t) = f(t, x_t), \qquad t \geqslant \sigma \geqslant 0, \quad x_\sigma = \phi \in X, \tag{2.1}$$

where f: $[0, \infty) \times X \to R^n$ is completely continuous and D is stable. Throughout this paper, we assume that the solution $x(\sigma, \phi)$ of the initial value problem for (2.1) is unique. Now, define the operators T, T_D and T_0 as

$$\begin{aligned} &T(\sigma, t)\phi := x_{\sigma+t}(\sigma, \phi), && \sigma \geqslant 0, \quad t \geqslant 0, \\ &T_D(\sigma, t)\phi := z_{\sigma+t}(\sigma, \phi, D(\sigma)\phi), && \sigma \geqslant 0, \quad t \geqslant 0, \\ &T_0(\sigma, t)\phi := z_{\sigma+t}(\sigma, 0, h_\phi), && \sigma \geqslant 0, \quad t \geqslant 0, \end{aligned} \tag{2.2}$$

where $h_\phi(u) := \int_\sigma^u f(s, T(\sigma, s-\sigma)\phi)ds$, $u \geqslant \sigma$. From the uniqueness of the solution of the initial value problem for (1.1), we have that, for all $\sigma \geqslant 0$, $t \geqslant 0$ and $\phi \in X$,

$$T(\sigma, t)\phi = T_D(\sigma, t)\phi + T_0(\sigma, t)\phi \tag{2.3}$$

and that $T_D(\sigma, t)\phi$ is linear in $\phi \in X$. Moreover, by Lemma 1.1, we get

$$\begin{aligned} &\|T_D(\sigma, t)\phi\| \leqslant be^{-at}\|\phi\|, && t \geqslant 0, \quad \sigma \geqslant 0, \quad \phi \in X, \quad D(\sigma)\phi = 0, \\ &\|T_0(\sigma, t)\phi\| \leqslant b \sup_{u \in [\sigma, \sigma+t]} |h_\phi(u)|, && t \geqslant 0, \quad \sigma \geqslant 0, \quad \phi \in X. \end{aligned} \tag{2.4}$$

LEMMA 2.1. *If $B \subset X$ is bounded with the property that $\bigcup_{0 \leqslant u \leqslant t} T(\sigma, u)B$ is bounded, then $T_0(\sigma, t)B$ is a precompact subset of X.*

PROOF. From the complete continuity of f, we can find $c > 0$ such that $|f(s, T(\sigma, s-\sigma)\phi)| \leqslant c$ for $(s, \phi) \in [\sigma, \sigma+t] \times B$. Then, we see easily that $\{h_\phi : \phi \in B\}$ is a precompact subset in $C([\sigma, \sigma+t], R^n)$. Now, (H_2) and Lemma 1.1 imply that

$$|z(\sigma, 0, h_\phi)(u)| \leqslant k_0\|z_u(\sigma, 0, h_\phi)\| \leqslant k_0 b \int_\sigma^u |f(s, T(\sigma, s-\sigma)\phi)|ds \leqslant k_0 bct \tag{2.5}$$

for $\phi \in B$ and $u \in [\sigma, \sigma+t]$, and that

(2.6)
$$\begin{aligned}&|z(\sigma, 0, h_1)(u_1)-z(\sigma, 0, h_2)(u_2)|\\ &\leqslant |z(\sigma, 0, h_1-h_2)(u_1)|+|z(\sigma, 0, h_2)(u_1)-z(\sigma, 0, h_2)(u_2)|\\ &\leqslant k_0 b \sup_{u\in[\sigma,\sigma+t]} |h_1(u)-h_2(u)|+|z(\sigma, 0, h_2)(u_1)-z(\sigma, 0, h_2)(u_2)|\end{aligned}$$

