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A STABLE MANIFOLD THEOREM FOR THE YANG-MILLS GRADIENT FLOW

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1. Introduction. The purpose of this paper is to study the asymptotic behavior of the gradient flow of the Yang-Mills functional near Yang-Mills connections.

In differential geometry, many subjects are defined as variational problems on Riemannian manifolds. Most of them, however, do not satisfy the Palais-Smale condition. (For the Palais-Smale condition, see Palais [P], Palais and Smale [PS], or Eells and Sampson [ES2]). If the variational problem defined by a functional $J(\cdot)$ on a function space X satisfies this condition, the equation for the gradient flow of J with initial value v:

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = -\operatorname{grad} J(u(t)) , \\ u(0) = v \end{cases}$$

must have a unique time-global solution. If J does not satisfy this condition, we do not know, in general, whether this property holds or not.

There are some results on the global existence of the gradient flow. In 1964, Eells and Sampson proved the existence theorem of harmonic maps by means of the asymptotic behavior of the gradient flow when the target manifold has non-positive curvature [ES1].

In studying the existence of a time-global solution for the gradient flow, we need pay attention to the relation between the Morse theoretic stability of a critical point and the asymptotic behavior of the solution of the gradient flow around the critical point. For harmonic maps, the first author studied the above relation in the case of a stable harmonic map [N2]. Concerning more general variational problems, we can refer to Simon [S]. Recently, the first author proved a stable manifold theorem for quasi-linear parabolic equations, and showed, as an application, the asymptotic behavior of the gradient flow even around an unstable critical point assuming the ellipticity of the Euler-Lagrange operator [N1].

Since the Yang-Mills functional is invariant under the gauge transformation group, the equation governing the gradient flow of the Yang-Mills functional is not parabolic.

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To avoid this difficulty, the second and third authors considered a gauge condition and the gradient flow under the condition that the second variation is strictly positive [KM]. Similar idea is found in Kono and Nagasawa [KN]. These results, however, assume the Yang-Mills connection to be strictly stable.

In this paper, we prove that the global existence of the gradient flow for the Yang-Mills functional with the initial value near a Yang-Mills connection without assuming that it is strictly stable. Our basic set-up is the following. (This set-up was introduced by Bourguignon and Lawson [BL]). Let (M, h) be a compact Riemannian manifold without boundary and P be a principal G-bundle over M, where G is a compact Lie group.

We consider a G-vector bundle $E := P \times_\rho \boldsymbol{R}^N$, associated to P by a faithful orthogonal representation $\rho: G \to O(N)$. The group of all inner automorphisms is called the *gauge group* of P and will be denoted by $\mathscr{G}_P$. It can be easily identified with the group of smooth sections of the bundle $G_P := P \times_{\mathrm{Ad}} G$, i.e., $\mathscr{G}_P = \Gamma(M; G_P)$. Related to $\mathscr{G}_P$ is the infinitesimal gauge group (gauge algebra) which will be denoted by $\mathfrak{G}_P$. It is the Lie algebra of smooth sections of the bundle of Lie algebras $\mathfrak{g}_P := P \times_{\mathrm{Ad}} \mathfrak{g}$, i.e., $\mathfrak{G}_P := \Gamma(M; \mathfrak{g}_P)$, where $\mathfrak{g}$ is the Lie algebra of the Lie group G.

The gauge group can be easily re-expressed in terms of E. Let O_E be the orthogonal frame bundle of E over M, i.e., the fiber at $x \in M$ is the group of orthogonal transformations in E_x. Let $\mathfrak{so}_E$ be the bundle over M whose fiber at $x \in M$ is the Lie algebra of skew-symmetric transformations of E_x. Then the representation ρ gives embedding $G_P \subsetneq O_E$ and $\mathfrak{g}_P \subsetneq \mathfrak{so}_E$. We denote the images by G_E and $\mathfrak{g}_E$, respectively. Clearly, $G_E \cong G_P$ and $\mathfrak{g}_E \cong \mathfrak{g}_P$.

We also denote by $\mathscr{G}_E := \Gamma(M; G_E)$ and $\mathfrak{G}_E := \Gamma(M; \mathfrak{g}_E)$ the spaces of smooth sections of G_E and of $\mathfrak{g}_E$, respectively.

We now introduce some notation; Given a smooth vector bundle F over M, let $\Omega^p(F) := \Gamma(\bigwedge^p T^*M \otimes F)$ be the space of exterior differential p-forms on M with values in F. Indeed $\Omega^0(F)$ is just the space of smooth sections of F and $\mathfrak{G}_E = \Omega^0(\mathfrak{g}_E)$. We study the space $\mathscr{C}_P$ of connections on P, or equivalently, $\mathscr{C}_E$ on E. (For the relation between a connection on P and a connection on E, we can refer to [BL]). It is easily shown that, for two connections ∇ and $\nabla' \in \mathscr{C}_E$, the difference $A = \nabla - \nabla'$ is an element of $\Omega^1(\mathfrak{g}_E)$. In particular, if we fix $\nabla \in \mathscr{C}_E$, then there is a canonical identification

$$T_\nabla(\mathscr{C}_E) \cong \Omega^1(\mathfrak{g}_E)\,. \tag{1.1}$$

To each connection $\nabla \in \mathscr{C}_E$, there is associated a curvature 2-form R^∇ in $\Omega^2(\mathfrak{g}_E)$ given by

$$R^\nabla_{X,Y} := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for tangent vectors X and Y.

The Yang-Mills functional $\mathscr{YM}$ on $\mathscr{C}_E$ is defined by

$$(1.2) \qquad \mathscr{YM}(\nabla)=\frac{1}{2}\int_M \|R^\nabla\|^2 ,$$

where the norm is defined in terms of the Riemannian metric on M and a fixed Ad_G-invariant scalar product on the Lie algebra $\mathfrak{g}$ of G. Section 2 contains the precise definition of this norm.

Critical points of the smooth functional $\mathscr{YM}: \mathscr{C}_E \to \boldsymbol{R}$ are called *Yang-Mills connections* and their associated curvature tensors are called *Yang-Mills fields*. Clearly, a connection $\nabla \in \mathscr{C}_E$ is a Yang-Mills connection if and only if $\nabla \in \mathscr{C}_E$ satisfies the Euler-Lagrange equation $\mathrm{grad}(\mathscr{YM}(\nabla))=0$ of the Yang-Mills functional $\mathscr{YM}$. The Euler-Lagrange equation of $\mathscr{YM}$ is expressed as

$$(1.3) \qquad \delta^\nabla R^\nabla=0 ,$$

where δ^∇ is the formal adjoint operator of the exterior derivative d^∇ associated with $\nabla \in \mathscr{C}_E$ with respect to the global inner product induced by $\|\cdot\|$. In particular, the equation $\delta^\nabla R^\nabla=0$ is a second order partial differential equation with respect to ∇.

We remark that the Yang-Mills functional is invariant under the action of the gauge group $\mathscr{G}_E$ on $\mathscr{C}_E$. Therefore if $\nabla \in \mathscr{C}_E$ is a Yang-Mills connection, then $\nabla^g := g \circ \nabla \circ g^{-1}$ is also a Yang-Mills connection, because of $(\delta^\nabla R^\nabla)^g=\delta^{\nabla^g} R^{\nabla^g}$.

On the other hand, since $d^\nabla R^\nabla=0$ (the Bianchi identity), $\nabla \in \mathscr{C}_E$ is a Yang-Mills connection if and only if

$$(1.4) \qquad \Delta^\nabla R^\nabla=0 ,$$

where $\Delta^\nabla=d^\nabla\delta^\nabla+\delta^\nabla d^\nabla$ is the Hodge Laplacian on $\Omega^2(\mathfrak{g}_E)$.

We shall construct a solution of the Yang-Mills gradient flow with initial value close to a Yang-Mills connection. The Yang-Mills gradient flow with initial value ∇_1 on $\mathscr{C}_E$ is governed by the following equation:

$$(\mathrm{YMGF}) \qquad \begin{cases} \dfrac{\partial \nabla}{\partial t} = -\delta^\nabla R^\nabla , \\ \nabla(0)=\nabla_1 . \end{cases}$$

Unfortunately, the above (YMGF) is not a parabolic equation, since (YMGF) is invariant under the gauge group action on $\mathscr{C}_E$. A more finely treatment on (YMGF) in [KM] allows us to avoid this difficulty.

