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AFFINE HYPERSURFACES WITH PARALLEL CUBIC FORMS

NEDA BOKAN, KATSUMI NOMIZU AND UDO SIMON*

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In this note, we first investigate the symmetry properties of the tensors ∇C and $\nabla^2 C$, where ∇ is the induced affine connection and C is the cubic form of a nondegenerate affine hypersurface M^n in $\boldsymbol{R}^{n+1}$. In particular, we study hypersurfaces with parallel cubic form, i.e. $\nabla C=0$. In the case $n=2$, this condition is known to characterize a Cayley surface (Nomizu and Pinkall [3]). We obtain a certain class of more general affine surfaces and hypersurfaces.

On the other hand, for an affine hypersurface M^n, $n\geq 3$, condition $\nabla R=0$ (i.e. parallel curvature tensor field) implies that M^n is an improper affine hypersphere or a quadric (Verheyen and Verstraelen [7]). We shall provide a generalization of this result by proving that the condition $\nabla^2 R=0$ implies $\nabla R=0$ for an affine hypersurface. Recall that, for a Riemannian manifold, the condition $\nabla^2 R=0$ (in fact, $\nabla^k R=0$ for some integer k) implies $\nabla R=0$ but that such a result does not hold for an affine connection in general.

Our study shows the common background for these results on the covariant differentials of the cubic form and those of the curvature tensor field.

1. Preliminaries. Although we mostly follow the notation in [3] in this paper, we consider exclusively the classical theory of nondegenerate affine hypersurfaces M^n in $\boldsymbol{R}^{n+1}$ in the sense of Blaschke (see [1], [2], [5] and [6]).

The difference between the induced affine connection ∇ and the Levi-Civita connection $\hat{\nabla}$ for the affine metric h is denoted by K:

$$K_X=\nabla_X-\hat{\nabla}_X \tag{1}$$

and we also write

$$K_X Y=K_Y X=K(X, Y)\,. \tag{2}$$

The so-called apolarity condition can be expressed by

$$\operatorname{trace} K_X=0 \qquad \text{for every} \quad X\,. \tag{3}$$

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The cubic form C is defined by

$$C(X, Y, Z)=(\nabla_X h)(Y, Z), \tag{4}$$

which is symmetric in X, Y, and Z. This is related to K by

$$C(X, Y, Z)=-2h(K_X Y, Z), \tag{5}$$

which implies that the operator K_X is symmetric relative to h.

The Pick invariant J is defined by

$$J=h(K, K), \tag{6}$$

where h is extended to an inner product in the tensor space. In terms of components of the tensors involved we have

$$J=\sum K^{ijk}K_{ijk}. \tag{6'}$$

Apolarity can be also expressed by

$$\sum h^{ij}C_{ijk}=0 \qquad \text{for all} \quad k. \tag{5'}$$

We denote by S the shape operator. The curvature tensor R of ∇ can be expressed by the Gauss equation

$$R(X, Y)Z=h(Y, Z)SX-h(X, Z)SY$$

in terms of S and h. We say that M^n is an affine hypersphere if $S=\lambda I$, where λ is the affine mean curvature ($=\operatorname{trace} S/n$).

There are two known results which are closely related to our purpose.

THEOREM (Nomizu-Pinkall [4]). *Let $n=2$. If $\nabla C=0$ and $C\neq 0$, then M^2 is equiaffinely congruent to a Cayley surface.*

THEOREM (Verheyen and Verstraelen [7]). *If $n\geq 3$ and $\nabla R=0$, then $C=0$ (that is, M^n is a quadric) or $S=0$ (that is, M^n is an improper affine sphere).*

REMARK. They first prove that in the case where $n\geq 3$, $R(X, Y)\cdot R=0$ for all X, Y if and only if $S=\lambda I$. Of course, $\nabla^2 R=0$ implies $R(X, Y)\cdot R=0$ and hence $S=\lambda I$.

In Section 2 we study the symmetry conditions for ∇C and $\nabla^2 C$. In Section 3, we shall derive properties of affine hypersurfaces satisfying $\nabla C=0$ and discuss a certain class of affine hypersurfaces which may be considered as generalizations of Cayley surfaces. In Section 4, we shall prove that the condition $\nabla^2 R=0$ for an affine hypersurface implies $\nabla R=0$.

