Tôhoku Math. J.
42 (1990), 1–16

THE EXISTENCE OF LIMIT CYCLES OF NONLINEAR OSCILLATION EQUATIONS*

WANG KE

(Received August 6, 1988, revised August 1, 1989)

1. Introduction. Many mathematicians have done a lot of work on the existence of limit cycles of the Liénard equation

$$x''+f(x)x'+g(x)=0\,, \tag{1}$$

and many good results have been obtained (see [5]–[7]). It is worthwhile to generalize these results to more general nonlinear equation. Huang [3] [4] considered the existence of limit cycles of the equation

$$x''+f(x)\eta(x')x'+\psi(x')g(x)=0\,. \tag{2}$$

In this paper we use a new method to deal with the existence of limit cycles of the equation (2) and obtain some new results. Our results generalize the well-known theorems of Dragilev [1] and Filippov [2].

2. A system equivalent to (2). Our basic idea in this paper is to find a closed orbit of a two-dimensional system equivalent to the given equation. There are several ways of obtaining such an equivalent system. It is well-known that each of the systems

$$\begin{cases} x'=y \\ y'=-f(x)y-g(x) \end{cases} \tag{3}$$

and

$$\begin{cases} x'=y-F(x) \\ y'=-g(x) \end{cases}, \tag{4}$$

where $F(x)=\int_0^x f(s)ds$, is considered to be a system equivalent to the Liénard equation (1). For the general equation (2), it is quite easy to consider an equivalent system

$$\begin{cases} x'=y \\ y'=-f(x)\eta(y)y-\psi(y)g(x) \end{cases} \tag{5}$$

corresponding to (3), but until now, we have never seen any method generalizing the equivalent system (4) for the more complicated equation (2). In this paper, we offer

* Supported by the National Natural Science Foundation of China.

such a method.

We assume that the following conditions hold:

(i) $f(x)$, $g(x)$, $\eta(y)$, $\psi(y)$ are all continuous and locally Lipschitzian.

(ii) $\eta(y)>0$, $\psi(y)>0$ for all y; $xg(x)>0$ for all $x\neq 0$.

Letting $x'=y$, we have $y'=-f(x)\eta(y)y-\psi(y)g(x)$. Let

$$u=u(y)=\int_0^y \frac{ds}{\eta(s)}.$$

Then the inverse function of $u=u(y)$ exists and is denoted by $y=s(u)$. It is easy to see that $s(u)$ is monotone increasing and differentiable. Let $w=F(x)+u(y)$. Then we can reduce the system (5), and hence the equation (2), to

$$\begin{cases} x'=s(w-F(x)) \\ w'=-H(w-F(x))g(x), \end{cases} \tag{6}$$

where $H(u)=\psi(s(u))/\eta(s(u))$, which is positive and continuous.

If $\eta(y)\equiv 1$, $\psi(y)\equiv 1$, then (2) becomes the Liénard equation (1) and the equivalent system (6) becomes (4).

3. Limit cycles of (2).

THEOREM 1. *In addition to the condition* (i), *suppose the following conditions are satisfied*:

(1) $f(x)\geqslant 0$, *or* $f(x)\leqslant 0$; *but* $f(x)\neq 0$ *for* $0<|x|\ll 1$,

(2) $xg(x)>0,\ x\neq 0,\ \psi(y)>0$,

(3) $\eta(y)\geqslant 0$ *or* $\eta(y)\leqslant 0$, *but* $\eta(y)\not\equiv 0$ *for* $0<|y|\ll 1$.

Then the equation (2) *has no closed trajectory.*

PROOF. Consider the equivalent system (5). Letting

$$V(x,y)=G(x)+\int_0^y \frac{u\,du}{\psi(u)},$$

where $G(x)=\int_0^x g(s)ds$, we have

$$V'_{(5)}(x,y)=-\frac{f(x)\eta(y)y^2}{\psi(y)}.$$

It is easy to see that (5) has a unique singular point (0, 0). If (5) has a closed trajectory, it must encircle the point (0, 0). An integration will yield a contradiction, and the theorem is proved.

EXAMPLE 1. The equation

$$x''+(1+ax^2)x'+(1+bx'^2)x=0, \qquad a,b\geqslant 0$$

has no non-trivial periodic solutions.

Set $R(u)=\eta(s(u))s(u)/\psi(s(u))$, and suppose

(iii) $\displaystyle\int_0^{\pm\infty}\frac{ds}{\eta(s)}=\pm\infty$, and there are continuous, increasing, non-negative functions $E_1(r)$, $E_2(r)$ such that $E_1(r)\to+\infty$ as $r\to+\infty$ and that

$$E_2(u_2-u_1)>R(u_2)-R(u_1)>E_1(u_2-u_1)\qquad\text{for}\quad u_2>u_1\,,$$

or

(iii′) $\displaystyle\int_0^{\pm\infty}\frac{ds}{\eta(s)}=\pm\infty$, $R(u)\to\pm\infty$, $u\to\pm\infty$, and there is a number $E_2^*>0$ such that

$$E_2^*(u_2-u_1)>R(u_2)-R(u_1)>0\qquad\text{for}\quad u_2>u_1\,.$$

LEMMA 1. *If the conditions* (i)–(iii) *hold, then each of the posititive half-trajectory* L_A^+ *and the negative half-trajectory* L_A^- *of* (6) *passing through the point* $A(x, F(x))$ *must cross the w-axis or tend to the point* (0, 0).

PROOF. It is easy to see that the curve $w=F(x)$ and the w-axis are the vertical isocline and the horizontal isocline, respectively, in the direction filed defined by (6).