This idea is stated briefly as follows. Take and fix a Yang-Mills connection $\nabla_0 \in \mathscr{C}_E$. There is a natural splitting of the tangent space:

$$(1.5) \qquad T_{\nabla_0}(\mathscr{C}_E) \cong \Omega^1(\mathfrak{g}_E)=Z^1(\mathfrak{g}_E)\oplus\Omega^1_*(\mathfrak{g}_E) ,$$

where $Z^1(\mathfrak{g}_E):=\{V=d^{\nabla_0}\phi;\ \phi\in\Omega^0(\mathfrak{g}_E)\cap(\mathrm{Ker}\, d^{\nabla_0})^\perp\}$ and $\Omega^1_*(\mathfrak{g}_E):=\mathrm{Ker}(\delta^{\nabla_0})$. The exponential map of the Lie groups induces naturaly the map

$$\exp : \mathfrak{g}_E \to \mathscr{G}_E .$$

For smooth curves $d^{\nabla_0}\Phi(t)$ and $A(t)$ in $Z^1(\mathfrak{g}_E)$ and $\Omega^1_*(\mathfrak{g}_E)$, respectively, we define the map $\sigma : Z^1(\mathfrak{g}_E) \times \Omega^1_*(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ by

$$\sigma(d^{\nabla_0}\Phi(t), A(t)) := g(t) \circ (\nabla_0 + A(t)) \circ g(t)^{-1} - \nabla_0 , \tag{1.6}$$

where $g(t) := \exp \Phi(t)$.

Since $\sigma(0, 0) = 0$ and the Fréchet derivative $D\sigma(0, 0)$ is an isomorphism on $\Omega^1(\mathfrak{g}_E)$, σ is a local diffeomorphism near 0 in $\Omega^1(\mathfrak{g}_E)$. Then, by (1.5), we may regard σ as giving a coordinate around ∇_0 in $\mathscr{C}_E$. Now, taking $\nabla(t) = \nabla_0 + \sigma(d^{\nabla_0}\Phi(t), A(t))$ in (YMGF), we have the following equation (Eq) equivalent to (YMGF):

$$\text{(Eq)} \qquad \begin{cases} \dfrac{\partial A(t)}{\partial t} = -J^{\nabla_0}A(t) - Q(A(t)) + [A(t), S(t)] + d^{\nabla_0}S(t) , \\ \delta^{\nabla_0}A(t) = 0 , \\ A(0) = A_0 , \end{cases}$$

where $S(t) = g(t)^{-1}dg(t)/dt$. For precise notation, see Section 2. We here notice that the linear differential operator J^{∇_0} is the Jacobi operator (denoted in Section 2) of the Yang-Mills connection ∇_0.

Let $H^m(\Omega^1(\mathfrak{g}_E))$ be the m-th order Sobolev space using the coordinate covering of M (see Section 3 for the precise definition). A subset $\mathscr{S} \in H^m(\Omega^1(\mathfrak{g}_E))$ is said to be a *stable manifold* of a Yang-Mills connection ∇_0, whenever $\mathscr{S}$ satisfies the following conditions:

(1) $\mathscr{S}$ is a submanifold in $H^m(\Omega^1(\mathfrak{g}_E))$.

(2) If $\nabla_1 - \nabla_0$ is contained in $\mathscr{S}$, then there exists a time-global solution ∇ of (YMGF) with initial value ∇_1 such that $\nabla(t)$ tends to gauge equivalence to ∇_0 as $t \to \infty$ in H^m-topology.

A subset $\mathscr{U}$ in $H^m(\Omega^1(\mathfrak{g}_E))$ is said to be an *unstable manifold*, whenever $\mathscr{U}$ is a stable manifold with respect to backward (YMGF).

Here we do not assume the maximality of stable and unstable manifolds, and therefore dimensions of these are not maximal.

For the evolution equation (Eq), our main result, briefly stated, is:

THEOREM. *Let $m > \dim M/2 + 2$. For the Yang-Mills connection ∇_0, there exists a finite codimensional stable manifold and a finite dimensional unstable manifold of* (Eq) *in the m-th order Sobolev space $H^m(\Omega^1_*(\mathfrak{g}_E))$ on $\Omega^1_*(\mathfrak{g}_E)$.*

REMARK. (1) Viewing $\Omega^1_*(\mathfrak{g}_E)$ as a closed subspace of $\Omega^1(\mathfrak{g}_E)$, we can induce naturally the norm $\|\cdot\|_m$ on the $\Omega^1_*(\mathfrak{g}_E)$ and make Sobolev space of $\Omega^1_*(\mathfrak{g}_E)$. For the technical reason, we will introduce another norm $\|\cdot\|_{H^m(\Omega^1_*(\mathfrak{g}_E))}$ equivalent to $\|\cdot\|_m$ for $m > \dim M/2 + 2$ (see Section 3). This is a reason why we need the assumption in the

Theorem.

(2) In the above theorem, the assumption $m > \dim M/2 + 2$ also needs to embed the Sobolev space $H^m(\Omega^1_*(\mathfrak{g}_E))$ into $C^2(\Omega^1_*(\mathfrak{g}_E))$.

In [KM], they have shown the asymptotic stability of the Yang-Mills gradient flow near a strictly stable Yang-Mills connection. Our theorem asserts that we can get such behavior without assuming the stability of the critical point: a Yang-Mills connection.

COROLLARY. *Let $\nabla_0 \in \mathscr{C}_E$ be a Yang-Mills connection and $\nabla_1 = \nabla_0 + A_0$, where $A_0 \in \Omega^1(\mathfrak{g}_E)$. If A_0 belongs to the stable manifold as in the Theorem, there exists a unique solution ∇* of (YMGF) *with initial value ∇_1 such that $\nabla - \nabla_0 \in L^2([0, \infty)$; $H^m(\Omega^1(\mathfrak{g}_E)))$. Moreover the solution $\nabla(t)$ tends to a Yang-Mills connection up to the gauge group action as $t \to \infty$ in H^m-topology.*

Here is an outline of the contents. In Section 2 we recall the set-up of the Yang-Mills theory and introduce the evolution equation (Eq). In Section 3, we discuss the evolution equation (Eq) and reduce it to an abstract evolution equation. Section 4 is devoted to the proofs of the existence of a solution for the abstract evolution equation and of the main result.

2. Preliminalies. As we mentioned in Section 1, we consider the Yang-Mills functional on $\mathscr{C}_E$.

Let (M, h) be a closed Riemannian manifold and P a principal G-bundle over M, where G is a compact Lie group. Taking a faithful representation $\rho: G \to O(N)$, we can define a G-vector bundle $E := P \times_\rho \boldsymbol{R}^N$ associated to P. The Yang-Mills functional on the set $\mathscr{C}_E$ of connections on E is

$$\mathscr{Y}\mathscr{M}(\nabla) = \frac{1}{2}\int_M \|R^\nabla\|^2 . \tag{2.1}$$

The norm $\|\cdot\|$ is, in local coordinates, defined as follows. In a chart U of M, $\omega \in \Omega^k(\mathfrak{g}_E)$ is expressed as

$$\omega = \frac{1}{k!}\omega_{j_1 \cdots j_k}{}^b{}_a dx^{j_1} \wedge \cdots \wedge dx^{j_k} \otimes E^a_b , \tag{2.2}$$

where $\{E^a_b\}$ is a basis of the fiber of $\mathfrak{g}_E$ at $x \in M$ and $(x^1, \cdots, x^n)$ is a local coordinate on U ($n = \dim M$). For such above ω, we denote $\omega = (\omega_{j_1 \cdots j_k}{}^a_b)$ simply.

In local coordinates neighborhoods, the norm of ω is defined by

$$\|\omega\|^2 := k! h^{j_1 i_1} \cdots h^{j_k i_k} \omega_{j_1 \cdots j_k}{}^b{}_a \omega_{i_1 \cdots i_k}{}^a{}_b . \tag{2.3}$$

That is to say, the fiber metric on E is defined by

$$\langle A, B \rangle := \frac{1}{2} \operatorname{trace} {}^t\! AB \tag{2.4}$$

for two endomorphisms A, $B \in E_x$.

Now let us calculate the first and the second variation formulas. For a smooth one-parameter family $\nabla_t \in \mathscr{C}_E$ of connections with ∇_0 at $t=0$ and $A := (d/dt)\nabla_t|_{t=0}$, the first and second variation formulas are given by

$$\frac{d}{dt}\mathscr{YM}(\nabla_t)\Big|_{t=0} = \int_M \langle \delta^{\nabla_0} R^{\nabla_0}, A\rangle , \tag{2.5}$$

and

$$\frac{d^2}{dt^2}\mathscr{YM}(\nabla_t)\Big|_{t=0} = \int_M \langle \delta^{\nabla_0} d^{\nabla_0} A + [R^{\nabla_0}, A], A\rangle , \tag{2.6}$$

respectively. In particular, if A satisfies $\delta^{\nabla_0}A = 0$ then

$$\frac{d^2}{dt^2}\mathscr{YM}(\nabla_t)\Big|_{t=0} = \int_M \langle (d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0})A + [R^{\nabla_0}, A], A\rangle . \tag{2.7}$$

Therefore, the Euler-Lagrange equation for $\mathscr{YM}$ is $\delta^\nabla R^\nabla = 0$. For $A \in \Omega^1(\mathfrak{g}_E)$ and a Yang-Mills connection ∇_0, the operator $J^{\nabla_0}A := (d^{\nabla_0}\delta^{\nabla_0} + \delta^{\nabla_0} d^{\nabla_0})A + [R^{\nabla_0}, A] = \Delta^{\nabla_0}A + [R^{\nabla_0}, A]$ is called the *Jacobi operator* for ∇_0.

