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HARMONIC INNER AUTOMORPHISMS OF COMPACT CONNECTED SEMISIMPLE LIE GROUPS

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0. Introduction. Harmonic maps of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) are the extrema of the energy functional (cf. [1])

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 dV_g .$$

In this paper, we treat the case $(M, g)=(N, h)=(G, g)$ for a compact connected semisimple Lie group G with a left invariant Riemannian metric g. It is well known that every inner automorphism of G into itself is both isometric and harmonic with respect to a bi-invariant Riemannian metric g_0 on G. However, we here deal with an arbitrary left invariant metric g on G, and show which inner automorphisms of G into itself are harmonic maps of (G, g) into itself.

In §1, we introduce Guest's criterion (cf. Lemma A) for the map between reductive homogeneous spaces G/H and G'/H' induced by a Lie group homomorphism from G into G'.

In §2, using this criterion, we obtain a necessary and sufficient condition for an inner automorphism A_x of (G, g) to be harmonic (cf. Theorem 2.2).

In the particular case $G=SU(2)$, we then completely determine harmonic inner automorphisms of $(SU(2), g)$ for every left invariant Riemannian metric g (cf. Proposition 3.3–3.5).

Finally in Theorems 3.6 and 3.7, we show that for any left invariant and but not bi-invariant Riemannian metric g on $G=SU(2)$, there always exist on (G, g) both a non-harmonic inner automorphism and a non-isometric but harmonic inner automorphism.

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1. Preliminaries. In this section, we review Guest's work which gives a necessary and sufficient condition for the map induced by a homomorphism $\theta: G\to G'$ between reductive homogeneous spaces G/H, G'/H' with invariant Riemannian metrics to be harmonic (cf. [4]).

Let $\theta: G \to G'$ be a homomorphism of compact Lie groups G, G' such that $\theta(H) \subset H'$ for closed subgroups H, H'. We denote by $\mathfrak{g}$ (resp. $\mathfrak{h}$, $\mathfrak{g}'$ and $\mathfrak{h}'$) the Lie algebra of all left invariant vector fields on G (resp. H, G' and H'). Let $f_\theta: G/H \to G'/H'$ be the map between reductive homogeneous spaces G/H, G'/H' induced by θ, that is, $f_\theta(xH) = \theta(x)H'$, $(x \in G)$. Let $\mathfrak{m}$ be the subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum of vector spaces) and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. Then the subspace $\mathfrak{m}$ of $\mathfrak{g}$ can be identified with the tangent space of G/H at the origin $O := \{H\} \in G/H$.

The derivative df_θ of the induced map f_θ is determined by its restriction to $O \in G/H$, which is given in terms of the Lie algebra homomorphism $\theta: \mathfrak{g} \to \mathfrak{g}'$ by

$$df_\theta(X) = \theta(X)_{\mathfrak{m}'}, \qquad X \in \mathfrak{m}, \tag{1.1}$$

where $\theta(X)_{\mathfrak{m}'}$ denotes the $\mathfrak{m}'$-component of the element $\theta(X) \in \mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$.

Let $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle'$) be an inner product which is invariant with respect to $\mathrm{Ad}(H)$ (resp. $\mathrm{Ad}(H')$) on $\mathfrak{m}$ (resp. $\mathfrak{m}'$), where Ad denotes the adjoint representation of H (resp. H') in $\mathfrak{g}$ (resp. $\mathfrak{g}'$). This inner product $\langle\ ,\ \rangle$ (resp. $\langle\ ,\ \rangle'$) determines an invariant Riemannian metric g (resp. g') on G/H (resp. G'/H').

Then, the connection function α (cf. [6, p. 43]) on $\mathfrak{m} \times \mathfrak{m}$ corresponding to the invariant Riemannian connection of $(G/H, g)$ is given as follows (cf. [6, p. 52]):

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \qquad (X, Y \in \mathfrak{m}),$$

where $U(X, Y)$ is determined by

$$2\langle U(X, Y), Z\rangle = \langle [Z, X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z, Y]_{\mathfrak{m}}\rangle, \quad (X, Y, Z \in \mathfrak{m}), \tag{1.2}$$

and $X_{\mathfrak{m}}$ denotes the $\mathfrak{m}$-component of an element $X \in \mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

The invariant metric g' on G'/H', U' on $\mathfrak{m}' \times \mathfrak{m}'$, and the connection function α' are given similarly.