It is sufficient to prove that the positive half-trajectory L_A^+ must cross the negative half w-axis or tend to the point (0, 0) on the region G: $x>0$, $w<F(x)$.

Let L_A^+ be represented by $(x(t), w(t))$. Since the function $x(t)$ and $w(t)$ are decreasing as long as $(x(t), w(t))$ remains in G, if L_A^+ does not cross the negative half w-axis and if L_A^+ is bounded, then L_A^+ must have a unique limit point D. Then D should be a singular point, and D is the point (0, 0), because the point (0, 0) is the unique singular point of (6).

If L_A^+ is unbounded, it must have a vertical asymptote $x=a>0$. Since

$$\frac{dw}{dx}=\frac{-g(x)}{R(w-F(x))}\,,$$

from (iii), we have $dw/dx\to 0$ as $w\to-\infty$. This is a contradiction, and the lemma is proved.

LEMMA 2. *If the conditions* (i)–(iii) *hold and*

(7) $$\limsup_{x\to+\infty}F(x)=+\infty\,;\qquad \liminf_{x\to-\infty}F(x)=-\infty\,,$$

or

(8) $$G(\pm\infty)=\pm\infty\,,\quad F(x)>k_1\,,\quad x>0\,;\quad F(x)<k_2\,,\quad x<0\,,$$

then the positive half-trajectory L_P^+ *of* (6) *passing through any given point* $P(0, w_P)$, $w_P\neq 0$, *must cross the curve* $w=F(x)$.

PROOF. We only prove the case where $w_P>0$. Suppose that L_P^+ stays above the curve $w=F(x)$. Then from the given conditions, w is decreasing while x is increasing unboundedly along L_P^+.

If (7) holds, we have a contradiction, that is, L_P^+ crosses the curve $w=F(x)$, since $w_P<\limsup_{x\to+\infty} F(x)$.

If (8) holds and

$$w_P>\limsup_{x\to+\infty} F(x)\,,$$

then along L_P^+ we have

$$0<w-F(x)<w_P-F(x)<w_P-k_1\,.$$

Thus,

$$\frac{dw}{dx}=-\frac{H(w-F(x))g(x)}{s(w-F(x))}\leqslant-\sigma g(x)\,,$$

where

$$\sigma=\inf_{0<u\leqslant w_P-k_1}\frac{H(u)}{s(u)}>0\,.$$

It follows that

$$k_1-w_P<w-w_P<-\sigma G(x)\to-\infty \qquad \text{as} \quad x\to+\infty\,.$$

This is a contradiction, and it proves that L_P^+ crosses the curve $w=F(x)$.

By the same argument, we can prove this lemma in the case $w_P<0$.

THEOREM 2. *If the conditions* (i)–(iii) *hold and if*

(iv) $G(\pm\infty)=\pm\infty, f(0)<0$, *and there are* $M>0$, $k_1>k_2$ *such that* $F(x)\geqslant k_1$ *for* $x>M$ *and* $F(x)\leqslant k_2$ *for* $x<-M$,

then (6) *has at least one closed trajectory.*

PROOF. We can assume $k_1>0$, because under the transformation $z=-x$, the type of the equation (2) is unchanged.

First of all, we construct an annular region surrounded by two boundary curves. Let

$$V(x, w)=G(x)+\int_0^{w-F(x)}\frac{s(u)du}{H(u)}\,.$$

Then we have

$$V'_{(6)}(x, w)=\frac{-s^2(w-F(x))f(x)}{H(w-F(x))}\,. \tag{9}$$

Since $f(0)<0$, there is an $x_0>0$ such that $f(x)<0$ for $x\in[-x_0, x_0]$. Take a number c so that $0<c<\min\{G(x_0), G(-x_0)\}$. The closed curve L_1: $V(x, w)=c$ is contained in the region $\{(x, w)\colon -x_0\leqslant x\leqslant x_0, |w|<\infty\}$. From (9) we have $V'_{(6)}(x, w)\geqslant 0$ along L_1. Therefore, we can take L_1 as the interior boundary.

Now, let us construct the exterior boundary. Suppose $|F(x)|<e$, $|g(x)|<b$ for $|x|\leqslant M$. We have $w-F(x)>d-e$ for $w\geqslant d$.

From (iii), for any given $\varepsilon>0$, if $d>0$ is sufficiently large,

$$0<dw/dx<\varepsilon \qquad \text{for} \quad -M<x<0\,, \quad w>d\,,$$

$$-\varepsilon>dw/dx>0 \qquad \text{for} \quad 0<x<M\,, \quad w<d\,.$$

Take a point U on the line $x=-M$ so that $w_U>2d$. If d is sufficiently large, the positive half-trajectory L_U^+ passing through U must cross the positive w-axis at a point P, cross the line $x=M$ at a point Q. The value of w on the section $\widehat{UPQ}$ of L_U^+ is more than d, and the value of $|w_Q-w_U|$ could be sufficiently small.

From Lemmas 1 and 2, as t increases, L_U^+ must cross the curve $w=F(x)$ at a point R, and cross the line $x=M$ again at a point S. Take a point T on the line $x=M$ so that $w_T<\min(-2d, w_S)$ (See Figure 1).