for $h_i \in C([\sigma, \sigma+t], R^n)$ with $h_i(\sigma)=0$ and $u_i \in [\sigma, \sigma+t]$, $i=1, 2$. The inequality (2.6) implies that $z(\sigma, 0, h)(u)$ is continuous in $(u, h) \in [\sigma, \sigma+t] \times \{h \in C([\sigma, \sigma+t], R^n): h(\sigma)=0\}$; hence $z(\sigma, 0, h)(u)$ is continuous uniformly in $(u, h) \in [\sigma, \sigma+t] \times \{h_\phi: \phi \in B\}$. Therefore, the inequality (2.5) implies that $\{z(\sigma, 0, h_\phi)(\cdot): \phi \in B\}$ is uniformly bounded and equicontinuous in $[\sigma, \sigma+t]$. Then, the compactness of the closure of the set $T_0(\sigma, t)B=\{z_{\sigma+t}(\sigma, 0, h_\phi): \phi \in B\}$ follows from the same argument as in [2, Lemma 2.1].

According to [4], for each $\sigma \geqslant 0$, there is an $n\times n$ matrix of functions $\Phi^\sigma=\{\phi_1, \cdots, \phi_n\}$, $\phi_j \in X$, and

(2.7)
$$\|\phi_j\| \leqslant M_1 ,$$

for $j=1, \cdots, n$, such that $A^{-1}(\sigma)D(\sigma)\Phi^\sigma=I$, where M_1 is a positive constant independent of σ and I is the $n\times n$ unit matrix.

THEOREM 2.2. *The solution operator* $T(\sigma, t)$ *of* (2.1) *can be written as* $T(\sigma, t)=T_1(\sigma, t)+T_2(\sigma, t)$, $\sigma \geqslant 0$, $t \geqslant 0$, *where*

$$T_1(\sigma, t):=T_D(\sigma, t)(I-\Phi^\sigma A^{-1}(\sigma)D(\sigma)) .$$
$$T_2(\sigma, t):=T_D(\sigma, t)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))+T_0(\sigma, t) ,$$

and T_D, T_0 *are the same as in* (2.2). *Furthermore,* T_1 *is a linear, bounded operator and is a contraction for large* t; T_2 *has the property that* $T_2(\sigma, t)B$ *is precompact for a bounded* $B\subset X$ *if* $\bigcup_{0\leqslant s\leqslant t} T(\sigma, s)B$ *is bounded.*

PROOF. Since

$$D(\sigma+t)x_{\sigma+t}(\sigma, \phi)=D(\sigma)\phi+\int_\sigma^{\sigma+t} f(s, T(\sigma, s-\sigma)\phi)ds ,$$

we have

$$\begin{aligned}T(\sigma, t)\phi&=x_{\sigma+t}(\sigma, \phi)=z_{\sigma+t}(\sigma, \phi, D(\sigma)\phi+h_\phi)=z_{\sigma+t}(\sigma, \phi, D(\sigma)\phi)+z_{\sigma+t}(\sigma, 0, h_\phi)\\ &=z_{\sigma+t}(\sigma, (I-\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi, 0)+z_{\sigma+t}(\sigma, \Phi^\sigma A^{-1}(\sigma)D(\sigma)\phi, D(\sigma)\phi)+z_{\sigma+t}(\sigma, 0, h_\phi)\\ &=T_D(\sigma, t)(I-\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi+T_D(\sigma, t)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi+T_0(\sigma, t)\phi .\end{aligned}$$

