Tôhoku Math. J.
42 (1990), 205–216

BUBBLING OUT OF EINSTEIN MANIFOLDS

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(Received March 4, 1989)

In [1], [8], and [4] the following compactness theorem of the space of Einstein metrics is obtained in the spirit of Gromov theory.

THEOREM A. *Let (X_i, g_i) be a sequence of n-dimensional $(n \geq 4)$ smooth manifolds and Einstein metrics on them with uniformly bounded Einstein constants $\{e_i\}$ satisfying*

$$\operatorname{diam}(X_i, g_i) \leq D, \quad \operatorname{vol}(X_i, g_i) \geq V \quad \text{and} \quad \int_{X_i} |R_{g_i}|^{n/2} dV_i \leq R$$

for some positive constants D, V and R, where we denote the curvature tensor of a metric g by R_g. Then there exist a subsequence $\{j\} \subset \{i\}$ and a compact Einstein orbifold (X_∞, g_∞) with a finite set of singular points $S = \{x_1, x_2, \cdots, x_s\} \subset X_\infty$ (possibly empty) for which the following statements hold:

(1) *(X_j, g_j) converges to (X_∞, g_∞) in the Hausdorff distance.*

(2) *There exists an into diffeomorphism $F_j : X_\infty \setminus S \to X_j$ for each j such that $F_j^* g_j$ converges to g_∞ in the C^∞-topology on $X_\infty \setminus S$.*

(3) *For every $x_a \in S$ $(a = 1, 2, \cdots, s)$ and j, there exist $x_{a,j} \in X_j$ and a positive number r_j such that*

(3.a) *$B(x_{a,j}; \delta)$ converges to $B(x_a; \delta)$ in the Hausdorff distance for all $\delta > 0$.*

(3.b) $\lim_{j \to \infty} r_j = 0$.

(3.c) *$((X_j, r_j^{-2} g_j), x_{a,j})$ converges to $((M_a, h_a), x_{a,\infty})$ in the pointed Hausdorff distance, where (M_a, h_a) is a complete, non-compact, Ricci-flat, non-flat n-manifold which is* ALE, *of order $n-1$ in general, of order n if (M_a, h_a) is Kähler or $n = 4$.*

(3.d) *There exists an into diffeomorphism $G_j : M_a \to X_j$ such that $G_j^*(r_j^{-2} g_j)$ converges to h_a in the C^∞-topology on M_a.*

(4) *It holds that*

$$\lim_{j \to \infty} \int_{X_j} |R_{g_j}|^{n/2} dV_j \geq \int_{X_\infty} |R_{g_\infty}|^{n/2} dV_\infty + \sum_a \int_{M_a} |R_{h_a}|^{n/2} dV_{h_a} .$$

Moreover if (X_i, g_i) are Kähler, then (X_∞, g_∞) and (M_a, h_a) are also Kähler.

Here we call a smooth n-dimensional complete Riemannian orbifold (X, g) asymptotically locally Euclidean (ALE, for short) of order $\tau > 0$, if there exists a compact subset $K \subset X$ such that $X \setminus K$ has coordinates at infinity; namely there are $R > 0$, $0 < \alpha < 1$, a finite subgroup $\Gamma \subset O(n)$ acting freely on $\boldsymbol{R}^n \setminus B(0; R)$, and a C^∞-diffeomorphism

$\mathscr{X}: X\setminus K\to(\boldsymbol{R}^n\setminus B(0;R))/\Gamma$ such that $\varphi=\mathscr{X}^{-1}\circ\mathrm{proj}$ satisfies (where proj is the natural projection of $\boldsymbol{R}^n$ to $\boldsymbol{R}^n/\Gamma$)

$$(\varphi^*g)_{ij}(z)=\delta_{ij}+O(|z|^{-\tau}), \qquad \partial_k(\varphi^*g)_{ij}(z)=O(|z|^{-\tau-1}),$$

$$\frac{|\partial_k(\varphi^*g)_{ij}(z)-\partial_k(\varphi^*g)_{ij}(w)|}{|z-w|^\alpha}=O(\min\{|z|,|w|\}^{-\tau-1-\alpha}) \qquad \text{for} \quad z, w\in\boldsymbol{R}^n\setminus B(0;R).$$

(For simplicity we assumed that (X, g) has only one end. In our Ricci-flat case this assumption is satisfied.)

Kronheimer classified all ALE hyper-Kähler surfaces of order 4 in his thesis [6], and called such manifolds ALE gravitational instantons. In particular, he proved the following:

THEOREM B. *An* ALE *gravitational instanton is diffeomorphic to a minimal resolution of $\boldsymbol{C}^2/\Gamma$, where Γ is a finite subgroup of $SU(2)$.*

We remark that a simply connected Ricci-flat Kähler surface is hyper-Kähler. Thus in the case of Einstein-Kähler surfaces we have rather good understanding of the nature of degeneration. Only missing point is the knowledge on the neck $B(x_{a,j};\delta)\setminus B(x_{a,j};r_j)$, i.e., how an instanton is glued to a singular point on X_∞. The purpose of this paper is to clarify the situation, namely, we get the following theorem stated in terms of the above notation.