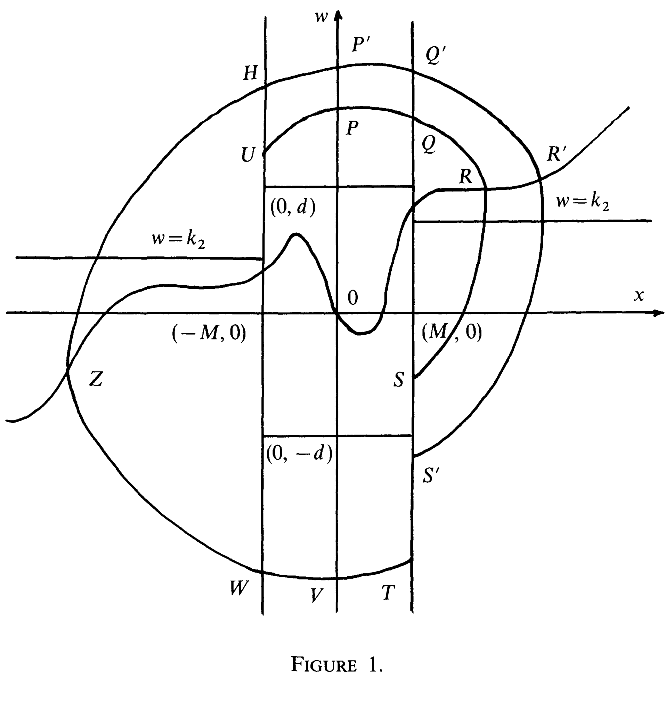

FIGURE 1.

By the same argument we can prove that L_T^+ must cross the negative w-axis at a point V, cross the line $x=-M$ at a point W provided d is sufficiently large. The value of w on the section $\widehat{TVW}$ of L_T^+ is less than $-d$, and the value of $|w_T - w_W|$ could be sufficiently small. From Lemmas 1 and 2, as t increases, L_T^+ will cross the curve $w=F(x)$ at a point Z, cross the line $x=-M$ at a point H and $w_H>0$.

If $w_H \leqslant w_U$, we can take

$$L_2 = \widehat{UPQRS} \cup \overline{ST} \cup \widehat{TVWZH} \cup \overline{HU}$$

as the exterior boundary.

If $w_H > w_U$, then $w_H > d$, and hence as t increases, L_T^+ will cross the positive w-axis at a point P', cross the line $x=M$ at a point Q', cross the curve $w=F(x)$ at a point R', and cross the line $x=M$ again at a point S'. Let

$$q(x, w) = G(x) + \int_0^{w-k_2} \frac{s(u)du}{H(u)}.$$

We have

$$\left.\frac{dq}{dw}\right|_{(6)} = \frac{-g(x)s(w-F(x))}{g(x)H(w-F(x))} + \frac{s(w-k_2)}{H(w-k_2)} = -R(w-F(x)) + R(w-k_2) \geqslant E_1(k_1-k_2),$$

if $x \geqslant M$. Therefore

$$q(Q') - q(S') \geqslant \int_{w_{S'}}^{w_{Q'}} E_1(F(x)-k_2)dw \geqslant \int_{w_{S'}}^{w_{Q'}} E_1(k_1-k_2)dw = E_1(k_1-k_2)(w_{Q'}-w_{S'}) \geqslant 0.$$

Similarly,

$$\left.\frac{dq}{dw}\right|_{(6)} \leqslant -E_1(k_2-F(x)) \qquad \text{if} \quad x \leqslant -M,$$

and hence

$$q(W) - q(H) \geqslant \int_{w_H}^{w_W} (-E_1(k_2-F(x)))dw \geqslant \int_{w_W}^{w_H} E_1(k_2-F(x))dw \geqslant 0.$$

We have

$$\left.\frac{dq}{dx}\right|_{(6)} = \frac{g(x)(1-R(w-k_2))}{R(w-F(x))} = \frac{g(x)(R(w-F(x))-R(w-k_2))}{R(w-F(x))}.$$

Therefore,

$$\left|\left.\frac{dq}{dw}\right|_{(6)}\right| \leqslant \frac{|g(x)(R(w-F(x))-R(w-k_2))|}{|R(w-F(x))|} \leqslant \frac{bE_2(e+k_2)}{E_1(|w|-e)} \qquad \text{for} \quad |x| \leqslant M, \quad |w| > d > e.$$

Thus, if $d<0$ is sufficiently large, the values of $|q(T)-q(W)|$ and $|q(H)-q(Q')|$ can be sufficiently small, and we can take $d>0$ so large that

$$q(T)-q(S')>\frac{1}{2}E_1(k_1-k_2)(w_{Q'}-w_{S'})>0\,.$$

Hence, $w_{S'}>w_T$. We can take

$$L_2=\overparen{TVWZHP'Q'R'S'}\cup\overline{S'T}$$

as the exterior boundary (See Figure 1). Thus the theorem follows from the Poincare-Bendixson theorem.

4. The Filippov Transformation. It is well-known that the Filippov transformation is a powerful tool in the study of the Liénard equation. By the aid of the equivalent form (6), we can use the Filippov transformation to investigate the equation (2).

Let

$$z=z_i(x)=\int_0^x g(s)ds \qquad \text{when} \quad (-1)^{i+1}x\geqslant 0\,, \quad i=1,2\,;$$

and denote their inverse functions by $x=x_i(z)$, $i=1$, 2, $z\geqslant 0$. Substituting them in to (6), we get two new systems:

$$\frac{dz}{dw}=-R(w-F_1(z))\,, \qquad z\geqslant 0\,; \tag{10}$$

and

$$\frac{dz}{dw}=-R(w-F_2(z))\,, \qquad z\geqslant 0\,; \tag{11}$$

where $F_i(z)=F(x_i(z))$, $z\geqslant 0$, $i=1$, 2.