Then, from (2.4), we get

$$\begin{aligned}\|T_D(\sigma, t)(I-\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi\| &\leqslant be^{-at}\|(I-\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi\|\\ &\leqslant be^{-at}(1+M_1 l+M_1 lLA)\|\phi\| ,\end{aligned}$$

where a, b are as given in Lemma 1.1, M_1 is given in (2.7) and $l>0$ is a constant such that $\sum_{j=1}^{n}|u_j|\leqslant l|u|$ for all $u=(u_1, \cdots, u_n)\in R^n$. This means that $T_1(\sigma, t)$ is a linear, bounded operator and is a contraction for large t. Since

$$T_D(\sigma, t)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi=(T_D(\sigma, t)\Phi^\sigma)(A^{-1}(\sigma)D(\sigma)\phi),$$

the operator $T_D(\sigma, t)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))$ is an operator from X to a subset of X, which is spanned by $\{T_D(\sigma, t)\phi_j{:}\, j=1, \cdots, n\}$ and is bounded on bounded subsets of X. Hence, $T_D(\sigma, t)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))$ takes bounded subsets of X into precompact sets. The last assertion then follows from Lemma 2.1.

3. Application. In this section, we suppose that D and f in (2.1) are ω-periodic in t and $\omega>0$ is a constant. In this case, the set $\{f(t, \phi)\colon t\geqslant 0, \phi\in B\}$ is bounded for any bounded set $B\subset X$; hence $\bigcup_{t\geqslant 0}T(\sigma, t)B$ is bounded if B is bounded and the solutions of (2.1) are uniformly bounded. In the same way as in [4], one can show the following:

LEMMA 3.1. *If $D(\cdot)$ is ω-periodic, then $A(\cdot)$ is ω-periodic and there is an $n\times n$ matrix $\Phi^\sigma=\{\phi_1, \cdots, \phi_n\}$, $\phi_j\in X$, $\|\phi_j\|\leqslant M$, $\Phi^{\sigma+\omega}=\Phi^\sigma$, for $\sigma\geqslant 0, j=1, \cdots, n$, such that $A^{-1}(\sigma)D(\sigma)\Phi^\sigma=I$, for $\sigma\geqslant 0$.*

For the statement below, some definitions and notation are needed. The δ-neighborhood of a set $K\subset X$ will be denoted by $O(K, \delta)$ or $O(K)$. Let $\alpha(K)$ be the Kuratowski measure of noncompactness of a bounded set $K\subset X$. For fixed $\sigma\geqslant 0$, a family $\{T(\sigma, t), t\geqslant 0\}$ of mappings from X to X is an ω-periodic flow, if $T(\sigma, t)x$ is continuous in (t, x), $T(\sigma, 0)x=x$ and $T(\sigma, t+\omega)=T(\sigma,t)T(\sigma, \omega)$. If the system (2.1) is ω-periodic, then so is the solution operator $T(\sigma, \cdot)$. $\{T(\sigma, t), t\geqslant 0\}$ is point (resp. compact, resp. locally) dissipative if there is a bounded set $B\subset X$ of attracting each point x (resp. each compact set H, resp. a neighborhood $O(x)$ of each point x) in X, by which we mean that for each x (resp. each H, resp. a neighborhood $O(x)$ of each x), there is an $N>0$ such that $T(\sigma, t)x\in B$ (resp. $T(\sigma, t)H\subset B$, resp. $T(\sigma, t)O(x)\subset B$) for $t\geqslant N$. $\{T(\sigma, t), t\geqslant 0\}$ is said to be conditionally completely continuous, if for any bounded set $B\subset X$ with the property that $\bigcup_{0\leqslant s\leqslant t}T(\sigma, s)B$ is bounded, the set $T(\sigma, t)B$ is precompact. $\{T(\sigma,t), t\geqslant 0\}$ is a conditional α-contraction if there is a constant $k\in[0, 1)$ such that $\alpha(T(\sigma, t)B)\leqslant k\alpha(B)$ for any bounded set $B\subset X$ with the property that $\bigcup_{0\leqslant s\leqslant t}T(\sigma, s)B$ is bounded. If $T(\sigma, \cdot)(\cdot)$ takes bounded subsets of $[0, \infty)\times X$ into bounded sets, then a conditional α-contraction is an α-contraction. The same definitions can be given for a continuous function $T\colon X\to X$.