Recall that for Riemannian manifolds (M, g), (N, h), a smooth map $f: M \to N$ is said to be *harmonic* if $\operatorname{tr} \nabla(df) = 0$, namely, the tension field $\tau(f)$ vanishes identically (cf. [1, 2]).

Guest [4, Lemma 2.1] obtained the following:

LEMMA A. *The induced map $(G/H, g)$ into $(G'/H', g')$ is harmonic if and only if*

$$\textstyle\sum_{i=1}^{m} \{[\theta(X_i)_{\mathfrak{h}'}, df_\theta(X_i)] + U'(df_\theta(X_i), df_\theta(X_i)) - df_\theta(U(X_i, X_i))\} = 0,$$

where $\{X_i\}_{i=1}^m$ is an orthonormal basis of $\mathfrak{m}$ with respect to $\langle\ ,\ \rangle$, and $m :=$ $\dim(G/H) = \dim \mathfrak{m}$.

2. Harmonic maps between compact semisimple Lie groups.

2.1. Let G be a compact semisimple Lie group and T be a maximal torus of G.

We denote by $\mathfrak{g}$ (resp. $\mathfrak{t}$) the Lie algebra of G (resp. T). Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$. We denote by Δ the set of all nonzero roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$, and by Δ^+ the set of all positive roots with respect to a fixed linear order in the dual space of $\{H\in\mathfrak{t}^c \mid \alpha(H)\in\boldsymbol{R} \text{ for any } \alpha\in\Delta\}_{\boldsymbol{R}}$. Let B be the Killing form of $\mathfrak{g}^c$. We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{g}$ by $\langle X, Y\rangle_0 := -B(X, Y)$, $(X, Y\in\mathfrak{g})$.

We choose an orthonormal basis of $\mathfrak{g}$ with respect to the inner product $\langle\ ,\ \rangle_0$ as follows: For $\alpha\in\Delta$, let E_α be a root vector such that $B(E_\alpha, E_{-\alpha})=-1$ and $N_{\alpha,\beta}=N_{-\alpha,-\beta}$ for α, $\beta\in\Delta$ $(\alpha+\beta\neq 0)$, where $N_{\alpha,\beta}$ are real numbers defined by

$$(2.1)\qquad \begin{cases} [E_\alpha, E_\beta]=N_{\alpha,\beta}E_{\alpha+\beta} & \text{if } \alpha, \beta, \alpha+\beta\in\Delta\,, \quad\text{and} \\ N_{\alpha,\beta}=0 & \text{if } 0\neq\alpha+\beta\notin\Delta\,. \end{cases}$$

Hence, $[E_\alpha, E_{-\alpha}]=-H_\alpha$, H_α being determined by $B(H, H_\alpha)=\alpha(H)$ for any $H\in\mathfrak{t}$. For $\alpha\in\Delta$, put $U_\alpha=E_\alpha+E_{-\alpha}$, $V_\alpha=\sqrt{-1}(E_\alpha-E_{-\alpha})$ which belong to $\mathfrak{g}$. Let $\{H_i\}_{i=1}^s$ be an orthonormal basis of $\mathfrak{t}$ with respect to $\langle\ ,\ \rangle_0$, where $s=\dim T$. Then

$$(2.2)\qquad \{(1/\sqrt{2})U_\alpha, (1/\sqrt{2})V_\alpha, H_i \mid \alpha\in\Delta^+, 1\leqq i\leqq s\}$$

is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle_0$.

On the other hand, we take another inner product $\langle\ ,\ \rangle$ on $\mathfrak{g}$ such that

$$(2.3)\qquad \{a_\alpha^{-1}\cdot(U_\alpha/\sqrt{2}), b_\alpha^{-1}\cdot(V_\alpha/\sqrt{2}), c_i^{-1}\cdot H_i \mid \alpha\in\Delta^+, 1\leqq i\leqq s\}$$

is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$, where a_α, b_α and c_i are positive constants. Then $\langle\ ,\ \rangle_0$ (resp. $\langle\ ,\ \rangle$) determines a left invariant Riemannian metric g_0 (resp. g) on G. In fact, g_0 becomes a bi-invariant metric on G.