2. Total symmetry of ∇C and $\nabla^2 C$. The covariant differential ∇C of the cubic form C is denoted by

$$(\nabla C)(U, V, W; X)=(\nabla_X C)(U, V, W),$$

where $\nabla_X C$ is the covariant derivative of C defined by

$$(\nabla_X C)(U, V, W) = X(C(U, V, W)) - C(\nabla_X U, V, W) - C(U, \nabla_X V, W) - C(U, V, \nabla_X W),$$

U, V, W on the right-hand side being vector fields that extend vectors U, V, W, respectively. In terms of components, we have

$$\nabla C = (C_{ijk;t}),$$

where t is an index corresponding to the vector X.

Similarly, the second covariant differential $\nabla^2 C$ is denoted by

$$(\nabla^2 C)(U, V, W; X, Y) = (\nabla_Y(\nabla_X C))(U, V, W) - (\nabla_{\nabla_Y X} C)(U, V, W),$$

where X on the right-hand side is a vector field extending the vector X, and $\nabla_Y(\nabla_X C)$ is the covariant derivative relative to Y of the tensor field $\nabla_X C$. In terms of components, we have

$$\nabla^2 C = (C_{ijk;t;s}),$$

where t and s are indices corresponding to X and Y, respectively.

This being said, we are now concerned with the symmetry properties of ∇C and $\nabla^2 C$.

First, we say that ∇C is totally symmetric if $(\nabla C)(U, V, W; X)$ is symmetric relative to its four variables. (This is to say that $C_{ijk;m}$ are symmetric in all four indices.) Since ∇C is symmetric in U, V, W just like C, total symmetry follows if it is symmetric in X and U.

Similarly, we may consider the covariant differential $\hat{\nabla} C$ relative to the Levi-Civita connection $\hat{\nabla}$ for the affine metric and consider its symmetry properties. In fact, we prove:

LEMMA 1. *∇C is totally symmetric if and only if $\hat{\nabla} C$ is totally symmetric.*

PROOF. Using (1) we get

$$\begin{aligned}(\nabla_X C)(U, V, W) &= (\hat{\nabla}_X C)(U, V, W) + (K_X C)(U, V, W)\\ &= (\hat{\nabla}_X C)(U, V, W) - C(K_X U, V, W) - C(U, K_X V, W) - C(U, V, K_X W)\\ &= (\hat{\nabla}_X C)(U, V, W) - C(K(X, U), V, W) + 2h(K(X, V), K(U, W)) + 2h(K(X, W), K(U, V))\end{aligned}$$

by using symmetry of C and (5). The term $C(K(X, U), BV, W)$ is symmetric in X and U. The sum $2h(K(X, V), K(U, W)) + 2h(K(X, W), K(U, V))$ is symmetric in X and U as well. Thus

$$(\nabla_X C)(U, V, W) - (\hat{\nabla}_X C)(U, V, W) = (\nabla_U C)(X, V, W)(\hat{\nabla}_U C)(X, V, W),$$

that is,

$$(\nabla_X C)(U, V, W) - (\nabla_U C)(X, V, W) = (\hat{\nabla}_X C)(U, V, W) - (\hat{\nabla}_U C)(X, V, W),$$

proving our assertion. □

We note that

$$(\hat{\nabla}_X C)(U, V, W) = -2h((\hat{\nabla}_X K)(U, V), W) \tag{7}$$

as follows from (5).

On the other hand, the following formula is found in the proof of Proposition 5.4 of [2].

$$2(\hat{\nabla}_X K)(Y, Z) - 2(\hat{\nabla}_Y K)(X, Z) = h(Y, Z)SX - h(X, Z)SY - [h(SY, Z)X - h(SX, Z)Y]\,. \tag{8}$$

REMARK. The right-hand side is nothing but $R(X, Y)Z - R^*(X, Y)Z$, where R is the curvature tensor of ∇ and R^* is the curvature tensor for the induced connection of the dual affine hypersurface.