LEMMA 3. *Consider the equation*

$$\frac{dz}{dw}=-R(w-F(z))\,, \qquad z\geqslant 0\,, \tag{12}$$

and assume that $R(u)$ *is continuous and satisfies the condition* (iii′), *and that* $F(z)$ *is continuous,* $F(0)=0$, *and* $F(z)<a\sqrt{z}$ $(F(z)>-a\sqrt{z})$ *for* $0<z<\delta$, *where* $a<\sqrt{8/E_2^*}$. *Then the solution curve of* (12) *passing through the point* $B(F(z),z)$ *must cross the* w*-axis at points* A *and* C *with* $w_A\geqslant 0$, $w_C<0$ $(w_A>0$, $w_C\leqslant 0)$, when $z>0$ *is arbitrarily given.*

PROOF. From the condition (iii′), we have

$$-R(w-F(z))\geqslant E_2^*(F(z)-w) \qquad \text{when} \quad w-F(z)\geqslant 0\,, \tag{13}$$

and

$$E_2^*(F(z)-w)\geqslant -R(w-F(z)) \qquad \text{when} \quad w-F(z)\leqslant 0\,. \tag{14}$$

If $F(z)<a\sqrt{z}$, consider the equation

$$\frac{dz}{dw}=E_2^*(a\sqrt{z}-w)\,. \tag{15}$$

It is obvious that below the curve $w=F(z)$ we have

$$E_2^*(a\sqrt{z}-w)>E_2^*(F(z)-w)\geqslant -R(w-F(z))\,.$$

Under the transformation $z=u^2$, the equation (15) becomes

$$\begin{cases} \dfrac{du}{dt}=aE_2^*u-E_2^*w \\ \dfrac{dw}{dt}=2u\,. \end{cases} \tag{16}$$

It is easy to see that the unique singular point $(0,0)$ is a spiral provided $a<\sqrt{8/E_2^*}$. Therefore, all trajectories of (16) below the curve $w=F(z)$ intersect the negative w-axis, and all trajectories of (15) below the curve $w=F(z)$ have the same property. Thus, it is easy to see from the comparison theorem that the solution curve of (12) passing through the point B must cross the negative w-axis at a point C with $w_C<0$.

By an argument similar to that in Lemma 1, we can prove that L_B must cross the positive w-axis at a point A with $w_A\geqslant 0$.

By the same argument we can prove the case in which $F(z)>-a\sqrt{z}$.

LEMMA 4. *Consider the system* (6) *and its equivalent equations* (10), (11), *and suppose the following condition is satisfied*:

(v) *There exists a number $\delta>0$ such that $F_2(z)\geqslant F_1(z)$ but $F_2(z)\not\equiv F_1(z)$, $F_1(z)<a\sqrt{z}$, $F_2(z)>-a\sqrt{z}$ when $0<z<\delta$ for $0<a<\sqrt{8/E_2^*}$.*

Then we can construct a simple closed curve L_1 such that

(a) $(0,0)\in S$, *the region encircled by L_1*;

(b) *each trajectory of* (6) *passing through a point on L_1 will enter S as t increases.*

PROOF. From (v) and Lemma 3, taking $0<z<\delta$ and the points $B(F_1(z),z)$, $E(F_2(z),z)$, we see that the solution curve $L_B|_{(10)}$ of (10) passing through the point B, and the solution curve $L_E|_{(11)}$ of (11) passing through the point E will both cross the w-axis at points A, C and F, D, respectively, with $w_A\geqslant 0$, $w_C<0$, $w_F>0$, $w_D\leqslant 0$.

Consider the equations

$$\frac{dw}{dz}=\frac{-1}{R(w-F_1(z))} \tag{17}$$

and

$$\frac{dw}{dz}=\frac{-1}{R(w-F_2(z))}\,. \tag{18}$$

From (v) we have

$$F_2(z)-w \geqslant F_1(z)-w \qquad \text{for} \quad 0 \leqslant z \leqslant \delta$$

and

$$F_2(z)-w \not\equiv F_1(z)-w \qquad \text{for} \quad 0 \leqslant z \leqslant \delta .$$

Therefore, on $\widehat{EF}$ we have

$$0 > \left.\frac{dw}{dz}\right|_{(17)} \geqslant \left.\frac{dw}{dz}\right|_{(18)},$$

and hence the curve $\widehat{BA}$ will be below $\widehat{EF}$. It follows that $w_A < w_F$. By the same argument we can prove that $w_C < w_D$ (See Figure 2).

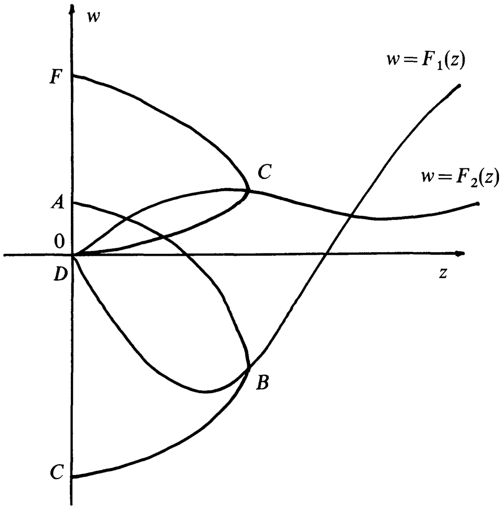

FIGURE 2.

Returning to the (x, w)-plane, let the points B', E' be the image points of B, E, respectively. Let G be the first intersection point of the positive half-trajectory L_F^+ passing through F of (6) and the line $x=x_B$, and then $w_G > w_{B'}$. Let H be the first intersection point of L_C^+ of (6) and the line $x=x_E$, and then $w_H < w_{E'}$.

Take $L_1 = \widehat{E'FG} \cup \overline{GB'} \cup \widehat{B'CH} \cup \overline{HE'}$ (See Figure 3). The lemma is proved.