An inner automorphism $A_x\colon (G, g)\to(G, g)$, $(x\in G)$, is harmonic if and only if

$$(2.4)\qquad \begin{aligned} &\textstyle\sum_{i=1}^s c_i^{-2}\{U(\mathrm{Ad}(x)H_i, \mathrm{Ad}(x)H_i)-\mathrm{Ad}(x)U(H_i, H_i)\} \\ &\quad+\textstyle\sum_{\alpha\in\Delta^+}(a_\alpha^{-2}/2)\{U(\mathrm{Ad}(x)U_\alpha, \mathrm{Ad}(x)U_\alpha)-\mathrm{Ad}(x)U(U_\alpha, U_\alpha)\} \\ &\quad+\textstyle\sum_{\alpha\in\Delta^+}(b_\alpha^{-2}/2)\{U(\mathrm{Ad}(x)V_\alpha, \mathrm{Ad}(x)V_\alpha)-\mathrm{Ad}(x)U(V_\alpha, V_\alpha)\}=0\,. \end{aligned}$$

This follows from the case $H=\{e\}$ of the reductive homogeneous space G/H in Lemma A of §1.

Now, we analyze the formula (2.4) further.

Lemma 2.1.
(i) $U(H_i, H_j)=0$, $\quad(1\leqq i, j\leqq s)$,
(ii) $U(U_\alpha, U_\alpha)=U(V_\alpha, V_\alpha)=0$,
(iii) $U(U_\alpha, V_\alpha)=\sqrt{-1}\sum_{i=1}^s c_i^{-2}\alpha(H_i)(a_\alpha^2-b_\alpha^2)H_i$, *and*
(iii′) $\langle U(U_\alpha, V_\alpha), c_i^{-1}H_i\rangle=\sqrt{-1}\,c_i^{-1}\alpha(H_i)(a_\alpha^2-b_\alpha^2)$,
where $\alpha\in\Delta^+$ *in* (ii), (iii) *and* (iii′).

Proof. From (1.2), we have

$$(2.5)\qquad 2\langle U(U_\alpha, V_\alpha), Z\rangle = \langle [Z, U_\alpha], V_\alpha\rangle + \langle U_\alpha, [Z, V_\alpha]\rangle\,, \qquad Z\in\mathfrak{g}\,.$$

Using (2.1), we obtain the following equations:

$$(2.6)\qquad \begin{cases} [H_i, U_\alpha] = -\sqrt{-1}\,\alpha(H_i)\cdot V_\alpha\,, \qquad [H_i, V_\alpha] = \sqrt{-1}\,\alpha(H_i)\cdot U_\alpha\,, \\ [U_\beta, U_\alpha] = N_{\beta,\alpha}U_{\beta+\alpha} + N_{\beta,-\alpha}U_{\beta-\alpha}\,, \\ [U_\beta, V_\alpha] = N_{\beta,\alpha}V_{\beta+\alpha} - N_{\beta,-\alpha}V_{\beta-\alpha}\,, \\ [V_\beta, V_\alpha] = N_{\beta,-\alpha}U_{\beta-\alpha} - N_{\beta,\alpha}U_{\beta+\alpha}\,, \\ U_\alpha = U_{-\alpha}\,, \qquad V_{-\alpha} = -V_\alpha\,, \\ \langle U_\alpha, U_\alpha\rangle = 2a_\alpha^2\,, \qquad \langle V_\alpha, V_\alpha\rangle = 2b_\alpha^2\,, \end{cases}$$

where α, $\beta\in\Delta^+$, $1\leqq i\leqq s$. From (2.6), we get

$$(2.7)\ [Z, U_\alpha] = \begin{cases} -\sqrt{-1}\,c_i^{-1}\alpha(H_i)V_\alpha & \text{if } Z = c_i^{-1}H_i\,, \\ (a_\beta^{-1}/\sqrt{2})\{N_{\beta,\alpha}U_{\beta+\alpha} + N_{\beta,-\alpha}U_{\beta-\alpha}\} & \text{if } Z = a_\beta^{-1}(U_\beta/\sqrt{2})\,, \\ (b_\beta^{-1}/\sqrt{2})\{N_{\beta,\alpha}V_{\beta+\alpha} + N_{\beta,-\alpha}V_{\beta-\alpha}\} & \text{if } Z = b_\beta^{-1}(V_\beta/\sqrt{2}) \end{cases}$$