We now prove:

LEMMA 2. *∇C is totally symmetric if and only if M^n is an affine hypersphere.*

PROOF. By Lemma 1 and formulas (7), (8) we see that ∇C is totally symmetric if and only if

$$h(Y, Z)SX - h(X, Z)SY - [h(SY, Z)X - h(SX, Z)Y] = 0\,. \tag{9}$$

The trace of the linear mapping taking X to the left-hand side of (9) is

$$h(Y, Z)\operatorname{tr} S - h(SY, Z) - nh(SY, Z) + h(SY, Z) = 0$$

and hence

$$SY = \lambda Y \qquad \text{for every } Y, \quad \text{where} \quad \lambda = \operatorname{tr} S/n\,.$$

Conversely, if $S = \lambda I$, we have (9) and so total symmetry of ∇C. □

REMARK. We may also say that M^n is an affine hypersphere if and only if $R = R^*$. This condition is also equivalent to $R(X, Y)\cdot h = 0$.

Now assume that ∇C is totally symmetric and consider $\nabla^2 C$. From its definition we have

$$(\nabla^2 C)(U, V, W; X; Y) - (\nabla^2 C)(U, V, W; Y; X) = -(R(X, Y)\cdot C)(U, V, W)\,, \tag{9}$$

where $R(X, Y)$ acts on C as a derivation.

It follows that $(\nabla^2 C)(U, V, W; X; Y)$ is symmetric in X and Y if and only if $R(X, Y)\cdot C = 0$. This is the case, in particular, if $R = 0$ (which is known to be equivalent to $S = 0$).

As we assume that ∇C is totally symmetric, $(\nabla^2 C)(U, V, W; X; Y)$ is symmetric in U, V, W, X. Combined with symmetry in X and Y, we have total symmetry of $\nabla^2 C$. Thus we get:

LEMMA 3. *Assume ∇C is totally symmetric. Then $\nabla^2 C$ is totally symmetric if and only if $R(X, Y)\cdot C = 0$.*

We now prove:

THEOREM 1. *Both ∇C and $\nabla^2 C$ are totally symmetric if and only if $C=0$ (M^n is a quadric) or $S=0$ (M^n is an improper affine hypersphere).*

PROOF. If $C=0$, obviously, $\nabla C=0$ and $\nabla^2 C=0$. If $S=0$, we already know that ∇C is totally symmetric (Lemma 2). Moreover, $S=0$ implies $R=0$ and hence $\nabla^2 C$ is totally symmetric.

Assume that both ∇C and $\nabla^2 C$ are totally symmetric. Lemma 2 implies that $S=\lambda I$, with $\lambda=\operatorname{tr} S/n$. The Gauss equation says

$$R(X, Y)Z=\lambda[h(Y, Z)X-h(X, Z)Y].$$

We use this expression and evaluate $(R(X, Y)\cdot C)(U, V, W)=0$ (which holds by Lemma 3). If $\lambda\neq 0$, then we get

$$h(Y, U)C(X, V, W)-h(X, U)C(Y, V, W)+h(Y, V)C(U, X, W)-h(X, V)C(U, Y, W)$$
$$+h(Y, W)C(U, V, X)-h(X, W)C(U, V, Y)=0.$$

Using (5) we may write this in the following form

$$(10)\qquad -2h(Y, U)K(X, V)+2h(X, U)K(Y, V)-h(Y, V)K(U, X)+2h(X, V)K(U, Y)$$
$$+C(U, V, X)Y-C(U, V, Y)X=0.$$

Taking the trace of the linear mapping which takes X to the left-hand side of (10) we obtain by using apolarity

$$h(K(Y, V), U)-h(K(U, Y), V)+C(U, V, Y)-nC(U, V, Y)=0,$$

namely,

$$(n+1)C(U, V, Y)=0, \quad \text{that is,} \quad C=0. \qquad \square$$

3. Affine hypersurfaces with parallel cubic forms. We now prove:

THEOREM 2. *If $\nabla C=0$ for an affine hypersurface M^n, $n\geq 2$, then $C=0$ or $S=0$ and we have the following consequences*:

1) $(\nabla_Y K)_X=2K_Y K_X$ *for all* X, Y;
2) $\operatorname{tr}\{Y\to(\nabla_Y K)_X Z\}=0$ *for all* X, Z;
3) $\operatorname{tr}\{Y\to(\hat{\nabla}_Y K)_X Z\}=0$ *for all* X, Z;
4) $\operatorname{tr}(K_X K_Z)=0$ *for all* X, Z; *thus h is an Einstein metric.*
5) *The Pick invariant $J=0$.*

PROOF. The first assertion follows from Theorem 1. To prove 1), we have by (5)

$$h(K(X, U), Z)=-\frac{1}{2}C(X, U, Z).$$

Taking ∇_Y of both sides and using $\nabla C=0$, we obtain

$$(\nabla_Y h)(K(X, U), Z)+h((\nabla_Y K)(X, U), Z)=0\,.$$

Again using (5) we have

$$-2h(K(K(X, U), Y), Z)+h((\nabla_Y K)(X, U), Z)=0$$

and hence

$$2K(K(X, U), Y)=(\nabla_Y K)(X, U)\,.$$

Writing the left-hand side as $2K_Y K_X U$, we obtain $(\nabla_Y K)_X=2K_Y K_X$.