Since there is a natural $\mathscr{G}_E$ action on $\mathscr{C}_E$, it is easily shown that $\mathscr{YM}(\nabla) = \mathscr{YM}(\nabla^g)$ and $\delta^{\nabla^g} R^{\nabla^g} = (\delta^\nabla R^\nabla)^g$, where $\nabla^g := g \circ \nabla \circ g^{-1}$.

For a Yang-Mills connection ∇_0, we define the stability of it. Since the restriction of the Jacobi operator J^{∇_0} to $\Omega^1_*(\mathfrak{g}_E)$ maps $\Omega^1_*(\mathfrak{g}_E)$ into itself and has the discrete spectrum: $\{\lambda_1 \le \lambda_2 \le \cdots \nearrow +\infty\}$, we can define the *index*: Index(∇_0) of ∇_0 and the *nullity*: Null(∇_0) of ∇_0 as

$$\begin{aligned} \mathrm{Index}(\nabla_0) &:= \#\{\text{negative eigenvalues}\} , \\ \mathrm{Null}(\nabla_0) &:= \#\{\text{zero eigenvalues}\} . \end{aligned} \tag{2.8}$$

A Yang-Mills connection ∇_0 is said to be *weakly stable* whenever Index$(\nabla_0) = 0$. This definition is equivalent to

$$\frac{d^2}{dt^2}\mathscr{YM}(\nabla_t)\Big|_{t=0} \ge 0 \tag{2.9}$$

for any one-parameter family ∇_t of connections with ∇_0.

Furthermore, we define a strictly stable Yang-Mills connection. A Yang-Mills connection ∇_0 is said to be *strictly stable* whenever for any one-parameter family ∇_t of connections with ∇_0 and $(d/dt)\nabla_t|_{t=0} \ne 0 \in \Omega^1_*(\mathfrak{g}_E)$,

$$\frac{d^2}{dt^2}\mathscr{YM}(\nabla_t)\Big|_{t=0} > 0 . \tag{2.10}$$

(c.f. [BL], [KM].) In terms of the Jacobi operator, a weakly stable Yang-Mills connection ∇_0 is strictly stable if and only if $\mathrm{Ker}\, J^{\nabla_0} \subset Z^1(\mathfrak{g}_E)$.

In this situation, we try to avoid the difficulty that the Euler-Lagrange equation of $\mathcal{YM}$ is not elliptic.

In what follows, a Yang-Mills connection ∇_0 is fixed. As we mentioned in Section 1, there are a canonical identification of the tangent space $T_{\nabla_0}(\mathscr{C}_E)$ with $\Omega^1(\mathfrak{g}_E)$, and the splitting of $\Omega^1(\mathfrak{g}_E)$:

$$T_{\nabla_0}(\mathscr{C}_E) \cong \Omega^1(\mathfrak{g}_E) = Z^1(\mathfrak{g}_E) \oplus \Omega^1_*(\mathfrak{g}_E) . \tag{2.11}$$

Concerning (2.11), we define the map $\sigma\colon Z^1(\mathfrak{g}_E) \times \Omega^1_*(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ as follows:

$$\sigma(d^{\nabla_0}\Phi, A) := g \circ (\nabla_0 + A) \circ g^{-1} - \nabla_0 , \tag{2.12}$$

where $g := \exp \Phi$ and $\Phi \in \Omega^0(\mathfrak{g}_E) \cap (\mathrm{Ker}\, d^{\nabla_0})^\perp$. It is easily shown that there exist three neighborhoods U_1, U_2 and U in $Z^1(\mathfrak{g}_E)$, $\Omega^1_*(\mathfrak{g}_E)$ and $\Omega^1(\mathfrak{g}_E)$, respectively, such that the map σ induces a diffeomorphism from $U_1 \times U_2$ onto U. In particular, taking $A = 0$ in (2.12), we get

$$\sigma(d^{\nabla_0}\Phi, 0) = g \circ \nabla_0 \circ g^{-1} - \nabla_0 .$$

Therefore the tangent space at ∇_0 of orbits under the action of the gauge group coincides with $Z^1(\mathfrak{g}_E)$.

In the above formulation, the Yang-Mills gradient flow is rewritten as the evolution equation (Eq) (in Section 1).

For a smooth curve $\nabla(t)$ in $\mathscr{C}_E$ with $\nabla(0) = \nabla_0$ a Yang-Mills connection, we can find smooth curves $\Phi(t)$ and $A(t)$ in $\Omega^0(\mathfrak{g}_E)$ and $\Omega^1_*(\mathfrak{g}_E)$, respectively, such that

$$\nabla(t) - \nabla(0) = \sigma(d^{\nabla_0}\Phi(t), A(t)) \tag{2.13}$$

using the diffeomorphism $U_1 \times U_2 \overset{\sigma}{\cong} U$. These U, U_1 and U_2 give local coordinate neighborhoods around zero, respectively, and we call the system $(\sigma; U, U_1, U_2)$ an *admisible coordinate system*. For the sake of simplicity, we denote $\sigma(d^{\nabla_0}\Phi(t), A(t))$ by $\sigma(t)$.

The differentiation of σ in t gives:

PROPOSITION 2.1. ([KM, (3.8)]). *For the map σ defined by* (2.12), *we obtain*

$$\begin{aligned} \frac{d\sigma(t)}{dt} &= g(t) \circ \left(\frac{dA(t)}{dt} - [\nabla_0 + A(t), S(t)] \right) \circ g(t)^{-1} \\ &= g(t) \circ \left(\frac{dA(t)}{dt} - d^{\nabla_0} S(t) - [A(t), S(t)] \right) \circ g(t)^{-1} , \end{aligned} \tag{2.14}$$

where $S(t) = g(t)^{-1} dg(t)/dt$ and $g(t) = \exp \Phi(t)$.

On the other hand, the Euler-Lagrange operator at $\nabla(t)=\nabla_0+\sigma(t)$ is $\delta^{\nabla_0+\sigma(t)}R^{\nabla_0+\sigma(t)}$. For $g\in\mathscr{G}_E$ and $\nabla\in\mathscr{C}_E$, the formulas

$$R^{\nabla^g}=g\circ R^{\nabla}\circ g^{-1}\,, \tag{2.15}$$

and

$$\delta^{\nabla^g}R^{\nabla^g}=(\delta^{\nabla}R^{\nabla})^g=g\circ\delta^{\nabla}R^{\nabla}\circ g^{-1} \tag{2.16}$$

guarantee

$$\begin{aligned}\delta^{\nabla_0+\sigma(t)}R^{\nabla_0+\sigma(t)}&=g(t)\circ(\delta^{\nabla_0+A(t)}R^{\nabla_0+A(t)})\circ g(t)^{-1}\\&=g(t)\circ(\delta^{\nabla_0}R^{\nabla_0}+\delta^{\nabla_0}d^{\nabla_0}A(t)+[R^{\nabla_0},A(t)]+Q(A(t)))\circ g(t)^{-1}\,,\end{aligned} \tag{2.17}$$

where

$$Q(A)=\delta^{\nabla_0}\frac{1}{2}[A,A]+[d^{\nabla_0}A,A]+\frac{1}{2}[[A,A],A]\,. \tag{2.18}$$

(cf. [KM, Section 3].)

Here, we use the bracket $[\cdot,\cdot]$ in (2.18) for $\mathfrak{g}_E$-valued forms. Namely, using local coordinates, for any $\alpha=(1/2)(\alpha_{j\,b}^{\,a})\in\Omega^1(\mathfrak{g}_E)$ and $\beta=(1/2)(\beta_{j\,b}^{\,a})\in\Omega^1(\mathfrak{g}_E)$, we put

$$[\alpha,\beta]_{ij\,b}^{\;\;a}=\frac{1}{2}(\alpha_{i\,c}^{\,a}\beta_{j\,b}^{\,c}-\beta_{j\,c}^{\,a}\alpha_{i\,b}^{\,c})$$

and extend this for general $\mathfrak{g}_E$-valued forms.

Therefore we obtain:

PROPOSITION 2.2. *Let ∇_0 be a Yang-Mills connection, and $(\sigma; U, U_1, U_2)$ be an admisible coordinate system. If a smooth curve $\nabla(t)$ in U is the solution of*

$$\begin{cases}\dfrac{\partial\nabla}{\partial t}=-\delta^{\nabla}R^{\nabla}\,,\\ \nabla(0)=\nabla_1\,,\end{cases}$$

then a smooth curve $\{S(t), A(t)\}$ in $U_1\times U_2$ satisfying $\nabla(t)-\nabla_0=\sigma(d^{\nabla_0}\Phi(t), A(t))$ is the solution of

$$\begin{cases}\dfrac{\partial A(t)}{\partial t}=-\{(d^{\nabla_0}\delta^{\nabla_0}+\delta^{\nabla_0}d^{\nabla_0})A(t)+[R^{\nabla_0},A(t)]+Q(A(t))-[A(t),S(t)]-d^{\nabla_0}S(t)\}\,,\\ \delta^{\nabla_0}A(t)=0\,,\\ A(0)=A_1\,,\end{cases} \tag{2.19}$$

where $A_1=\nabla_1-\nabla_0$ (note that $\sigma(0,A_1)=\nabla_1$) and $S(t)=g(t)^{-1}dg(t)/dt$.