and

$$(2.8)\ [Z, U_\alpha] = \begin{cases} \sqrt{-1}\,c_i^{-1}\alpha(H_i)U_\alpha & \text{if } Z = c_i^{-1}H_i \\ (a_\beta^{-1}/\sqrt{2})\{N_{\beta,\alpha}V_{\beta+\alpha} + N_{-\beta,\alpha}V_{-\beta+\alpha}\} & \text{if } Z = a_\beta^{-1}(U_\beta/\sqrt{2})\,, \\ (b_\beta^{-1}/\sqrt{2})\{N_{\beta,-\alpha}U_{\beta-\alpha} - N_{\beta,\alpha}U_{\beta+\alpha}\} & \text{if } Z = b_\beta^{-1}(V_\beta/\sqrt{2})\,. \end{cases}$$

Hence, from (2.3) and (2.5)–(2.8), we obtain (iii). The assertion (iii′) follows immediately from (iii). Similarly, using (1.2), (2.1) and (2.6), we can prove (i) and (ii). q.e.d.

THEOREM 2.2. *An inner automorphism A_t, $(t\in T)$, of a compact connected semisimple Lie group (G, g) is a harmonic map if and only if*

$$(2.9)\qquad \textstyle\sum_{\alpha\in\Delta^+}(b_\alpha^{-2} - a_\alpha^{-2})(b_\alpha^2 - a_\alpha^2)\sin(2\sqrt{-1}\,\alpha(H))\alpha = 0\,,$$

where $t = \exp H$, $(H\in\mathfrak{t})$.

PROOF. For $t = \exp H\in T$, we have

$$(2.10)\qquad \begin{cases} \mathrm{Ad}(t)U_\alpha = \cos(\sqrt{-1}\,\alpha(H))U_\alpha - \sin(\sqrt{-1}\,\alpha(H))V_\alpha\,, \\ \mathrm{Ad}(t)V_\alpha = \sin(\sqrt{-1}\,\alpha(H))U_\alpha + \cos(\sqrt{-1}\,\alpha(H))V_\alpha\,. \end{cases}$$

Theorem 2.2 is obtained from (2.4), Lemma 2.1 and (2.10). q.e.d.

REMARK. Let x be an arbitrary point of a compact connected semisimple Lie group G. Then there exists a maximal torus T of G containing x (cf. [8, Th. 3.9.4, p.

72]). Therefore the criterion for A_x to be harmonic can be obtained by a direct application of Theorem 2.2.

2.2. The Lie algebra $\mathfrak{sl}_n(C)$ of $SL_n(C)$ is the complexification of the real Lie algebra $\mathfrak{su}(n)$ of $SU(n)$. Let E_{ij} denote a square matrix with the (i,j)-entry being 1, and all the other entries being 0. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_n(C)$ which consists of the diagonal matrices of trace 0. Then we have the direct some decomposition

$$\mathfrak{sl}_n(C)=\mathfrak{h}+\textstyle\sum_{i\neq j}CE_{ij}\,. \tag{2.11}$$

If $e_i(H)$, $(H\in\mathfrak{h},\ 1\leqq i\leqq n)$, is the diagonal matrix with the (i,i)-entry 1 and the other entries 0, we get

$$[H, E_{ij}]=(e_i-e_j)(H)\cdot E_{ij}\,. \tag{2.12}$$

Here, the non-zero roots of $\mathfrak{sl}_n(C)$ with respect to $\mathfrak{h}$ are

$$e_i-e_j\,,\qquad (1\leqq i,j\leqq n,\ i\neq j)\,. \tag{2.13}$$

Let B be the Killing form of $\mathfrak{sl}_n(C)$ which is given by

$$B(X, Y)=2n\,\mathrm{Trace}(XY)\,,\qquad (X, Y\in\mathfrak{sl}_n(C))\,. \tag{2.14}$$