LEMMA 3.2 (cf. [7]). *Let $\{T(\sigma, t), t\geqslant 0\}$ be an ω-periodic flow. If $T(\sigma, t)=S(\sigma, t)+U(\sigma, t)$, where $S(\sigma, t)$ is a bounded linear operator such that $S^n(\sigma, \omega)=S(\sigma, n\omega)$ for any integer $n>0$, $S(\sigma, \omega)$ has the spectral radius less than one, and $\{U(\sigma, t), t\geqslant 0\}$ is conditionally completely continuous, then $T(\sigma, \omega)$ has a fixed point if $\{T(\sigma, t), t\geqslant 0\}$ is compact dissipative.*

THEOREM 3.3. *If the solution operator* $\{T(\sigma, t), t \geqslant 0\}$ *of* (2.1) *is compact dissipative, the operator* $T(\sigma, \omega)$ *has a fixed point.*

PROOF. For each $\sigma \geqslant 0$, let $S(\sigma, t) = T_1(\sigma, t)$. Then

$$\begin{aligned} S^2(\sigma, \omega)\phi &= T_1^2(\sigma, \omega)\phi = T_D(\sigma, \omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))(T_D(\sigma, \omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma)))\phi \\ &= T_D^2(\sigma, \omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi \\ &\quad - T_D(\sigma, \omega)(\Phi^\sigma A^{-1}(\sigma)D(\sigma))(T_D(\sigma, \omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi) \\ &= T_D(\sigma, 2\omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi \\ &\quad - (T_D(\sigma, \omega)\Phi^\sigma)(A^{-1}(\sigma)D(\sigma)T_D(\sigma, \omega)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi) \\ &= S(\sigma, 2\omega)\phi - (T_D(\sigma, \omega)\Phi^\sigma)(A^{-1}(\sigma)D(\sigma)(I - \Phi^\sigma A^{-1}(\sigma)D(\sigma))\phi) = S(\sigma, 2\omega)\phi\,, \end{aligned}$$

and in general, we have $S^n(\sigma, \omega) = S(\sigma, n\omega)$ for any integer $n > 0$. The assertion now follows from Theorem 2.2 and Lemma 3.2.

COROLLARY 3.4. *If the equation* (2.1) *has a bounded solution* $x(\phi)(\cdot)$, *which is uniformly stable and asymptotically stable in the large, then* (2.1) *has an* ω*-periodic solution.*

PROOF. Under the conditions in the corollary, the closure of the set $\{x_t(\phi)\colon t \geqslant 0\}$ is a compact set and it attracts each compact subset of X, since it is uniformly stable and asymptotically stable in the large (for the details, we refer to [6, pp. 95–98]).

LEMMA 3.5 (cf. [5]). *Suppose* $T\colon X \to X$ *is point dissipative, is a conditional* α*-contraction, and satisfies the condition that for any* $x \in X$, *there is a neighborhood* $O(x)$ *such that* $\bigcup_{j=1} T^j O(x)$ *is bounded. Then* T *is locally dissipative.*

THEOREM 3.6. *If the solutions of* (2.1) *are uniformly bounded and ultimately bounded for a bound* b, *i.e., for any* $A > 0$ *there is a* $\beta(A) > 0$ *such that* $\|T(\sigma, t)\phi\| \leqslant \beta(A)$, *for* $t \geqslant 0$ *and* $\phi \in X$ *with* $\|\phi\| \leqslant A$, *and for each* $(\sigma, \phi) \in [0, \infty) \times X$ *there is an* $n(\sigma, \phi) > 0$ *such that* $\|T(\sigma, t)\phi\| \leqslant b$ *for* $t \geqslant n(\sigma, \phi)$, *then* (2.1) *has an* ω*-periodic solution.*