2) Using 1) we write

$$(\nabla_Y K)_X Z=2K_Y K_X Z=2K_{K_X Z} Y\,.$$

Since trace $K_{K_X Z}=0$ by apolarity, we get the assertion.

3) *From Proposition* 5.4, [2], *we get*

$$\mathrm{tr}\{Y\to(\hat{\nabla}_Y K)_X Z\}=(n/2)\{h(X, Z)\,\mathrm{tr}\, S/n-h(SX, Z)\}\,,$$

which is 0 since $S=\lambda I$.

4) Using (1) we get

$$(\nabla_Y K)_X Z=(\hat{\nabla}_Y K)_X Z+(K_Y\cdot K)_X Z\,,$$

where

$$(K_Y\cdot K)_X Z=K_Y(K_X Z)-K_{K_Y X}Z-K_X(K_Y Z)=K_{K_X Z}Y-K_Z K_X Y-K_X K_Z Y.$$

Thus by apolarity and $\mathrm{trace}(K_Z K_X)=\mathrm{trace}(K_X K_Z)$ we get

$$\mathrm{trace}\{Y\to(K_Y\cdot K)_X Z\}=-2\,\mathrm{trace}\, K_X K_Z\,.$$

Now using 2) and 3) we see $\mathrm{trace} K_X K_Z=0$.

From the formula for the Ricci tensor $\widehat{\mathrm{Ric}}$ for $\hat{\nabla}$ (Proposition 5.2, [2]):

$$\widehat{\mathrm{Ric}}(Y, Z)=\frac{1}{2}[h(Y,Z)\,\mathrm{trace}\, S+(n-2)h(SY, Z)]+\mathrm{trace}(K_Y K_Z)$$

we see that h is an Einstein metric.

5) From 4) we have $h(K_X, K_Z)=0$, where h extends the affine metric to the tensor space. If $\{X_1, \cdots, X_n\}$ is an orthonormal basis with $h(X_i, X_i)=\varepsilon_i=\pm 1$, we get $J=h(K, K)=\sum \varepsilon_i h(K_{X_i}, K_{X_i})=0$. □

REMARK. The proof for 4) establishes the general formula

$$L(X, Z)=\hat{L}(X, Z)-2\,\mathrm{trace}(K_X K_Z)\,, \tag{11}$$

where

$$L(X, Z) = \mathrm{trace}\{Y \to (\nabla_Y K)_X Z\} \tag{12}$$

and

$$\hat{L}(X, Z) = \mathrm{trace}\{Y \to (\hat{\nabla}_Y K)_X Z\}\,. \tag{13}$$

COROLLARY. *If $\nabla C = 0$ and $C \neq 0$, then $S = 0$, ∇ is flat ($R = 0$), $J = 0$, and h is a hyperbolic metric with zero Ricci tensor.*

REMARK 1. Even for $n = 2$, there are many nondegenerate affine surfaces such that $S = 0$, $J = 0$, $C \neq 0$ (thus h is hyperbolic). In fact, such surfaces are essentially the same as ruled surfaces which are improper affine spheres and can locally be represented as the graph of $z = xy + \varphi(y)$, where φ is an arbitrary differentiable function of one variable (see [1, p. 221]).

We may consider this class of surfaces as generalized Cayley surfaces (for which $\varphi(y) = y^3$). A simple computation shows that for the affine metric h we have (writing x^1, x^2 for x, y)

$$h_{11} = 0\,, \qquad h_{12} = 1\,, \qquad h_{22} = \varphi''(y)\,.$$

The affine normal ξ is given by (0, 0, 1). The cubic form C has all components 0 except possibly C_{222}, and similarly for ∇C, $\nabla^2 C$, $\cdots$.