We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{su}(n)$ by

$$\langle X, Y\rangle_0:=-B(X, Y)\,,\qquad (X, Y\in\mathfrak{su}(n))\,.$$

We choose an orthonormal basis of $\mathfrak{su}(n)$ with respect to $\langle\ ,\ \rangle_0$ as follows: For i, j such that $1\leqq i<j\leqq n$, let $E_{e_i-e_j}$ (resp. $E_{e_j-e_i}$) denote the root vectors with the (i,j)-entry being $1/\sqrt{2n}$ (resp. the (j,i)-entry being $-1/\sqrt{2n}$) and all the other entries being 0. Then $B(E_{e_i-e_j}, E_{e_j-e_i})=-1$, and $H_{e_i-e_j}$, $(i<j)$, is the diagonal matrix

$$(0,\cdots,0,\overset{\overset{i}{\vee}}{1/2n},0,\cdots,0,\overset{\overset{j}{\vee}}{-1/2n},0,\cdots,0)$$

of order n. We put

$$U_{e_i-e_j}:=E_{e_i-e_j}+E_{e_j-e_i}\,,\quad V_{e_i-e_j}:=\sqrt{-1}(E_{e_i-e_j}-E_{e_j-e_i})$$

and

$$H_{i,j}:=\sqrt{-n}\,H_{e_i-e_j}\,,$$

where $1\leqq i, j\leqq n$ and $i\neq j$. Then,

$$\{U_{e_i-e_j}/\sqrt{2}\,, V_{e_i-e_j}/\sqrt{2}\,, H_{i,i+1}\,|\,1\leqq i\leqq n-1,\ 1\leqq i<j\leqq n\} \tag{2.15}$$

is an orthonormal basis of $\mathfrak{su}(n)$ with respect to $\langle\ ,\ \rangle_0$.

On the other hand, we take another inner product $\langle\ ,\ \rangle$ on $\mathfrak{su}(n)$ such that

(2.16) $\{a_{ij}^{-1}(U_{e_i-e_j}/\sqrt{2}),\, b_{ij}^{-1}(U_{e_i-e_j}/\sqrt{2}),\, c_i^{-1}H_{i,i+1}\,|\,1\leqq i\leqq n-1,\, 1\leqq i<j\leqq n\}$,

(a_{ij}, b_{ij}, c_i: positive constants), is an orthonormal basis of $\mathfrak{su}(n)$ with respect to $\langle\ ,\ \rangle$. Then $\langle\ ,\ \rangle$ determines a left invariant Riemannian metric g on $\mathfrak{su}(n)$. Let T be a maximal torus of $SU(n)$ whose Lie algebra is $\mathfrak{t} := \{H_{i,i+1}\,|\,1\leqq i\leqq n-1\}_{\boldsymbol{R}}$. Then, we get the following from Theorem 2.2:

COROLLARY 2.3. *An inner automorphism A_t, $(t\in T)$, of $(SU(n), g)$ is a harmonic map if and only if*

$$\sum_{1\leqq i<j\leqq n}(b_{ij}^{-2}-a_{ij}^{-2})(b_{ij}^2-a_{ij}^2)\sin(2\sqrt{-1}(e_i-e_j)(H))(e_i-e_j)=0\,, \tag{2.17}$$

where $t=\exp(H)$, $(H\in\mathfrak{t})$.

3. The case of $SU(2)$. In this section, we get necessary and sufficient conditions for inner automorphisms A_x, $(x\in SU(2))$, of $SU(2)$ to be harmonic with respect to any left invariant Riemannian metric.

The Lie algebra $\mathfrak{sl}_2(\boldsymbol{C})$ of $SL_2(\boldsymbol{C})$ is the complexification of the real Lie algebra $\mathfrak{su}(2)$. The Killing form B of $\mathfrak{sl}_2(\boldsymbol{C})$ satisfies

$$B(X, Y)=4\,\mathrm{Trace}(XY)\,, \qquad (X, Y\in\mathfrak{sl}_2(\boldsymbol{C}))\,. \tag{3.1}$$

We define an inner product $\langle\ ,\ \rangle_0$ on $\mathfrak{su}(2)$ by

$$\langle X, Y\rangle_0 := -B(X, Y)\,, \qquad (X, Y\in\mathfrak{su}(2))\,.$$

In this section, g denotes any left invariant Riemannian metric of $SU(2)$.

The following lemma is known (cf. [7, Lemma 1.1, p. 154]):

Lemma 3.1. *Let g be a left invariant Riemannian metric. Let $\langle\ ,\ \rangle$ be an inner product on $\mathfrak{su}(2)$ defined by $\langle X, Y\rangle := g_e(X_e, Y_e)$, where X, $Y\in\mathfrak{su}(2)$ and e is the identity matrix of $SU(2)$. Then there exist an orthonormal basis (X_1, X_2, X_3) of $\mathfrak{su}(2)$ with respect to $\langle\ ,\ \rangle_0$ such that*

$$\begin{cases}[X_1, X_2]=(1/\sqrt{2})X_3,\ [X_2, X_3]=(1/\sqrt{2})X_1\,,\\ [X_3, X_1]=(1/\sqrt{2})X_2,\ \langle X_i, X_j\rangle=\delta_{ij}a_i^2\,,\end{cases} \tag{3.2}$$

where a_i, $(1\leqq i\leqq 3)$, are positive real numbers determined by the given left invariant Riemannian metric g of $SU(2)$.