The convese is also true.

The first equation of (2.19) is parabolic since $J^{\nabla_0}A=(d^{\nabla_0}\delta^{\nabla_0}+\delta^{\nabla_0}d^{\nabla_0})A+[R^{\nabla_0}, A]$ is an elliptic operator on $\Omega^1_*(\mathfrak{g}_E)$. These ideas were introduced by [KM]. In this section, we showed that the Yang-Mills gradient flow yields the evolution equation (2.19) by taking a gauge condition into account. Hence, our main purpose is to solve the evolution equation (2.19) when A_1 is small. If the Yang-Mills connection ∇_0 is strictly stable, J^{∇_0} is a strictly positive operator in $\Omega^1_*(\mathfrak{g}_E)$. In this case, [KM] has shown the asymptotic stability of the solution of (2.19) using the method in Sections 3 and 4. We would like to extend such a result for (2.19), without assuming the (strict) stability of ∇_0. In our case, J^{∇_0} restricted $\Omega^1_*(\mathfrak{g}_E)$ may have finite dimensional eigenspaces with non-positive eigenvalues. To solve (2.19) in such a case, we need to introduce a new evolution equation which is equivalent to (2.19), and have some preliminary analysis: definitions of Banach spaces, some basic inequalities.

3. Reduction of (Eq). In this section, the equation (Eq) will be reduced to a new evolution equation.

Let P be a projection from $\Omega^1(\mathfrak{g}_E)$ to $\Omega^1_*(\mathfrak{g}_E)$:

$$P\colon \Omega^1(\mathfrak{g}_E)\to\Omega^1_*(\mathfrak{g}_E)\,. \tag{3.1}$$

The operator G^{∇_0} denotes the Green operator of the Hodge Laplacian $\Delta^{\nabla_0}=\delta^{\nabla_0}d^{\nabla_0}$ acting on $\Omega^0(\mathfrak{g}_E)$. See [KM, Section 3.3].

Our main purpose of this section is to prove the following theorem.

THEOREM 3.1 ([KM, (3.3)]). *Let be ∇_0 a Yang-Mills connection. A smooth curve $\{S(t), A(t)\}$ in $\Omega^0(\mathfrak{g}_E)\times\Omega^1_*(\mathfrak{g}_E)$ is a solution of (2.19) with $A(0)=A_1$ if and only if $\{S(t), A(t)\}$ is the solution of*

$$\begin{cases} \dfrac{\partial A(t)}{\partial t}+PJ^{\nabla_0}A(t)+P(Q(A(t))-[A(t), S(t)])=0\,, \\ S(t)=G^{\nabla_0}\delta^{\nabla_0}(Q(A(t))-[A(t), S(t)])+G^{\nabla_0}\mathscr{R}A(t)\,, \\ A(0)=A^1\,, \end{cases} \tag{3.2}$$

where $\mathscr{R}A:=\delta^{\nabla_0}[R^{\nabla_0}, A]+(\delta^{\nabla_0})^2d^{\nabla_0}A$.

REMARK. In (3.2), we define a operator $\mathscr{R}$, which seems to contains a third order differential operator $(\delta^{\nabla_0})^2d^{\nabla_0}$. However, by using the Ricci formula, this term can be reduce to a first order differential operator. Then, for $A\in H^m(\Omega^1(\mathfrak{g}_E))$, we have $(\delta^{\nabla_0})^2d^{\nabla_0}A\in H^{m-1}(\Omega^0(\mathfrak{g}_E))\subset C^1(\Omega^0(\mathfrak{g}_E))$.

PROOF. The evolution equation (2.19) consists of

$$\frac{\partial A}{\partial t}=-(\Delta^{\nabla_0}A+[R^{\nabla_0}, A]+Q(A)-[A, S]-d^{\nabla_0}S) \tag{3.3}$$

and

$$\delta^{\nabla_0} A = 0 . \tag{3.4}$$

Applying the projection P to both sides of (3.3), we get

$$\frac{\partial A}{\partial t} = -P(\Delta^{\nabla_0} A + [R^{\nabla_0}, A]) - P(Q(A) - [A, S]) , \tag{3.5}$$

since $Pd^{\nabla_0} S = 0$. Note that $PA = A$ since $\delta^{\nabla_0} A = 0$. Furthermore applying δ^{∇_0} to both sides of (3.3), we have

$$(\delta^{\nabla_0})^2 d^{\nabla_0} A + \delta^{\nabla_0}([R^{\nabla_0}, A] + Q(A) - [A, S]) - \delta^{\nabla_0} d^{\nabla_0} S = 0 . \tag{3.6}$$

Since $\delta^{\nabla_0} d^{\nabla_0} S = \Delta^{\nabla_0} S$, we can rewrite (3.6) by in terms of the Green operator G^{∇_0} as

$$S = G^{\nabla_0} \delta^{\nabla_0}([R^{\nabla_0}, A] + Q(A) - [A, S]) + G^{\nabla_0}(\delta^{\nabla_0})^2 d^{\nabla_0} A . \tag{3.7}$$

Therefore the assertion of this theorem follows from (3.5) and (3.7). ∎

Recall that the Sobolev spaces $H^m(\Omega^k(\mathfrak{g}_E))$, $(k = 0, 1, \cdots, \dim M)$ are defined as the completion of $\Omega^k(\mathfrak{g}_E)$ with respect to the norm

$$\|\omega\|_m^2 := \sum_{l=0}^{m} \sum_{|I|=l} \int_M (D_{i_1} \cdots D_{i_l} \omega_{j_1 \cdots j_k}{}^b_a(x) D_{i_1} \cdots D_{i_l} \omega^{j_1 \cdots j_k}{}^a_b(x)) dx ,$$

where $I = (i_1, \cdots, i_k)$ is a multi-index and $D_{i_1} \cdots D_{i_k} := \partial^{|I|}/\partial x_{i_1} \cdots \partial x_{i_k}$.

For the sake of simplicity, we abbreviate $-P(Q(A) - [A, S])$ as $N(A, S)$. The following proposition shows that the first equation of (3.2) is a parabolic equation.

PROPOSITION 3.1. *For $L^{\nabla_0} := PJ^{\nabla_0}|_{\Omega^1_*(\mathfrak{g}_E)}$, we have*

$$\|u\|_{m+2} \le C(\|L^{\nabla_0} u\|_m + \|u\|_m) \qquad \textit{for all} \quad u \in \Omega^1_*(\mathfrak{g}_E) \tag{3.8}$$

with C independent of u.

PROOF. We can express the projection P as follows:

$$Pu = u - d^{\nabla_0} G^{\nabla_0} \delta^{\nabla_0} u , \tag{3.9}$$

where G^{∇_0} is the Green operator. From (3.9), the operator $L^{\nabla_0} = PJ^{\nabla_0}$ with its domain in $\Omega^1_*(g_E)$ is denoted by

$$L^{\nabla_0} u = J^{\nabla_0} u - d^{\nabla_0} G^{\nabla_0} \delta^{\nabla_0} J^{\nabla_0} u = J^{\nabla_0} u - d^{\nabla_0} G^{\nabla_0} (\delta^{\nabla_0})^2 u \qquad \text{for} \quad u \in \Omega^1_*(\mathfrak{g}_E) . \tag{3.10}$$

Here note that $d^{\nabla_0} G^{\nabla_0} (\delta^{\nabla_0})^2$ is a bounded operator from $H^m(\mathfrak{g}_E)$ to $H^m(\mathfrak{g}_E)$ by the Ricci formula. Since J^{∇_0} is an elliptic operator, we get the desired results. ∎

As we mentioned in Section 2, $-L^{\nabla_0}$ has discrete spectra: $\{\lambda_1 \ge \lambda_2 \ge \cdots \searrow -\infty\}$. Now we re-number these spectra, and denote by $\{\lambda_N \ge \lambda_{N-1} \ge \cdots \ge \lambda_1\}$ the positive spectra and $\{\lambda_{-1} \ge \lambda_{-2} \ge \cdots \searrow -\infty\}$ the negative spectra. Moreover, π_+, π_0 and π_-

denote the projection operators onto the eigenspaces of $-L^{\nabla_0}$ with positive, zero and negative eigenvalues, respectively. The space $L^2(\Omega^1_*(\mathfrak{g}_E))$ is naturally defined by considering L^2-norm on $\Omega^1_*(\mathfrak{g}_E)$. We remark that the operator L^{∇_0} is self-adjoint on $L^2(\Omega^1_*(\mathfrak{g}_E))$.