THEOREM 3. *In addition to the conditions* (i), (ii), (iii′), (v), *suppose*

(iv′) $G(\pm\infty) = +\infty$; *there are* $M > 0$, $k_1 > k_2$ *such that* $F(x) \geqslant k_1$ *for* $x > M$, *and* $F(x) \leqslant k_2$ *for* $x < -M$; *and there are* $x_2 < 0 < x_1$ *such that* $F(x) < 0$ *for* $0 < x < x_1$, *and* $F(x) > 0$ *for* $x_2 < x < 0$.

Then (6) *has at least one closed trajectory.*

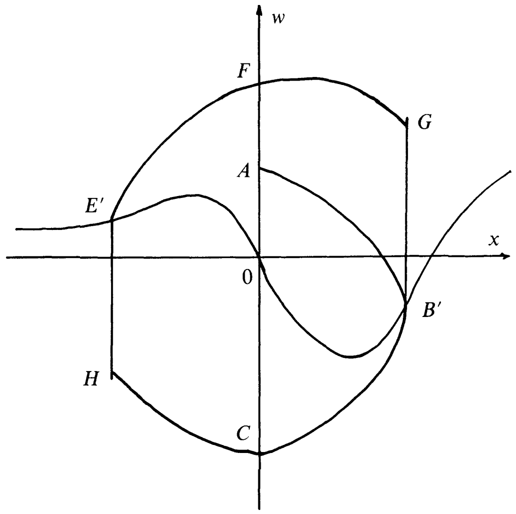

FIGURE 3.

PROOF. It is easy to verify that the equivalent equations (10) and (11) satisfy the conditions of Lemma 4. Let L_1 be the closed curve given in Lemma 4. We can take the closed curve L_1 as interior boundary of an annular region mentioned in the proof of Theorem 2. The construction of the exterior boundary is the same as that in Theorem 2. So we complete the proof of this theorem by the Poincare-Bedixson theorem.

REMARK. Theorem 3 is a generalization of Dragilev's Theorem in [1].

LEMMA 5. *Consider the equation* (12), *and suppose that* $R(u)$ *is continuous and satisfies the condition* (iii′) *and that* $F(z)$ *is continuous with* $F(0)=0$ *and that there is a number* $z_0>0$ *such that* $F(z)<a\sqrt{z}$ *(resp.* $F(z)>-a\sqrt{z}$*) for* $z\geqslant z_0$ *where* $a<\sqrt{8/E_2^*}$. *Then the solution curve* L_K *(resp.* L_M*) passing through any point* $K(w_K, 0)$, $w_K<0$ *(resp.* $M(w_M, 0)$, $w_M>0$*) must cross the w-axis at a point R with* $w_R\geqslant 0$ *(resp. N with* $w_N\leqslant 0$*).*

PROOF. We only prove the case where $F(z)<a\sqrt{z}$.

If L_K does not cross the line $z=z_0$, then it must cross the curve $w=F(z)$, and next, cross the positive w-axis at point R with $w_R\geqslant 0$.

If L_K crosses the line $z=z_0$ at a point P, then the trajectory $L_P|_{(15)}$ of (15) passing through P must cross the line $z=z_0$ again at another point P'. The point P' will be above the curve $w=F(z)$, because for a point (z, w) below the curve $w=F(z)$, we have

$$\left.\frac{dz}{dw}\right|_{(15)}>0 .$$

Furthermore, we have

$$\left.\frac{dz}{dw}\right|_{(15)} > \left.\frac{dz}{dw}\right|_{(12)},$$

therefore, $L_P|_{(12)}$ will be on the left hand side of $\widehat{PP'}$ and does not cross $\widehat{PP'}$. It follows that $L_P|_{(12)}$ must cross the curve $w=F(z)$ at a point Q. When $w>F(z)$,

$$\left.\frac{dz}{dw}\right|_{(12)} < 0,$$

so $L_P|_{(12)}$ will cross the positive w-axis at a point R with $w_R \geq 0$. The lemma is proved.

LEMMA 6. *Under the conditions* (i), (ii), (iii′). *Suppose that*

(vi) $R(u)=(L+\varepsilon(u))u$, $\varepsilon(u)\in C^1(R\to R)$, $\varepsilon(u)\to 0$, $\varepsilon'(u)u\to 0$ *as* $|u|\to\infty$;

and that

(vii) *there is a number* $z_0>\delta$ *such that*

$$\int_0^{z_0}(F_1(z)-F_2(z))dz>0,$$

and $F_1(z)\geq F_2(z)$, $F_1(z)>-a\sqrt{z}$, $F_2(z)<a\sqrt{z}$ *for* $z>z_0$, $0<a<\sqrt{8/E_2^*}$.

If $w_1(z)$, $w_2(z)$ *are the solutions of* (17) *and* (18) *satisfying the initial value condition* $w_1(0)=w_2(0)=w_0$, *respectively, then there is a number* $d>0$ *such that* $w_1(z_0)<w_2(z_0)$ *for* $|w_0|>d$.