THEOREM. *Assume that the sequence (X_i, g_i) consists of Einstein-Kähler surfaces. If we fix a sufficiently small constant $\delta>0$, then for sufficiently large j, the geodesic ball $B(x_{a,j};\delta)$ in X_j is diffeomorphic to a cyclic quotient of an* ALE *gravitational instanton.*

REMARK. In the 4-dimensional case, for a compact Einstein manifold X the curvature integral $\int_X|R|^2=\mathrm{const}\,\chi(X)$ is a topological invariant.

The author would like to express his thanks to Professor M. T. Anderson. He got the idea of the proof in an enlightening discussion with him. He would also like to thank Professors Furushima and Ohnita for helpful discussions. He would like to acknowledge his gratitude to Max-Planck-Institut für Mathematik for hospitality. This work was done during his stay at MPI.

1. Preparation from analysis. Let M be a complete n-dimensional $(n\geq 3)$ Riemannian manifold with a fixed point $o\in M$. For $0<r_1<r_2$ we denote $B(o;r_2)\setminus B(o;r_1)$ by $D(r_1, r_2)$. We assume that there is a domain $D=D(r_0, r_\infty)$ in M with $0\leq r_0<r_\infty$ which satisfies the following conformally invariant conditions:

$$\left\{\int_D|v|^{2\gamma}\right\}^{1/\gamma}\leq S\int_D|\nabla v|^2 \qquad \text{for all} \quad v\in C_c^1(D),$$

$$\operatorname{vol}(D(r_1, r_2)) \leq V r_2^n \qquad \text{for all} \quad r_0 \leq r_1 \leq r_2 \leq r_\infty$$

with some positive constants S, V and $\gamma = n/(n-2)$. Let u be a non-negative function defined on D which satisfies

$$\Delta u \geq -fu \qquad \text{on} \quad D$$

with a non-negative function f. Then we have the following lemmas. The proofs, which are essentially the same as those for the corresponding lemmas in [4; §4], are omitted.

LEMMA 1. *Suppose* $f \in L^{n/2}$, *and* $u \in L^p$ *for some* $p \in [p_0, p_1]$ *where* $p_0 > 1$. *Then* $u \in L^q$ *for all* $q \geq p$, *and there exists* $\varepsilon_1 = \varepsilon_1(S, V, p_0, p_1) > 0$ *such that if*

$$\int_{D(r,8r)} f^{n/2} \leq \varepsilon_1 \qquad \text{with} \quad r_0 \leq r < 8r \leq r_\infty ,$$

then we have

$$\left\{ \int_{D(2r,4r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{D(r,8r)} u^p ,$$

where $C_1 = C_1(S, V, p_0)$. *Moreover if* $r_0 = 0$ *and*

$$\int_{B(o;2r)} f^{n/2} \leq \varepsilon_1 \qquad \text{with} \quad 2r \leq r_\infty ,$$

then it holds that

$$\left\{ \int_{B(o;r)} u^{p\gamma} \right\}^{1/\gamma} \leq C_1 r^{-2} \int_{B(o;2r)} u^p .$$

LEMMA 2. *Suppose* $f \in L^{n/2}$, *and* $u \in L^p$ *for some* $p \in [p_0, p_1]$ *where* $p_0 > \gamma$. *Then there exists* $\varepsilon_2 = \varepsilon_2(S, V, p_0, p_1) > 0$ *such that if*

$$\int_D f^{n/2} \leq \varepsilon_2 ,$$

then it holds that for $r_0 \leq r_1 < 2r_1 < r_2 < 2r_2 \leq r_\infty$

$$\int_{D(2r_1, r_2)} u^p \leq C_2 \int_{D(r_1, 2r_1) \cup D(r_2, 2r_2)} u^p ,$$

$$\int_{D(r_1, r_2)} u^p \leq C_2 \max\left\{ \left(\frac{r_0}{r_1} \right)^{\varepsilon_3}, \left(\frac{r_2}{r_\infty} \right)^{\varepsilon_3} \right\} \int_D u^p ,$$

where $C_2 = C_2(S, V, p_0)$, $\varepsilon_3 = \varepsilon_3(S, V, p_0) > 0$.

LEMMA 3. *If* $f \in L^q$ *for some* $q > n/2$, $u \in L^p$ *for some* $p > 1$, *and if*

$$\int_{D(r,8r)} f^q \leq A r^{-(2q-n)}$$

with some constant A for any r such that $r_0 \leq r < 8r \leq r_\infty$, then we have

$$\sup_{D(2r,4r)} u^p \leq C_3 r^{-n} \int_{D(r,8r)} u^p ,$$

where $C_3 = C_3(A, S, V, p, q)$. Moreover if $r_0 = 0$ and

$$\int_{B(o;2r)} f^q \leq A r^{-(2q-n)} ,$$

then

$$\sup_{B(o;r)} u^p \leq C_3 r^{-n} \int_{B(o;2r)} u^p .$$

Let (M, g) be an n-dimensinal Einstein manifold. Then applying the Weitzenböck formula we get

$$\Delta |R| \geq -C_4 |R|^2 .$$

Moreover, we have the following inequality using Yau's trick. For the proof see [2], [4] and [9].