We now introduce the Sobolev space $H^m(\Omega^1_*(\mathfrak{g}_E))$ of $\Omega^1_*(\mathfrak{g}_E)$ by defining the norm $\|\cdot\|_{H^m(\Omega^1_*(\mathfrak{g}_E))}$ as

$$\|A\|^2_{H^m(\Omega^1_*(\mathfrak{g}_E))}:=\|(L^{\nabla_0})^{m/2}\pi_-A\|^2_{L^2}+\|\pi_0 A\|^2_{L^2}+\|\pi_+A\|^2_{L^2}\,.$$

As we mention before, the norm $\|\cdot\|_{H^m(\Omega^1_*(\mathfrak{g}_E))}$ is equivalent to the norm $\|\cdot\|_m$ on $\Omega^1_*(\mathfrak{g}_E)$ for $m>\dim M/2+2$. Since L^{∇_0} can be considered as a positive definite self-adjoint operator on $\pi_-(L^2(\Omega^1_*(\mathfrak{g}_E)))$, the first term on the right hand side is well-defined.

The Banach spaces $L^2(\boldsymbol{R}_+; H^m(\Omega^1_*(\mathfrak{g}_E)))$ and $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$ with norms $|||\cdot|||_{m,1}$ and $|||\cdot|||_{m,0}$, respectively, are defined by

$$|||A|||^2_{m,1}:=\int_0^\infty \|A(t)\|^2_{H^m(\Omega^1_*(\mathfrak{g}_E))}dt$$

and

$$|||S|||^2_{m,0}:=\int_0^\infty \|S(t)\|^2_m dt\,.$$

Furthermore for $\mu>0$, the Banach space $\mathscr{B}_{m,\mu}\subset L^2(\boldsymbol{R}_+; H^{m+1}(\Omega^1_*(\mathfrak{g}_E)))\cap L^\infty(\boldsymbol{R}_+; H^m(\Omega^1_*(\mathfrak{g}_E)))$ with the norm $|\cdot|_{\mu,m}$ is defined by

$$|A|^2_{\mu,m}:=|||A|||^2_{m+1,1}+\sup_{t>0}\,[e^{2\mu t}\|A(t)\|^2_{H^m(\Omega^1_*(\mathfrak{g}_E))}]\,.$$

For the sake of simplicity, we abbreviate $|||S|||_{m,0}$, $|||A|||_{m,1}$ and $\|A\|_{H^m(\Omega^1_*(\mathfrak{g}_E))}$ as $|S|_m$, $|||A|||_m$ and $\|A\|_{H^m}$ for $S\in\Omega^0(\mathfrak{g}_E)$ and $A\in\Omega^1_*(\mathfrak{g}_E)$, respectively.

For the proof of the existence of a solution of (3.2), we need some inequalities.

PROPOSITION 3.2. *Let $m>\dim M/2+2$. For A_1, $A_2\in\Omega^1_*(\mathfrak{g}_E)$ and S_1, $S_2\in\Omega^0(\mathfrak{g}_E)$ satisfying $|A_i|_{\mu,m}<1$, and $|S_i|_m<1$, we have*

$$\begin{aligned}\|N(A_1,S_1)-N(A_2,S_2)\|_{H^{m-1}}\le C(&\|A_1-A_2\|_{H^{m+1}}\|A_1\|_{H^m}\\ &+\|A_1-A_2\|_{H^m}\|A_2\|_{H^{m+1}}+\|A_1-A_2\|_{H^m}\|S_1\|_m+\|A_2\|_{H^m}\|S_1-S_2\|_m)\,,\end{aligned}\tag{3.11}$$

with $C>0$ independent of A_i and S_i $(i=1,2)$.

PROOF. In local coordinate neighborhoods, brackets $[A,A]$ and $[A,S]$ are expressed as

$$[A,A]_{ij}{}^a_b=\frac{1}{2}(A_i{}^a_e A_j{}^e_b-A_j{}^a_e A_i{}^e_b)$$

and

$$[A, S]_{ib}^{a} = A_{ie}^{a} S_{b}^{e} - S_{e}^{a} A_{ib}^{e} .$$

The $\mathfrak{g}$-valued 1-form $Q(A) = \delta^{\nabla_0}[A, A]/2 + [d^{\nabla_0}A, A] + [[A, A], A]/2$ is estimated as

$$\|Q(A_1) - Q(A_2)\|_{H^{m-1}} \le C(\|A_1 - A_2\|_{H^{m+1}} \|A_1\|_{H^m} + \|A_1 - A_2\|_{H^m} \|A_2\|_{H^{m+1}}) . \tag{3.12}$$

Note that $\|A_i\|_{H^{m+1}} < 1$ $(i = 1, 2)$. Similarly, we have

$$\|[A_1, S_1] - [A_2, S_2]\|_{H^{m-1}} \le C(\|A_1 - A_2\|_{H^{m-1}} \|S_1\|_{m-1} + \|A_2\|_{H^{m-1}} \|S_1 - S_2\|_{m-1}) . \tag{3.13}$$

Therefore (3.12) and (3.13) guarantee (3.11). ■

Here note that the assumptions $|A_i|_{\mu,m} < 1$ and $|S_i|_m < 1$ in Proposition 3.2 are used in the proof of (3.12).

Since $(\delta^{\nabla_0})^2 d^{\nabla_0}$ is a first order differential operator by the remark immediately after Theorem 3.1, we see the $\mathscr{R}$ in (3.2) is of first order. Therefore using a property of the Green operator, we obtain

$$\|G^{\nabla_0} \mathscr{R} A\|_{H^m} \le C \|A\|_{H^{m-1}} \tag{3.14}$$

and

$$\|G^{\nabla_0} \delta^{\nabla_0} A\|_{H^m} \le C \|A\|_{H^{m-1}} , \tag{3.15}$$

for all $A \in H^{m-1}(\Omega^1(\mathfrak{g}_E))$ with $C > 0$ independent of A. These estimates (3.11)–(3.15) play very important roles in the proof of the main result.

The following lemma is a basic inequality for linear partial differential equations.

LEMMA 3.1. *For* $u \in L^2(\boldsymbol{R}_+; H^{m+1}(\Omega^1_*(\mathfrak{g}_E)))$ *and* $v \in L^2(\boldsymbol{R}_+; H^{m-1}(\Omega^1_*(\mathfrak{g}_E)))$, *we assume that*

$$\begin{cases} \dfrac{\partial u}{\partial t} = -L^{\nabla_0} u + \pi_- v , \\ u(0) \in \mathrm{Im}(\pi_-) . \end{cases}$$

Then we obtain

$$\int_0^\infty \|u(t)\|^2_{H^{m+1}} dt \le \|u(0)\|^2_{H^m} + \int_0^\infty \|v(t)\|^2_{H^{m-1}} dt \tag{3.16}$$

and

$$e^{2\mu t} \|u(t)\|^2_{H^{m+1}} \le \|u(0)\|^2_{H^m} + C \int_0^\infty \|v(t)\|^2_{H^{m-1}} dt , \qquad \text{for all} \quad t > 0 , \tag{3.17}$$

where $0 < \mu < \min\{|\lambda_1|, |\lambda_{-1}|\}$.

For the proof see [N1, Lemmas 2.1 and 2.3].

In the next section, we will prove the main result.

4. Proof of the main result. In this section, we show our main result by solving (3.2), which is reduced to solving the following initial value problem:

$$
(4.1) \qquad \begin{cases} \dfrac{\partial A(t)}{\partial t} = -PJ^{\nabla_0}A(t) + PN(A(t), S(t)), \\ S(t) = G^{\nabla_0}\delta^{\nabla_0}N(A(t), S(t)) + G^{\nabla_0}\mathscr{R}A(t), \\ A(0) = A^1. \end{cases}
$$

For this purpose, we adopt the following iteration scheme:

$$
(4.2) \qquad \begin{cases} \dfrac{\partial A_n}{\partial t} = -L^{\nabla_0}A_n + PN(A_{n-1}, S_{n-1}) \qquad (n \geq 1), \\ S_n = G^{\nabla_0}\delta^{\nabla_0}(N(A_{n-1}, S_{n-1})) + G^{\nabla_0}\mathscr{R}A_n \qquad (n \geq 1), \\ A_0(t) = e^{-tL^{\nabla_0}}\pi_- A^1, \\ S_0 = 0, \end{cases}
$$

where A^1 is in $H^m(\Omega^1_*(\mathfrak{g}_E))$ for $m > \dim M/2 + 2$. For the sake of simplicity, we abbreviate $N(A(t), S(t))$ as $N(A, S)(t)$. This iteration scheme is given as the following system of equations:

$$
(4.3) \qquad \begin{cases} A_n(t) = e^{-tL^{\nabla_0}}\pi_- A^1 + \displaystyle\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1}, S_{n-1})(s)ds \\ \qquad - \displaystyle\int_t^\infty \pi_0 PN(A_{n-1}, S_{n-1})(s)ds - \displaystyle\int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ PN(A_{n-1}, S_{n-1})(s)ds \\ S_n(t) = G^{\nabla_0}\delta^{\nabla_0}N(A_{n-1}, S_{n-1})(t) + G^{\nabla_0}\mathscr{R}A_n(t), \qquad \text{for} \quad t > 0 \\ S_0 = 0. \end{cases}
$$

If this iteration scheme converges in H^m-topology, the initial value of a solution is expressed as

$$
(4.4) \qquad A(0) = \pi_- A^1 - \int_0^\infty \pi_0 PN(A, S)(s)ds - \int_0^\infty e^{sL^{\nabla_0}}\pi_+ PN(A, S)(s)ds.
$$

Therefore the initial value A^1 of (4.1) is expressed as (4.4). This implies that for a Yang-Mills connection ∇_0, if A^1 is expressed as (4.4) and satisfies a suitable condition (this condition will be mentioned in the proof of the result), then there exists a solution of (4.1) with the initial value A^1 which tends to zero as $t \to \infty$ in H^m-topology.