PROOF. For the ω-periodic flow $\{T(\sigma, t), t \geqslant 0\}$ generated by (2.1), there is an equivalent norm $\|\cdot\|_1$ in X,

$$\|\phi\| \leqslant \|\phi\|_1 \leqslant K\|\phi\|\,, \qquad \text{for} \quad \phi \in X$$

such that $\|T_1(\sigma, \omega)\|_1 < 1$ (cf. [6, p. 92]). The assumptions on the solutions of (2.1) imply that $\{T(\sigma, t), t \geqslant 0\}$ is point dissipative with $B := \{\phi \in X\colon \|\phi\|_1 \leqslant b\}$ attracting points of X, and for any $\phi \in X$ there is a neighborhood $O(\phi)$ such that $\bigcup_{j=1}^{\infty} T(\sigma, t)O(\phi)$ is bounded. Furthermore, $T(\sigma, \omega)$ is an α-contraction, since $\bigcup_{0 \leqslant t \leqslant \omega} T(\sigma, t)H$ is always bounded for bounded subsets $H \subset X$ and $\{T(\sigma, t), t \geqslant 0\}$ is a conditional α-contraction with respect to the new norm $\|\cdot\|_1$. Then by Lemma 3.5, $T(\sigma, \omega)$ is locally dissipative. Thus the existence of an ω-periodic solution follows from Theorem 4.4 in [6, p. 92].

The following theorem is a generalization of Theorem 6.4 in [6, p. 98].

THEOREM 3.7. *For an ω-periodic linear nonhomogeneous* NFDE *with infinite delay, the existence of a solution bounded for $t \geqslant \sigma$ implies the existence of an ω-periodic solution.*

PROOF. For a linear NFDE

$$\text{(N)} \qquad \frac{d}{dt}(D(t)x_t) = F(t)x_t + h(t)\,,$$

we have $T(\sigma, t)\phi = L(\sigma, t)\phi + x_{\sigma+t}(\sigma, 0, h)$, where L is the solution operator generated by the equation

$$\text{(H)} \qquad \frac{d}{dt}(D(t)x_t) = F(t)x_t$$

and $x(\sigma, 0, h)$ is the solution of (N) with $x_\sigma = 0$. Since each solution $x(\sigma, \phi)$ of (N) is the solution of (1.1) with

$$H(t) = D(\sigma)\phi + \int_\sigma^t F(s)x_s ds + \int_\sigma^t h(s)ds\,,$$

it follows from Lemma 1.1 that

$$\begin{aligned} \|x_t(\sigma, \phi)\| &\leqslant be^{-a(t-\sigma)}\|\phi\| + b\Big(\sup_{u\in[\sigma,t]}\Big|D(\sigma)\phi + \int_\sigma^u F(s)x_s ds + \int_\sigma^u h(s)ds\Big|\Big) \\ &\leqslant (be^{-a(t-\sigma)} + b(A+L))\|\phi\| + \int_\sigma^t \|F(s)\|\cdot\|x_s\|ds + \int_\sigma^t |h(s)|ds\,, \end{aligned}$$

hence

$$\|x_t(\sigma, \phi)\| \leqslant \Big(b(A+L+e^{-a(t-\sigma)})\|\phi\| + \int_\sigma^t |h(s)|ds\Big)e^{\int_\sigma^t \|F(s)\|ds}\,, \qquad t \geqslant \sigma\,.$$

which implies that $T(\sigma, t)$ takes bounded subsets of $[0, \infty) \times X$ into bounded sets in X. In particular, $L(\sigma, t)$ has this property. Therefore, Theorem 2.2 implies that $L(\sigma, t) = U_1(\sigma, t) + U_2(\sigma, t)$, where $U_2(\sigma, t)$ takes bounded subsets of X into precompact sets, $U_1(\sigma, n\omega) = U_1^n(\sigma, \omega)$ and $U_1(\sigma, t)$ is linear, bounded and $\|U_1(\sigma, t)\| \leqslant c_1 e^{-at}$. Then, as in the proof of Theorem 3.6, we can find an equivalent norm $\|\,.\,\|_1$ such that $\|U_1(\sigma, \omega)\|_1 < 1$. Since $T(\sigma, \omega)$ is only a translation of $L(\sigma, \omega)$, it follows that $T(\sigma, \omega)$ is an α-contraction. Repeating the same reasoning as in [6, p. 98], one can complete the proof.