PROOF. Since $w_1(0)=w_2(0)=w_0$, from (17), (18) we have

$$\begin{aligned} w_0^2(w_2(z_0)-w_1(z_0)) &= w_0^2\int_0^{z_0}\frac{R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z))}{R(w_2(z)-F_2(z))R(w_1(z)-F_1(z))}dz \\ &= w_0^2\int_0^{z_0}(L+\varepsilon(\xi)+\varepsilon'(\xi)\xi)\frac{(w_2(z)-w_1(z))+(F_1(z)-F_2(z))}{R(w_2(z)-F_2(z))R(w_1(z)-F_1(z))}dz, \end{aligned}$$

where ξ is a value between $w_1(z)-F_1(z)$ and $w_2(z)-F_2(z)$. Let $L(\xi)=L+\varepsilon(\xi)+\varepsilon'(\xi)\xi$, we have

$$\begin{aligned} w_0^2(w_2(z_0)-w_1(z_0)) &= (1/L)\int_0^{z_0}(F_1(z)-F_2(z))dz+\int_0^{z_0}\frac{L(\xi)w_0^2(w_2(z)-w_1(z))dz}{R(w_1(z)-F_1(z))R(w_2(z)-F_2(z))} \\ &+\int_0^{z_0}(F_1(z)-F_2(z))\left(\frac{L(\xi)w_0^2}{R(w_1(z)-F_1(z))R(w_2(z)-F_2(z))}-\frac{1}{L}\right)dz=:I_1+I_2+I_3. \end{aligned}$$

Suppose that $|F_i(z)|<a$, $i=1, 2$, $z\in[0,z_0]$. Take $M>a$ so large that

$$\frac{z_0}{R(M-a)}<M, \quad \text{and} \quad \frac{z_0}{|R(-M+a)|}<M.$$

Then, if $|w_0|>2M$, we have

$$\left|\frac{dw_i(z)}{dz}\right| = \frac{1}{|R(w_i(z)-F_i(z))|} < \frac{1}{\min(R(M-a), |R(-M+a)|)}$$

for $z \in [0, z_0]$, $i=1, 2$.

It follows from (iii′) that

$$\left|\frac{dw_i(z)}{dz}\right| \to 0 \qquad \text{as} \quad |w_0| \to \infty\,, \qquad i=1, 2\,,$$

and then, $|w_i(z)| \to \infty$ as $|w_0| \to \infty$, $i=1, 2$, $|w_2(z)-w_1(z)| \to 0$ as $|w_0| \to \infty$ for $z \in [0, z_0]$. Therefore, $L(\xi) \to L$ and $L(\xi)w_0^2/R(w_1(z)-F_1(z))R(w_2(z)-F_2(z)) \to 1/L$ as $|w_0| \to \infty$. This implies that $I_2 \to 0$, $I_3 \to 0$ as $|w_0| \to \infty$. From (vii) we have $I_1 > 0$. Thus the lemma is proved.

THEOREM 4. *If the conditions* (i), (ii), (iii′), (v)–(vii) *are satisfied, then the system* (6) *has at least one closed trajectory.*

PROOF. From Lemma 4 we can construct a closed curve which is the interior boundary of an annular region. Now, let us construct the exterior boundary.

Taking a point $K(w_K, 0)$, $w_K < 0$, from Lemmas 3 and 5, we conclude that the solution curve $L_K|_{(11)}$ of (11) passing through the point K must cross the w-axis at a point R with $w_R > 0$. Suppose that w_R converges to w_M as $w_K \to -\infty$.

If $w_M > \infty$, then we take point $M = (w_M, 0)$. From Lemmas 3 and 5 $L_M|_{(10)}$ must cross the w-axis at a point N with $w_N < 0$, and $L_N|_{(11)}$ must cross the w-axis at a point P with $w_P < 0$, $w_P < w_M$. Returnning to the (x, w)-plant, we take $L_2 = \overset{\frown}{MNP} \cup \overline{PN}$ which can be an exterior boundary.

When $w_M = \infty$, suppose $w_1(z)$, $\bar{w}_1(z)$ and $w_2(z)$, $\bar{w}_2(z)$ are solutions of (17), (18) satisfying the initial value conditions $w_1(0) = w_2(0) = w_K$, $\bar{w}_1(0) = \bar{w}_2(0) = w_R$, respectively. From Lemma 6 we have $w_1(z_0) < w_2(z_0)$, $\bar{w}_1(z_0) < \bar{w}_2(z_0)$ provided $|w_K|$ is sufficiently large. Thus, the point $S(\bar{w}_1(z_0), z_0)$ is below the point $D(\bar{w}_2(z_0), z_0)$, and the point $V(w_1(z_0), z_0)$ is below the point $U(w_2(z_0), z_0)$. The solution curve $L_S|_{(10)}$ of (10) passing through the point S must cross the line $z = z_0$ again at a point Q with $w_Q < w_S$. From (vii), $F_2(z) \leqslant F_1(z)$ for $z > z_0$. By the comparison theorem, the section $\overset{\frown}{SQ}$ of the trajectory of (10) will be on the left hand side of $\overset{\frown}{DU}$, the section of trajectory of (11), and $\overset{\frown}{SQ}$ does not cross $\overset{\frown}{DU}$ (See Figure 4). Returning to the (x, w)-plane, let the points D', S', Q', U', V' be the images of the points D, S, Q, U, V, respectively. Let

$$L_2 = \overset{\frown}{V'KU'D'RS'Q'} \cup \overline{Q'V'}$$

Then we can take L_2 as the exterior boundary (See Figure 5). From the Poincare-Bendixson theorem, the theorem is proved.

REMARK. Theorem 4 is a generalization of a theorem due to Filippov [2], which corresponds to the case where $\eta(y) \equiv 1$ and $\psi(y) \equiv 1$.