We assume $m > \dim M/2 + 2$. Then the Sobolev spaces $H^m(\Omega^1_*(\mathfrak{g}_E))$ and $H^m(\Omega^0(\mathfrak{g}_E))$

are compactly embedded in $C^2(\Omega^1_*(\mathfrak{g}_E))$ and $C^2(\Omega^0(\mathfrak{g}_E))$, respectively. Moreover, we choose a positive number μ satisfying $0<\mu<\min\{|\lambda_1|, |\lambda_{-1}|\}$. We prove that the iteration scheme (4.3) converges in $\mathscr{B}_{\mu,m}$ and $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$. Put $M_n := |A_n|^2_{\mu,m}$ and $K_n := |S_n|^2_m$.

THEOREM 4.1. *For the iteration scheme* (4.3), *there exist positive constants* C_1, C_2 *and* C_3 *depending only* μ *and* m *such that if* $M_n<1$ *and* $K_n<1$ *then*

$$\begin{cases} M_n \le \|\pi_- A^1\|^2_{H^m} + C_1(M^2_{n-1} + M_{n-1}K_{n-1}), \\ K_n \le C_2(\|\pi_- A^1\|^2_{H^m} + M^2_{n-1} + M_{n-1}K_{n-1}), \end{cases} \tag{4.5}$$

and

$$|A_n - e^{-tL^{\nabla_0}}\pi_- A^1|^2_{\mu,m} \le C_3(M^2_{n-1} + M_{n-1}K_{n-1}). \tag{4.6}$$

PROOF. We will construct $\mathscr{B}_{\mu,m}$ and $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$-estimates of (4.5)–(4.6) by separating (4.3) freely.

Step 1. The $H^m(\Omega^1_*(\mathfrak{g}_E))$-estimate.

(i) To estimate the π_--part of the first equation, we apply (3.17) in Lemma 3.1 to

$$f_-(t) := e^{-tL^{\nabla_0}}\pi_- A^1 + \int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- PN(A_{n-1}, S_{n-1})(s)ds.$$

This function $f_-(t) = \pi_- A_n(t)$ satisfies

$$\begin{cases} \dfrac{\partial f_-}{\partial t} = -L^{\nabla_0} f_- + \pi_- PN(A_{n-1}, S_{n-1}) \\ f_-(0) = \pi_- A^1. \end{cases}$$

Remark that by (3.9) and (3.15), we have $\|Pu\|_{H^m} \le C\|u\|_{H^m}$. Lemma 3.1 and Proposition 3.2 imply that

$$\begin{aligned} e^{2\mu t}\|\pi_- A_n(t)\|^2_{H^m} &\le \|\pi_- A_1\|^2_{H^m} + C\int_0^\infty \|\pi_- PN(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}} ds \\ &\le \|\pi_- A^1\|^2_{H^m} + C\int_0^\infty (\|A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m} + \|A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_m) ds \\ &\le \|\pi_- A^1\|^2_{H^m} + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}} ds \\ &\quad + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_m ds. \end{aligned}$$

By the definitions of norms $|\cdot|_{\mu,m}$ and $|\cdot|_m$, we obtain

$$e^{2\mu t}\|\pi_- A_n(t)\|^2_{H^m} \le \|\pi_- A^1\|^2_{H^m} + C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m). \tag{4.7}$$

(ii) We will estimate the $\pi_+ A_n$ in the H^m-norm. Put

$$f_+(t) := -\int_t^\infty e^{-(t-s)L^\nabla_0}\pi_+ PN(A_{n-1}, S_{n-1})(s)ds .$$

Since $f_+(t)=\pi_+ A_n(t)$ and $\dim(\mathrm{Im}(\pi_+))<\infty$, we obtain

$$e^{2\mu t}\|\pi_+ A_n(t)\|^2_{H^m} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m) . \tag{4.8}$$

(iii) Since $\dim(\mathrm{Im}(\pi_0))<\infty$, we obtain

$$e^{2\mu t}\|\pi_0 A_n(t)\|^2_{H^m} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m) . \tag{4.9}$$

Combining (4.7)–(4.9), we conclude that

$$e^{2\mu t}\|A_n(t)\|^2_{H^m} \le \|\pi_- A^1\|^2_{H^m} + C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m) , \tag{4.10}$$

and

$$e^{2\mu t}\|A_n(t) - e^{-tL^\nabla_0}\pi_- A^1\|^2_{H^m} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m) . \tag{4.11}$$

Step 2. The $L^2(\boldsymbol{R}_+; H^{m+1}(\Omega^1_*(\mathfrak{g}_E)))$-estimate.

As is remarked in Step 1, since the dimension of $\mathrm{Im}(\pi_+ + \pi_0)$ is finite, there exists a constant C such that

$$\|\pi_+ A_n(t)\|^2_{H^{m+1}} + \|\pi_0 A_n(t)\|^2_{H^{m+1}} \le C(\|\pi_+ A_n(t)\|^2_{H^m} + \|\pi_0 A_n(t)\|^2_{H^m}) . \tag{4.12}$$

Integrating (4.12) in t, and then applying (4.8)–(4.9), we get

$$|||\pi_+ A_n|||^2_{m+1} + |||\pi_0 A_n|||^2_{m+1} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m) . \tag{4.13}$$

For $\pi_- A_n(t)$, we apply (3.16) in Lemma 3.1 and Proposition 3.2

$$f_-(t) := e^{-tL^\nabla_0}\pi_- A^1 + \int_0^t e^{-(t-s)L^\nabla_0}\pi_- PN(A_{n-1}, S_{n-1})(s)ds ,$$

and get (note that $\pi_- A_n(t) = f_-(t)$)

$$\begin{aligned}
|||\pi_- A_n|||^2_{m+1} &\le \|\pi_- A^1\|^2_{H^m} + \int_0^\infty \|N(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}ds \\
&\le \|\pi_- A^1\|^2_{H^m} + \int_0^\infty (\|A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m} + \|A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_m)ds \\
&\le \|\pi_- A^1\|^2_{H^m} + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}ds \\
&\quad + C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_m ds .
\end{aligned}$$

Hence it follows that

$$(4.14)\qquad |||\pi_- A_n|||^2_{m+1} \le \|\pi_- A^1\|^2_{H^m} + C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)\,.$$

Combining (4.11) and (4.12), we obtain

$$(4.15)\qquad |||A_n|||^2_{m+1} \le \|\pi_- A^1\|^2_{H^m} + C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)$$

and

$$(4.16)\qquad |||A_n - e^{-tL^{\nabla_0}}\pi_- A^1|||^2_{m+1} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)\,.$$

Therefore (4.12)–(4.15) and (4.13)–(4.16) yield

$$(4.17)\qquad |A_n|^2_{\mu,m} \le \|\pi_- A^1\|^2_{H^m} + C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)$$

and

$$(4.18)\qquad |A_n - e^{-tL^{\nabla_0}}\pi_- A^1|^2_{\mu,m} \le C(|A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)\,,$$

respectively, which gives (4.6).

Step 3. The $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$-estimate.

We estimate both sides of the second equation of (4.3) in $H^m(\Omega^0(\mathfrak{g}_E))$-norm, we have

$$\|S_n(t)\|_m \le \|G^{\nabla_0}\delta^{\nabla_0}(N(A_{n-1}, S_{n-1})(t))\|_{H^m} + \|G^{\nabla_0}\mathscr{R}A_n(t)\|_{H^m}\,.$$

Using (3.12)–(3.15), we get

$$(4.19)\qquad \|S_n(t)\|^2_m \le C(\|A_n(t)\|^2_{H^m} + \|A_{n-1}(t)\|^4_{H^m} + \|A_{n-1}(t)\|^2_{H^m}\|S_{n-1}(t)\|^2_m)\,.$$

Integrating both sides of (4.19) in t, we see

$$\int_0^\infty \|S_n(t)\|^2_m dt \le C\left[\int_0^\infty \|A_n(t)\|^2_{H^m}dt + \left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^m}ds\right.$$
$$\left. + \left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|_m ds\right].$$

This yields

$$(4.20)\qquad |S_n|^2_m \le C(|A_n|^2_{\mu,m} + |A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)\,.$$

By (4.17) and (4.20), we obtain

$$(4.21)\qquad |S_n|^2_m \le C(\|\pi_- A^1\|^2_{H^m} + |A_{n-1}|^4_{\mu,m} + |A_{n-1}|^2_{\mu,m}|S_{n-1}|^2_m)\,.$$

Then (4.5) follows from (4.17) and (4.21). Here one should note that the assumptions $M_n < 1$ and $K_n < 1$ are needed for the estimate of non-linear terms. ■

Theorem 4.1 leads to apriori estimates.