Now, putting $Y_1 := 2\sqrt{2}\,X_1$, $Y_2 := 2\sqrt{2}\,X_2$, and $Y_3 := 2\sqrt{2}\,X_3$ for the orthonormal basis (X_1, X_2, X_3) with respect to $\langle\ ,\ \rangle_0$ in Lemma 3.1, we have

$$[Y_1, Y_2]=2Y_3\,,\quad [Y_2, Y_3]=2Y_1\,,\quad [Y_3, Y_1]=2Y_2\,. \tag{3.3}$$

We know from Lemma A of §1 that an inner automorphism A_x, $(x\in SU(2))$, of $(SU(2), g)$ is harmonic if and only if

$$\begin{aligned}(3.4)\qquad &\textstyle\sum_{i=1}^{3}\{U(\mathrm{Ad}(x)\cdot Y_i/(2\sqrt{2}\,a_i),\ \mathrm{Ad}(x)\cdot Y_i/(2\sqrt{2}\,a_i))\\ &\quad-\mathrm{Ad}(x)\cdot U(Y_i/(2\sqrt{2}\,a_i),\ Y_i/(2\sqrt{2}\,a_i))\}\\ &=\textstyle\sum_{i=1}^{3}(a_i^{-2}/8)\{U(\mathrm{Ad}(x)Y_i,\ \mathrm{Ad}(x)Y_i)-\mathrm{Ad}(x)U(Y_i,\ Y_i)\}=0\,.\end{aligned}$$

In order to analyze (3.4) further, we need the following:

Lemma 3.2.

$$(3.5)\qquad \begin{cases} U(Y_1,\ Y_1)=U(Y_2,\ Y_2)=U(Y_3,\ Y_3)=0\,,\\ U(Y_1,\ Y_2)=(a_2^2-a_1^2)a_3^{-2}Y_3\,,\\ U(Y_2,\ Y_3)=(a_3^2-a_2^2)a_1^{-2}Y_1\,,\\ U(Y_3,\ Y_1)=(a_1^2-a_3^2)a_2^{-2}Y_2\,.\end{cases}$$

Proof. Using (1.2), we can prove this lemma in the same way as in the proof of Lemma 2.1 of §2. q.e.d.

PROPOSITION 3.3. *An inner automorphism* A_x, $(x=\exp(rY_1),\ r\in\boldsymbol{R})$, *of* $(SU(2), g)$ *is harmonic if and only if*

$$(a_3^2-a_2^2)(a_2^{-2}-a_3^{-2})\sin(4r)=0\,,$$

that is,

$$(3.6)\qquad a_2=a_3\quad or\quad r\in\{(n\pi)/4\mid n\ is\ an\ integer\}\,.$$

Proof. Using (3.3), we have

$$(3.7)\qquad \begin{cases}\mathrm{Ad}(x)Y_2=\cos(2r)Y_2+\sin(2r)Y_3\,,\\ \mathrm{Ad}(x)Y_3=\cos(2r)Y_3-\sin(2r)Y_2\,.\end{cases}$$

We know from (3.4), Lemma 3.2 and (3.7) that A_x is harmonic if and only if

$$(3.8)\qquad \sin(4r)(a_2^{-2}-a_3^{-2})(a_3^2-a_2^2)a_1^{-2}Y_1=0\,.$$

q.e.d.

PROPOSITION 3.4. *An inner automorphism* A_x, $(x=\exp(rY_2),\ r\in\boldsymbol{R})$, *of* $(SU(2), g)$ *is a harmonic map if and only if*

$$(a_1^2-a_3^2)(a_1^{-2}-a_3^{-2})\sin(4r)=0\,,$$

that is,

$$(3.9)\qquad a_1=a_3\quad or\quad r\in\{(n\pi)/4\mid n\ is\ an\ integer\}\,.$$

PROOF. Using (3.3), we have

$$(3.10)\qquad \begin{cases}\mathrm{Ad}(x)Y_1=\cos(2r)Y_1-\sin(2r)Y_3\,,\\ \mathrm{Ad}(x)Y_3=\cos(2r)Y_3+\sin(2r)Y_1\,.\end{cases}$$

Hence, we find from (3.4), Lemma 3.2 and (3.10) that A_x is harmonic if and only if

$$\sin(4r)(a_3^{-2}-a_1^{-2})a_2^{-2}(a_1^2-a_3^2)Y_2=0\,. \tag{3.11}$$

q.e.d.