EXAMPLE. Consider the neutral integrodifferential equation

$$\text{(3.1)} \qquad \frac{d}{dt}\Big(x(t) - \int_{-\infty}^0 c_1(t, t+s)x_t(s)ds\Big) = Ax(t) + \int_{-\infty}^0 c_2(t, t+s)x_t(s)ds + f(t)\,,$$

where $x \in R^n$, A is an $n \times n$ matrix, and f: $R^+ \to R$, c_i: $R^+ \times R \to R^{n^2}$ $(i=1,2)$ are continuous. (3.1) has at least one T-periodic solution, if the following conditions are satisfied:

(i) There is a positive constant $T>0$ such that $f(t+T)=f(t)$ and $c_i(t+T, t+T+s)=c_i(t, t+s)$, for all $t \geqslant 0$, $s \leqslant 0$;

(ii) There are an $n \times n$ positive definite symmetric matrix B and positive constant w_i $(i=1, 2)$ such that

$$A^T B + BA = -I,$$

$$w_1^2 |x|^2 \leqslant x^T Bx \leqslant w_2^2 |x|^2,$$

for all $x \in R^n$;

(iii) There are constants $\gamma>0$ and $m \in (0, 1)$ such that

$$\int_{-\infty}^{0} |c_1(t, t+s)| e^{-\gamma s} ds \leqslant m,$$

$$1 - \frac{2|A^T B| m w_2}{(1-m) w_1} - \frac{2|B| w_2}{(1-m) w_1} \int_{-\infty}^{0} |c_2(t, t+s)| ds \geqslant u > 0,$$

and that $\int_{-\infty}^{0} |c_i(t, t+s)| e^{-\gamma s} ds$ $(i=1, 2)$ are convergent uniformly for $t \geqslant 0$.

PROOF. Denote

$$D(t, \phi) = \phi(0) - \int_{-\infty}^{0} c_1(t, t+s) \phi(s) ds,$$

$$F(t, \phi) = A\phi(0) + \int_{-\infty}^{0} c_2(t, t+s) \phi(s) ds + f(t).$$

For the space C_γ of the continuous functions $\phi: (-\infty, 0] \to R^n$ with the property that $\lim_{s \to -\infty} e^{\gamma s} |\phi(s)|$ exists, the hypotheses (H_1)–(H_3) are satisfied and $K=1$, $M(t) \to 0$ $(t \to +\infty)$ (cf. [2]). One can prove that D and F are continuous on $R^+ \times C_\gamma$, T-periodic in t, and linear bounded in ϕ, and that D is stable. Moreover, we have

$$\|z_t(\sigma, \phi)\| \leqslant \left(\|\phi\| + \sup_{\sigma \leqslant u \leqslant t} |H(u)| \right) \Big/ (1-m), \qquad t \geqslant \sigma, \tag{3.2}$$

for the solution $z(\sigma, \phi)$ of (1.1) with $z_\sigma = \phi$ and $H(\sigma) = D(\sigma, \phi)$.