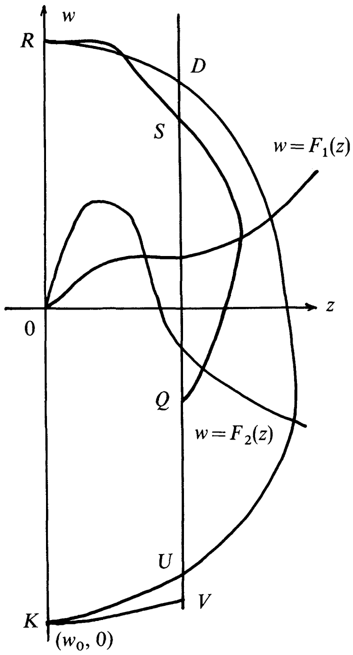

FIGURE 4.

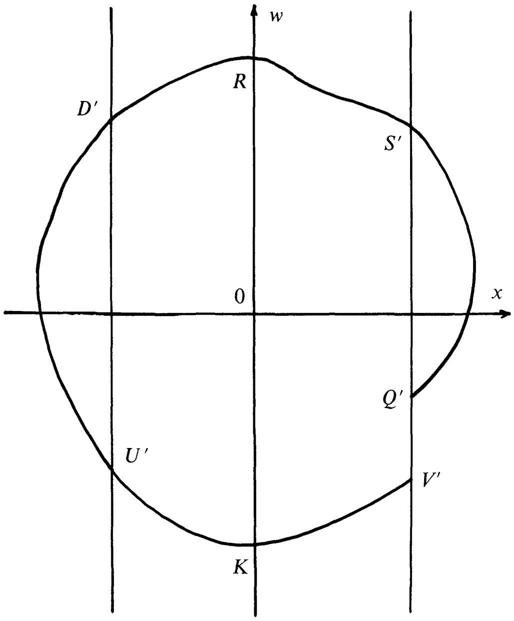

FIGURE 5.

EXAMPLE 2. *The equation*

$$x''+(x^2-1)\frac{1+x'^2}{2+x'^2}x'+(\pi-\arctan x')x=0$$

has a non-trivial periodic solution.

PROOF. Here, $f(x)=x^2-1$, $g(x)=x$, $\eta(y)=(1+y^2)/(2+y^2)$, $\psi(y)=\pi-\arctan y$, $u=u(y)=\int_0^y dt/\eta(t)=y+\arctan y$. It is easy to see that $1\leqslant u'(y)\leqslant 2$. Therefore, the inverse function of $u(y)$ is also an increasing function which is denoted by $s(u)$, and $1/2\leqslant s'(u)\leqslant 1$. We also have

$$\frac{d}{dy}\frac{\eta(y)y}{\psi(y)}=\psi^{-2}(y)J(y)$$

where

$$J(y)=\psi(y)\left(\frac{2+5y^2+y^4}{(2+y^2)^2}+\frac{y}{2+y^2}\right),$$

and we have the estimate

$$\pi/2\leqslant J(Y)\leqslant 17\pi/16\,.$$

It is not difficult to see that the condition (iii) is satisfied, and it is easy to verify that the conditions (i), (ii) and (iv) are also satisfied. Hence Theorem 2 yields the conclusion of this example.

Suppose $P(z)$ is a continuous differentiable, increasing, nonnegative function defined on $[0, \infty)$, and denote its inverse function by $p(u)$. Consider the equation

$$\frac{dz}{dw} = -E_2(w - P(z)), \tag{19}$$

where the function $E_2(r)$ is the function mentioned in (iii).

Let $u = P(z)$. Then

$$\frac{du}{dw} = \frac{dz}{dw} \cdot \frac{du}{dz} = \frac{-E_2(w-u)}{p'(u)},$$

and therefore, we obtain the equivalent system of (19)

$$\begin{cases} \dfrac{du}{dt} = -E_2(w-u) \\ \dfrac{dw}{dt} = p'(u). \end{cases} \tag{20}$$

If $P(0)=0$, $p'(0)=0$, and $p'(u) \neq 0$ for $u \neq 0$, then the point $(0, 0)$ is the unique singular point of (20).

The following assumptions are needed:

(iv″) $P(z)$ is a function satisfying the conditions above and such that the unique singular point $(0, 0)$ of (20) is a focus or a center or a center-focus.

(v′) There is a number $\delta > 0$ such that $F_2(z) \geqslant F_1(z)$ but $F_1(z) \not\equiv F_2(z)$, $F_1(z) < P(z)$, $F_2(z) > -P(z)$ for $0 < z < \delta$.

(vii′) There is a number $z_0^* > \delta$ such that $F_1(z) > F_2(z)$, $F_1(z) > -P(z)$, $F_2(z) < P(z)$ for $z > z_0^*$, and

$$\int_0^{z_0^*} E_1(F_1(z) - F_2(z))dz > 0,$$

where the function $E_1(r)$ is the function mentioned in (iii).

If we assume (iv″) holds, then the same argument applies with $a\sqrt{z}$ replaced by general $P(z)$. Corresponding to Lemmas 3, 4, 5, the following Lemmas 7, 8, 9 are obtained.

LEMMA 7. *Under the conditions* (iii) *and* (iv″), *if* $F(z) < P(z)$ (*resp.* $F(z) > -P(z)$) *for* $0 < z < \delta$, *then the solution curve* L_B *of* (12) *passing through the point* $B(F(z), z)$ *must cross the w-axis at points A and C with* $w_A \geqslant 0$, $w_C < 0$ (*resp.* $w_A > 0$, $w_C \leqslant 0$).

LEMMA 8. *If the conditions* (iii), (iv″) *and* (v′) *are satisfied, then we can construct a closed curve satisfying the same properties as there in Lemma* 4.