PROPOSITION 3.5. *An inner automorphism A_x, $(x=\exp(rY_3),\ r\in\boldsymbol{R})$, of $(SU(2), g)$ is a harmonic map if and only if*

$$a_1=a_2 \quad or \quad r\in\{(n\pi)/4 \mid n \text{ is an integer}\}\,. \tag{3.12}$$

PROOF. We get from (3.3)

$$\begin{cases} \mathrm{Ad}(x)Y_1=\cos(2r)Y_1+\sin(2r)Y_2\,, \\ \mathrm{Ad}(x)Y_2=\cos(2r)Y_2-\sin(2r)Y_1\,. \end{cases} \tag{3.13}$$

Using (3.4), Lemma 3.2 and (3.13), we obtain this proposition. q.e.d.

Thus, from Propositions 3.3, 3.4 and 3.5, we have:

THEOREM 3.6. *An inner automorphism A_x of $(SU(2), g)$ for any $x\in SU(2)$ is a harmonic map if and only if the metric g of $(SU(2), g)$ is bi-invariant.*

PROOF. If A_x for any $x\in SU(2)$ is harmonic, then $a_1=a_2=a_3$ by Propositions 3.3–3.5. Hence, $-B(X, Y)=c^2\langle X, Y\rangle$, and $\langle[Z, X], Y\rangle+\langle X, [Z, Y]\rangle=0$ for any $X, Y, Z\in\mathfrak{su}(2)$. The second equation implies that $\langle\mathrm{Ad}(\exp rZ)X, \mathrm{Ad}(\exp rZ)Y\rangle$ is a constant independent of $r\in\boldsymbol{R}$. Hence B is bi-invariant. Conversely, if g is bi-invariant, we know from (1.2) that $U(X, Y)=0$ for any $X, Y\in\mathfrak{su}(2)$. Thus, A_x for any $x\in SU(2)$ is harmonic. q.e.d.

Finally, we get:

THEOREM 3.7. *Assume that a left invariant metric g of $(SU(2), g)$ is not bi-invariant. Then, there always exist non-isometric harmonic inner automorphisms A_x of $(SU(2), g)$.*

PROOF. Since g is not bi-invariant by the assumption, there are two different numbers among $\{a_1, a_2, a_3\}$ by Theorem 3.6. Then, from (3.7), (3.10) and (3.13), there exist non-isometric but harmonic inner automorphisms A_x for $x\in SU(2)$ such that

$$x=\begin{cases} \exp(\pi/4)Y_1 & \text{when } a_2\neq a_3\,, \\ \exp(\pi/4)Y_2 & \text{when } a_3\neq a_1\,, \quad \text{and} \\ \exp(\pi/4)Y_3 & \text{when } a_1\neq a_2\,. \end{cases}$$

Indeed, we get $\langle Y_i, Y_i\rangle=8a_i^2$, $i=1, 2, 3$. On the other hand we obtain

$$\langle\mathrm{Ad}(\exp(\pi/4)Y_3)Y_1, \mathrm{Ad}(\exp(\pi/4)Y_3)Y_1\rangle=\langle Y_2, Y_2\rangle=8a_2^2\,,$$

$$\langle\mathrm{Ad}(\exp(\pi/4)Y_1)Y_2, \mathrm{Ad}(\exp(\pi/4)Y_1)Y_2\rangle=\langle Y_3, Y_3\rangle=8a_3^2\,,$$

and

$$\langle \mathrm{Ad}(\exp(\pi/4)Y_2)Y_3, \mathrm{Ad}(\exp(\pi/4)Y_2)Y_3\rangle = \langle Y_1, Y_1\rangle = 8a_1^2 .$$

Therefore, the inner automorphisms A_x, for the above elements x, are non-isometric but harmonic maps of $(SU(2), g)$ into itself. q.e.d.

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