COROLLARY 4.1. *Let $L_n := \max\{M_n, K_n\}$, $(n=0, 1, \cdots)$. There exist $\varepsilon_0 > 0$ and a monotone decreasing function $L(\varepsilon)$ of $0 < \varepsilon \le \varepsilon_0$ such that under the condition $L_0 < \varepsilon_0$ we have*

$$L_n \le L(\varepsilon_0) \quad \textit{for all} \quad n \ge 0 . \tag{4.22}$$

Moreover $L(\varepsilon)$ *satisfies*

$$\lim_{\varepsilon \downarrow 0} L(\varepsilon) = 0 . \tag{4.23}$$

PROOF. Theorem 4.1 yields

$$M_{n+1} \le \|\pi_- A^1\|_{H^m}^2 + CL_n^2 ,$$

$$K_{n+1} \le C(\|\pi_- A^1\|_{H^m}^2 + L_n^2) .$$

Remarking $\|\pi_- A^1\|_{H^m} \le |A_0|_{\mu,m}$, we have

$$L_{n+1} \le C(L_0 + L_n^2) . \tag{4.24}$$

Elementary calculus yields the assertion of this corollary. ∎

COROLLARY 4.2. *There is* $\varepsilon > 0$ *such that if* $\|\pi_- A^1\|_{H^m}^2 \le \varepsilon$, $|A_n|_{\mu,m}^2 < \varepsilon$ *and* $|S_n|_m^2 < \varepsilon$ *then* A_{n+1} *is contained in the* ε*-ball in* $\mathscr{B}_{\mu,m}$ *whose center is* $e^{-tL^{\nabla_0}}\pi_- A^1$.

The final step of the proof of the main result is to show the convergence of the iteration scheme (4.3). For this purpose, we may prove that sequence $\{A_n\}$ and $\{S_n\}$ are Cauchy sequences in $\mathscr{B}_{\mu,m}$ and $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$, respectively.

THEOREM 4.2. *For the iteration scheme* (4.3), *the following two inequalities hold*:

$$\begin{aligned} &|A_{n+1} - A_n|_{\mu,m}^2 \\ &\le C(|A_n|_{\mu,m}^2 + |A_{n-1}|_{\mu,m}^2 + |S_n|_m^2 + |S_{n-1}|_m^2)(|A_n - A_{n-1}|_{\mu,m}^2 + |S_n - S_{n-1}|_m^2) ; \end{aligned} \tag{4.25}$$

$$\begin{aligned} &|S_{n+1} - S_n|_m^2 \\ &\le C(|A_n|_{\mu,m}^2 + |A_{n-1}|_{\mu,m}^2 + |S_n|_m^2 + |S_{n-1}|_m^2)(|A_n - A_{n-1}|_{\mu,m}^2 + |S_n - S_{n-1}|_m^2) . \end{aligned} \tag{4.26}$$

PROOF. The proof of this theorem is essentially the same as that of Theorem 4.1. First we calculate successive differences:

$$\begin{aligned} A_{n+1}(t) - A_n(t) = &\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_- P(N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s))ds \\ &- \int_t^\infty \pi_0 P(N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s))ds \\ &- \int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+ P(N(A_n, S_n)(s) - N(A_{n-1}, S_{n-1})(s))ds \end{aligned} \tag{4.27}$$

and

$$S_{n+1}(t) - S_n(t) = G^{\nabla_0}\delta^{\nabla_0}(N(A_n, S_n)(t) - N(A_{n-1}, S_{n-1})(t)) + G^{\nabla_0}\mathscr{R}(A_{n+1}(t) - A_n(t)) . \tag{4.28}$$

As in the proof of Theorem 4.1, we will take three steps to show our assertion.

Step 1. The $H^m(\Omega^1_*(\mathfrak{g}_E))$-estimate.

(i) Taking

$$f_-(t):=\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_-P(N(A_n, S_n)(s)-N(A_{n-1}, S_{n-1})(s))ds\,, \tag{4.29}$$

we have $f_-(t)=\pi_-(A_n(t)-A_{n-1}(t))$ and

$$\begin{cases} \dfrac{\partial f_-}{\partial t}=-L^{\nabla_0}f_-+\pi_-P(N(A_n, S_n)-N(A_{n-1}, S_{n-1}))\,, \\ f_-(0)=0\,. \end{cases}$$

It follows from Lemma 3.1 and Proposition 3.2 that

$$\begin{aligned} e^{2\mu t}\|f_-(t)\|^2_{H^m}&\le C\int_0^t \|N(A_n, S_n)(s)-N(A_{n-1}, S_{n-1})(s)\|^2_{H^{m-1}}ds \\ &\le C\int_0^t (\|A_n(s)-A_{n-1}(s)\|^2_{H^{m+1}}\|A_{n-1}(s)\|^2_{H^m}+\|A_n(s)\|^2_{H^{m+1}}\|A_n(s)-A_{n-1}(s)\|^2_{H^m} \\ &+\|A_n(s)-A_{n-1}(s)\|^2_{H^m}\|S_{n-1}(s)\|^2_m+\|A_{n-1}(s)\|^2_{H^m}\|S_n(s)-S_{n-1}(s)\|^2_m)ds \\ &\le C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_n(s)-A_{n-1}(s)\|^2_{H^{m+1}}ds \\ &+C\left[\sup_{t>0} e^{2\mu t}\|A_n(t)-A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|A_{n-1}(s)\|^2_{H^{m+1}}ds \\ &+C\left[\sup_{t>0} e^{2\mu t}\|A_n(t)-A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_{n-1}(s)\|^2_m ds \\ &+C\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|^2_{H^m}\right]\int_0^\infty e^{-2\mu s}\|S_n(s)-S_{n-1}(s)\|^2_m ds\,. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} e^{2\mu t}\|\pi_-(A_{n+1}-A_n)\|^2_{H^m}&\le C[(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m})|A_n-A_{n-1}|^2_{\mu,m} \\ &+|A_n-A_{n-1}|^2_{\mu,m}|S_n|^2_m+|A_{n-1}|^2_{\mu,m}|S_n-S_{n-1}|^2_m] \\ &\le C[(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m) \\ &\times(|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m)]\,. \end{aligned}$$

(ii) As in the proof of Lemma 3.1 and the above (i), we can estimate the π_0- and π_+-parts of $A_{n+1}(t)-A_n(t)$, as follows:

$$\begin{aligned} e^{2\mu t}\|\pi_+(A_{n+1}(t)-A_n(t)\|^2_{H^m}&\le C[(|A_n|^2_{\mu,m}+|A_{n-1}|^2_{\mu,m}+|S_n|^2_m+|S_{n-1}|^2_m) \\ &\times(|A_n-A_{n-1}|^2_{\mu,m}+|S_n-S_{n-1}|^2_m)]\,, \end{aligned} \tag{4.31}$$

and

$$e^{2\mu t}\|\pi_0(A_{n+1}(t)-A_n(t))\|_{H^m}^2 \le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \times (|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)]\,. \tag{4.32}$$

Consequently, as for the $H^m(\Omega_*^1(g_E))$-norm of the successive difference, we see that

$$e^{2\mu t}\|A_{n+1}(t)-A_n(t)\|_{H^m}^2 \le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \times (|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)]\,. \tag{4.33}$$

Step 2. The $L^2(\boldsymbol{R}_+;\ H^{m+1}(\Omega_*^1(\mathfrak{g}_E)))$-estimate.