If we take $\varphi(y) = y^4$, then the surface satisfies $\nabla^2 C = 0$ but $\nabla C \neq 0$. Obviously, when φ is a polynomial of degree d, the surface has the property that $\nabla^{d-2} C = 0$.

REMARK 2. These surfaces have been rediscovered by M. Magid and P. Ryan as part of affine surfaces whose affine metrics are flat.

EXAMPLE FOR $n \geq 3$. We consider the graph M^n of

$$x^{n+1} = x^1 x^n + \frac{1}{2}\sum (x^i)^2 + \varphi(x^n)\,,$$

where the summation $\sum$ extends for $2 \leq i \leq n-1$. This hypersurface M^n has the following properties: it is an improper affine sphere ($S = 0$ and thus $R = 0$), the Pick invariant J is 0, ∇C, $\nabla^2 C$, $\cdots$ are all totally symmetric, the affine metric h is Lorentzian and flat (so $\hat{R} = 0$). These assertions follow from the information below:

$$h_{ij} = 0 \quad \text{except} \quad h_{1n} = h_{n1} = h_{22} = \cdots = h_{n-1\,n-1} = 1\,, \qquad h_{nn} = \varphi''(x^n)\,;$$

$$C_{ijk} = 0 \quad \text{except possibly} \quad C_{nnn} = \varphi^{(3)}\,;$$

$$C_{ijk;m} = 0 \quad \text{except possibly} \quad C_{nnn;n} = \varphi^{(4)}; \quad \text{and so on}\,.$$

4. Affine hypersurfaces with $\nabla^2 R = 0$. The following is our new result which extends the theorem of Verheyen and Verstraelen.

THEOREM 3. *If M^n, $n\geq 3$, is a nondegenerate hypersurface such that $\nabla^2 R=0$, then $C=0$ or $S=0$.*

PROOF. The assumption implies $R(X, Y)\cdot R=0$ and so $S=\lambda I$ by a result of Verheyen and Verstraelen. Thus

$$R(U, V)W=\lambda[h(V, W)U-h(U, W)V] \quad \text{and}$$

$$(\nabla_Y R)(U, V)W=\lambda[(\nabla_Y h)(V, W)U-(\nabla_Y h)(U, W)V]=\lambda[C(Y, V, W)U-C(Y, U, W)V].$$

Consequently, we get

$$(\nabla^2 R)(U, V; Y; X)W=\lambda[(\nabla_X C)(Y, V, W)U-(\nabla_X C)(Y, U, W)V] \tag{14}$$

where X, Y, U, V, W may be tangent vectors at a point.

If $\lambda\neq 0$, then by (14) we obtain

$$(\nabla_X C)(Y, V, W)U-(\nabla_X C)(Y, U, W)V=0 \quad \text{and therefore} \quad \nabla C=0.$$

We have thus shown that $S=0$ or $\nabla C=0$. By Theorem 1, we see that $S=0$ or $C=0$. □

REFERENCES

[1] W. BLASCHKE, Vorlesungen über Differentialgeometrie, II, Affine Differentialgeometrie, Berlin, Springer, 1923.

[2] K. NOMIZU, Introduction to Affine Differential Geometry, Part I, Lecture Notes, MPI preprint 88–37.

[3] K. NOMIZU AND U. PINKALL, On the geometry of affine immersions, Math. Z. 195 (1987), 165–178.

[4] K. NOMIZU AND U. PINKALL, Cayley surfaces in affine differential geometry, MPI preprint 88–20, Tôhoku Math. J. 41 (1989), 589–596.

[5] P. A. SCHIROKOW AND A. P. SCHIROKOW, Affine Differentialgeometrie, Teubner, Leipzig, 1962.

[6] U. SIMON, Zur Entwicklung der affinen Differentialgeometrie nach Blaschke, Gesammelte Werke W. Blaschke, Band 4, Thales, Essen, 1985.

[7] P. VERHEYEN AND L. VERSTRAELEN, Locally symmetric affine hypersurfaces, Proc. Amer. Math. Soc. 93 (1985), 101–103.

N. BOKAN
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF BELGRADE
YUGOSLAVIA

U. SIMON
FACHBEREICH MATHEMATIK MA8-3
TECHNISCHE UNIVERSITÄT BERLIN
STRASSE DES 17. JUNI 136
D-1000 BERLIN 12
FEDERAL REPUBLIC OF GERMANY

K. NOMIZU
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RI 02912
U.S.A.