LEMMA 9. *Under the conditions* (iii) *and* (iv″), *if the function* $F(z)$ *is continuous with* $F(0)=0$, *and if there is a number* $z_0>0$ *such that* $F(z)<P(z)$ *(resp.* $F(z)>-P(z)$*), then the solution curve* L_K *(resp.* L_M*) of* (12) *passing through the point* $K(w_K, 0)$, $w_K<0$ *(resp.* $M(w_M, 0)$, $w_M>0$*) must cross the w-axis at a point R with* $w_R\geqslant 0$ *(resp. N with* $w_N\leqslant 0$*).*

LEMMA 10. *Under the conditions* (i), (ii), (iii), (iv″) *and* (vii′), *if* $w_1(z)$, $w_2(z)$ *are the solutions of* (17), (18), *respectively, and* $w_1(0)=w_2(0)=w_0$, *then there are numbers* $z_0>z_0^*$ *and* $d>0$ *such that* $w_1(z_0)<w_2(z_0)$ *for* $|w_0|>d$.

PROOF. Let $J(w_0)=(R(w_1(z)-F_1(z)))/R(w_0)$. We have

$$1+\frac{E_2(|w_1(z)-w_0-F_1(z)|)}{R(w_0)}\geqslant J(w_0)\geqslant 1+\frac{E_1(|w_1(z)-w_0-F_1(z)|)}{R(w_0)}.$$

Since $w_1(z)-w_0\to 0$ as $|w_0|\to\infty$. Therefore, $J(w_0)\to 1$ as $|w_0|\to\infty$. By the same argument, we can prove that

$$\frac{R(w_2(z)-F_2(z))}{R(w_0)}\to 1 \qquad \text{as} \quad |w_0|\to\infty .$$

For any $z_0>z_0^*$, we have

$$\begin{aligned}(21)\quad R^2(w_0)(w_2(z_0)-w_1(z_0))&=\int_0^{z_0}(R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z))dz\\ &+\int_0^{z_0}(R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z)))\left(\frac{R^2(w_0)}{R(w_1(z)-F_1(z))R(w_2(z)-F_2(z))}-1\right)dz\\ &=: I_1+I_2 .\end{aligned}$$

Since

$$|R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z))|\leqslant E_2(|w_1(z)-w_2(z)|+|F_1(z)-F_2(z)|),$$

and $|w_1(z)-w_2(z)|\to 0$, $z\in[0, z_0]$, as $|w_0|\to\infty$, we have

$$|w_1(z)-w_2(z)|+|F_1(z)-F_2(z)|=O(1), \quad |w_0|\to\infty ,$$

and then

$$R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z))=O(1), \quad |w_0|\to\infty .$$

Thus, $I_2\to 0$ as $|w_0|\to\infty$. Let

$$K(r)=\min(E_1(r), E_2(r)) .$$

We have

$$(22)\ I_1=\int_0^{z_0}(R(w_2(z)-F_2(z))-R(w_1(z)-F_1(z))dz\geqslant\int_0^{z_0}K(F_1(z)-F_2(z)+w_2(z)-w_1(z))dz$$
$$=\int_0^{z_0}K(F_1(z)-F_2(z))dz+\int_0^{z_0}(K(F_1(z)-F_2(z)+w_2(z)-w_1(z))$$
$$-K(F_1(z)-F_2(z)))dz=:I_3+I_4\,.$$

Since the function $K(r)$ is uniformly continuous on $[\alpha,\beta]$, where $\alpha=-1+\inf_{0\leqslant z\leqslant z_0}(F_1(z)-F_2(z))$, $\beta=1+\sup_{0\leqslant z\leqslant z_0}(F_1(z)-F_2(z))$, and $|w_2(z)-w_1(z)|\to 0$ as $|w_0|\to\infty$, we get $I_4\to 0$ as $|w_0|\to\infty$. From (vii′) we have $I_3>0$. From (21) and (22), we get

$$R^2(w_0)(w_2(z_0)-w_1(z_0))\geqslant I_2+I_3+I_4\,.$$

Therefore,

$$R^2(w_0)(w_2(z_0)-w_1(z_0))>0$$

provided $|w_0|$ is sufficiently large. The lemma is proved.

THEOREM 5. *If the conditions* (i)–(iii), (iv″), (v′) *and* (vii′) *are satisfied*, then (6) *has at least one closed trajectory.*

By Lemmas 7 through 10, the proof of this theorem is similar to the one for Theorem 4.

REFERENCES

[1] A. V. DRAGILEV, Periodic Solutions of Differential Equations of Nonlinear Oscillation, Prikl. Mat. i Meh. (in Russian), 16 (1952), 85–88.

[2] A. F. FILIPPOV, Sufficient Conditions for the Existence of Stable Limit Cycles of Equations of Second Order, Math. Sbornik (in Russian), 30 (72), (1952), 171–180.

[3] HUANG QICHANG, On Limit Cycles of Equation $x''+\varphi(x')+\psi(x')\eta(x)=0$, Science Bulletin, 27 (1982), 366–369.

[4] HUANG QICHANG, On the Existence of Periodic Solutions of Nonlinear Oscillation Equations, Chin. Ann. of Math. 5B (3), (1984), 311–318.

[5] R. REISSIG, G. SANSONE AND R. CONTI, Qualitative Theorie nichtlinearer Differentialgleichungen, Edzioni Cremonese, Roma, 1963.

[6] YE YANQIAN, Theory of Limit Cycles (Second Edition), Shanghai Science and Technology Press, 1984.

[7] ZANG ZHIFEN, DING TONGREN, HUANG WANZAO AND DONG ZHENXI, Qualitative Theory of Differential Equations, Press of Science, Peking, 1985.

DEPARTMENT OF MATHEMATICS
NORTHEAST NORMAL UNIVERSITY
CHANGCHUN, JILIN
PEOPLE'S REPUBLIC OF CHINA