As we remarked in the proof of Theorem 4.1, $\operatorname{Im}(\pi_+ + \pi_0)$ is of finite dimension. Calculating similarly to (4.31) and (4.32), we have

$$\begin{aligned} &|||\pi_+(A_{n+1}-A_n)|||_{m+1}^2 + |||\pi_0(A_{n+1}-A_n)|||_{m+1}^2 \\ &\le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \\ &\quad\times (|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)]\,. \end{aligned} \tag{4.34}$$

For the estimate of the π_--part of the successive difference, we apply (3.16) of Lemma 3.1 to (4.29). We thus obtain

$$\begin{aligned} |||f(t)|||_{m+1}^2 &\le \int_0^t \|N(A_n, S_n)(s)-N(A_{n-1}, S_{n-1})(s)\|_{H^{m-1}}^2 ds \\ &\le C\Bigg[\left[\sup_{t>0} e^{2\mu t}\|A_n(t)\|_{H^m}^2\right]\int_0^t e^{-2\mu s}\|A_n(s)-A_{n-1}(s)\|_{H^{m+1}}^2 ds \\ &\quad+\left[\sup_{t>0} e^{2\mu t}\|A_n(t)-A_{n-1}(t)\|_{H^m}^2\right]\int_0^t e^{-2\mu s}\|A_{n-1}(s)\|_{H^{m+1}}^2 ds \\ &\quad+\left[\sup_{t>0} e^{2\mu t}\|A_n(t)-A_{n-1}(t)\|_{H^m}^2\right]\int_0^t e^{-2\mu s}\|S_n(s)\|_m^2 ds \\ &\quad+\left[\sup_{t>0} e^{2\mu t}\|A_{n-1}(t)\|_{H^m}^2\right]\int_0^t e^{-2\mu s}\|S_n(s)-S_{n-1}(s)\|_m^2 ds\Bigg]\,. \end{aligned}$$

Hence we have

$$|||\pi_-(A_{n+1}-A_n)|||_{m+1}^2 \le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \times (|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)]\,. \tag{4.35}$$

Therefore we get by (4.34)–(4.35)

$$|||A_{n+1}-A_n|||_{m+1}^2 \le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \times (|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)]\,. \tag{4.36}$$

Consequently, it follows from (4.33) and (4.36) that

$$
\begin{aligned}
(4.37)\qquad |A_{n+1}-A_n|_{\mu,m}^2 \le C[(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \\
\times(|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2)] .
\end{aligned}
$$

Step 3. The $L^2(\boldsymbol{R}_+; H^m(\Omega^0(\mathfrak{g}_E)))$-estimate.
We estimate (4.28) by $H^m(\Omega^0(\mathfrak{g}_E))$-norm. We obtain

$$
\begin{aligned}
(4.38)\qquad \|S_n(t)-S_{n-1}(t)\|_m^2 \le \|G^{\nabla_0}\delta^{\nabla_0}(N(A_n, S_n)(t)-N(A_{n-1}, S_{n-1}(t)))\|_{H^m}^2 \\
+\|G^{\nabla_0}\mathscr{R}(A_{n+1}(t)-A_n(t))\|_{H^m}^2 .
\end{aligned}
$$

By using estimates (3.12), (3.13) and (3.15), the first term on the right hand side of (4.38) is estimated as

$$
\begin{aligned}
&\|G^{\nabla_0}\delta^{\nabla_0}(N(A_n, S_n)(t)-N(A_{n-1}, S_{n-1})(t))\|_{H^m}^2 \\
(4.39)\qquad &\le C(\|A_n(t)-A_{n-1}(t)\|_{H^{m+1}}^2\|A_n(t)\|_{H^m}^2+\|A_{n-1}(t)\|_{H^{m+1}}^2\|A_n(t)-A_{n-1}(t)\|_{H^m}^2 \\
&+\|A_n(t)-A_{n-1}(t)\|_{H^m}^2\|S_n(t)\|_m^2+\|A_{n-1}(t)\|_{H^m}^2\|S_n(t)-S_{n-1}(t)\|_m^2) .
\end{aligned}
$$

By (3.14) we have

$$
(4.40)\qquad \|G^{\nabla_0}\mathscr{R}(A_{n+1}(t)-A_n(t))\|_{H^m}^2 \le C\|A_{n+1}(t)-A_n(t)\|_{H^{m-1}}^2 .
$$

Therefore, we can estimate $|S_{n+1}-S_n|_m^2$ as follows:

$$
\begin{aligned}
(4.41)\qquad |S_{n+1}-S_n|_m^2 \le C(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \\
\times(|A_{n+1}-A_n|_{\mu,m}^2+|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2) .
\end{aligned}
$$

Combining (4.37) and (4.41), we therefore obtain

$$
\begin{aligned}
(4.42)\qquad |S_{n+1}-S_n|_m^2 \le C(|A_n|_{\mu,m}^2+|A_{n-1}|_{\mu,m}^2+|S_n|_m^2+|S_{n-1}|_m^2) \\
\times(|A_n-A_{n-1}|_{\mu,m}^2+|S_n-S_{n-1}|_m^2) .
\end{aligned}
$$

■

REMARK. The solution of (Eq) in Introduction given by the above step is unique solution in $\{(S, A);\ S\in L^2([0, \infty); H^m(\Omega^0(\mathfrak{g}_E))),\ A\in\mathscr{B}_{\mu,m}\}$ (cf. see [KM, the proof of Proposition 4.3, (ii) uniqueness]).

By virtue of Theorem 4.2, we can conclude the existence of a solution to (4.1), therefore (YMGF), with initial value determinated by (4.4). Combining Corollaries 4.1 and 4.2 with Theorem 4.2, we have

COROLLARY 4.3. *For $m>\dim M/2+2$ and μ satisfying $0<\mu<\dim\{|\lambda_1|, |\lambda_{-1}|\}$, there exists an $\varepsilon>0$ such that for every $A^1\in \mathrm{Im}(\pi_-)$ with*

$$|e^{-tL^{\nabla_0}}A^1|_{\mu,m}<\varepsilon$$

there is a unique solution $\{S, A\}$ of

$$\left\{\frac{\partial A(t)}{\partial t}=-PJ^{\nabla_0}A(t)+PN(A(t), S(t))\right.$$

$$\begin{cases} S(t)=G^{\nabla_0}\delta^{\nabla_0}(N(A(t),S(t)))+G^{\nabla_0}\mathscr{R}A(t) \\ \pi_-A(0)=A^1 \end{cases}$$

with $A\in\mathscr{B}_{\mu,m}$ and $S\in L^2(\boldsymbol{R}_+;H^m(\Omega^0(\mathfrak{g}_E)))$ satisfying $|A|_{\mu,m}<\varepsilon$. Moreover such a solution A exponentially tends to zero in H^m-norm at $t\to\infty$.

Therefore the solution of (4.1) is expressed as

$$\begin{aligned} A(t)=e^{-tL^{\nabla_0}}\pi_-A^1+\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_-N(A(s),S(s))ds \\ -\int_t^\infty \pi_0 N(A(s),S(s))ds-\int_t^\infty e^{-(t-s)L^{\nabla_0}}\pi_+N(A(s),S(s))ds\,, \end{aligned} \tag{4.43}$$

and

$$S(t)=G^{\nabla_0}\delta^{\nabla_0}(Q(A(t))+[A(t),S(t)])+G^{\nabla_0}\mathscr{R}A(t)\,. \tag{4.44}$$

By using standard arguments for semigroup theory, we obtain that $N(A(t),S(t))$ is locally Hölder continuous in t (cf. [GM, Theorem 2.5]). Then we get $A(t)$ in (4.43) satisfies (4.1). Furthermore, by the form of (4.43) and (4.44), $A(t)\in H^m(\Omega^1_*(\mathfrak{g}_E))$ and $S(t)\in H^m(\Omega^0(\mathfrak{g}_E))$.

Thanks to this expression (4.43)–(4.44) of a solution, one can show that the solutions $A(t)$ and $S(t)$ depend smoothly on the initial value A^1 in the $H^m(\Omega^1_*(\mathfrak{g}_E))$- and $H^m(\Omega^0(\mathfrak{g}_E))$-topology, respectively. The stable manifold of the Yang-Mills connection ∇_0 is the set of all initial values A^1 satisfying the condition in Corollary 4.3. That is, the stable manifold of ∇_0 is the set of the initial values of the Yang-Mills gradient flow with which the solution is tends to ∇_0 up to gauge equivalence as $t\to\infty$. The stable manifold is clearly a submanifold of $H^m(\Omega^1_*(\mathfrak{g}_E))$ with codimension $\dim(\mathrm{Im}(\pi_++\pi_0))$.

To obtain the unstable manifold, we use the iteration scheme:

$$\begin{cases} A_n(t)=e^{-tL^{\nabla_0}}\pi_+A^1+\int_0^t e^{-(t-s)L^{\nabla_0}}\pi_+PN(A_{n-1}(s),S_{n-1}(s))ds \\ \qquad -\int_t^{-\infty} \pi_0 PN(A_{n-1}(s),S_{n-1}(s))ds-\int_t^{-\infty} e^{-(t-s)L^{\nabla_0}}\pi_-PN(A_{n-1}(s),S_{n-1}(s))ds \\ S_n(t)=G^{\nabla_0}\delta^{\nabla_0}(Q(A_{n-1}(t))+[A_{n-1}(t),S_{n-1}(t)])+G^{\nabla_0}\mathscr{R}A_n(t)\,, \qquad \text{for} \quad t<0\,. \end{cases} \tag{4.45}$$

As in Corollary 4.3, one can show the existence of a set of the initial values for which the solutions are asymptotically stable for the backward evolution equation. This set is clearly a submanifold of $H^m(\Omega^1_*(\mathfrak{g}_E))$ and has dimension $\dim(\mathrm{Im}(\pi_+))$.

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