Now, take $V(t, x) = x^T Bx$. We will prove that

$$V(t, D(t, x_t(0, 0))) \leqslant M, \qquad t \geqslant 0, \tag{3.3}$$

where $M > (2\bar{f}|B| w_2^2 / u w_1)^2$, $\bar{f} := \sup_{t \geqslant 0} |f(t)|$. Indeed, if (3.3) is not true, then there are a number $t_0 > 0$ and a sequence $\{t_n\}$, $t_n \to t_0 + (n \to \infty)$, such that

$$V(s, D(s, x_s)) \leqslant M = V(t_0, D(t_0, x_{t_0})), \qquad s \leqslant t_0, \tag{3.4}$$
$$V(t_n, D(t_n, x_{t_n})) > M, \qquad n = 1, 2, \cdots.$$

Then, we have

$$\dot{V}_{(3.1)}(t_0, D(t_0, x_{t_0})) \geqslant 0, \tag{3.5}$$

and

$$|D(s, x_s)| \leqslant \frac{\sqrt{M}}{w_1} = \frac{1}{w_1}\{V(t_0, D(t_0, x_{t_0}))\}^{1/2}, \qquad s \leqslant t_0. \tag{3.6}$$

From (3.2) and (3.6), we have

$$|x(s)| \leqslant \| x_s \| \leqslant \sup_{0 \leqslant u \leqslant t_0} \frac{|D(u, x_u)|}{1-m} \leqslant \frac{1}{(1-m)w_1}\{V(t_0, D(t_0, x_{t_0}))\}^{1/2} \tag{3.7}$$
$$\leqslant \frac{|D(t_0, x_{t_0})| w_2}{(1-m)w_1}, \qquad s \leqslant t_0,$$

It follows that

$$\dot{V}_{(3.1)}(t_0, D(t_0, x_{t_0})) = \dot{D}^T BD + D^T B\dot{D}$$
$$= \left(Ax + \int_{-\infty}^{0} c_2 x_{t_0} ds\right)^T BD + f^T BD + D^T B\left(Ax + \int_{-\infty}^{0} c_2 x_{t_0} ds\right) + D^T Bf$$
$$= x^T A^T BD + \int_{-\infty}^{0} x_{t_0}^T c_2^T ds BD + f^T BD + D^T BAx + D^T B\int_{-\infty}^{0} c_2 x_{t_0} ds + D^T Bf$$
$$= D^T A^T BD + \int_{-\infty}^{0} x_{t_0}^T c_1^T ds A^T ds A^T BD + \int_{-\infty}^{0} x_{t_0}^T c_2^T ds BD + f^T BD$$
$$+ D^T BAD + D^T BA\int_{-\infty}^{0} c_1 x_{t_0} ds + D^T B\int_{-\infty}^{0} c_2 x_{t_0} ds + D^T Bf$$
$$\leqslant -|D|^2 + 2|A^T B|\int_{-\infty}^{0} |c_1||x_{t_0}| ds |D| + 2|B|\int_{-\infty}^{0} |c_2||x_{t_0}| ds |D| + 2|f||B||D|$$
$$\leqslant -\left(1 - \frac{2|A^T B| m w_2}{(1-m)w_1} - \frac{2|B| w_2}{(1-m)w_1}\int_{-\infty}^{0} |c_2| ds\right)|D|^2 + 2\bar{f}|B||D|,$$

where D, c_i $(i=1, 2)$, f, x and x_{t_0} stand for $D(t_0, x_{t_0})$, $c_i(t_0, t_0+s)$ $(i=1, 2)$, $f(t_0)$, $x(t_0)$ and $x_{t_0}(s)$, respectively. Hence

$$\dot{V}_{(3.1)}(t_0, D(t_0, x_{t_0})) \leqslant -\frac{u}{w_2^2} V(t_0, D(t_0, x_{t_0})) + \frac{2\bar{f}|B|}{w_1}\{V(t_0, D(t_0, x_{t_0}))\}^{1/2}$$

$$= -uM/w_2^2 + 2\bar{f}|B|\sqrt{M}/w_1 < 0\,,$$

which is contrary to (3.5). Therefore, the solution $x_t(0, 0)$ of (N) is bounded. The assertion now follows from Theorem 3.7.

The author would like to thank Professors Junji Kato and Satoru Murakami for their valuable suggestions. Thanks are also due to the referee for helpful advice.

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