LEMMA 4. *There exist positive constants $\delta = \delta(n)$ and $C_5 = C_5(n)$ such that*

$$\Delta |R|^{1-\delta} \geq -C_5 |R|^{2-\delta} .$$

If $n = 4$ or if (M, g) is Kähler, then we can take $\delta = 4/(n+2)$.

One can show the following lemma via L^2-Hodge theory.

LEMMA 5. *Let (X, g) be an n-dimensional $(n \geq 4)$, complete, non-compact, Ricci-flat,* ALE *orbifold. Then its first cohomology group $H^1(X; \boldsymbol{R})$ vanishes.*

Here we recall the existence theorem of Ricci-flat Kähler metrics on open Kähler orbifolds in [3], which is stated in the case of manifolds but whose proof works equally for orbifolds.

DEFINITION. A complete n-dimensional Riemannian orbifold (X, g) is said to be of $C^{k,\alpha}$-asymptotically flat geometry if for each point $p \in X$ with distance from a fixed point o in X, there exists a quasi-coordinate map $\phi: B^n \to X$ centered at p from the unit ball B^n in the Euclidean space (i.e. ϕ gives a local uniformization and $\phi(0) = p$), such that with respect to the standard coordinates $x = (x^1, x^2, \cdots, x^n)$ of the Euclidean space it satisfies the following conditions:

(i) If we write $\phi^* g = \sum g_{ij}(x) dx^i dx^j$, then the matrix $(r^2+1)^{-1}(g_{ij})$ is bounded from below by a constant positive matrix independent of p.

(ii) The $C^{k,\alpha}$-norms of $(r^2+1)^{-1}g_{ij}$, as functions in x, are uniformly bounded.

On such an orbifold we can define the Banach space $C^{k,\alpha}_\delta$ of weighted $C^{k,\alpha}$-bounded functions: The norm of a function $u \in C^{k,\alpha}_\delta$ is given by the supremum of the $C^{k,\alpha}$-norms of $(r^2+1)^{\delta/2}u$ with respect to the coordinates x.

THEOREM C. *Let (X, ω) be an n-dimensional $(n \geq 2)$ complete open Kähler orbifold of $C^{k,\alpha}$-asymptotically flat geometry with $k \geq 2$, $0<\alpha<1$. Assume that the singuralities sit in a compact set and there exists a barrier function ρ. If X admits a Ricci-flat volume form V such that $\omega^n = e^f V$ with $f \in C^{k,\alpha}_{\delta+2}$ and $\delta > 0$, then X admits a complete Ricci-flat Kähler metric asymptotically equal to ω.*

Here a barrier function ρ means that outside a compact set ρ satisfies the following conditions:

(i) ρ is compatible with the distance function d from o; that is, there exists a positive constant c_1 such that $c_1 d \leq \rho \leq c_1^{-1} d$.

(ii) The function $\rho^{-\delta}$ belongs to $C^{k+2,\alpha}_\delta$.

(iii) There exists a positive constant c_2 such that

$$\Box \rho^{-\delta} \leq -c_2 \rho^{-2-\delta}\,.$$

(iv) There exists a positive constant c_3 such that for any positive number K and sufficiently large d

$$(\omega + \sqrt{-1}\partial\bar{\partial}K\rho^{-\delta})^n \leq (1 - c_3 K \rho^{-2-\delta})\omega^n\,,$$

$$(\omega + \sqrt{-1}\partial\bar{\partial} - K\rho^{-\delta})^n \geq (1 + c_3 K \rho^{-2-\delta})\omega^n\,.$$

2. Einstein manifolds. Let (X_j, g_j) be a sequence of Einstein manifolds which enjoyes the properties stated in Theorem A. Then by [5] we have the Sobolev inequality on (X_j, g_j) with uniform Sobolev constants, and the following proposition holds. For the proof see [1], [8].

PROPOSITION 1. *There exist constants ρ, C_6 and ε_4 such that if*

$$\int_{B(x;2r)} |R_{g_j}|^{n/2} \leq \varepsilon_4$$

with $2r \leq \rho$, then we have

$$\sup_{B(x;r)} |R_{g_j}| \leq C_6 r^{-2} \left\{ \int_{B(x;2r)} |R_{g_j}|^{n/2} \right\}^{2/n}.$$

Now we take a positive constant $r_\infty < \rho$ sufficiently small, so that we can assume that for all a

$$\sup_{B(x_{a,j};r_\infty)} |R_{g_j}|^2 = |R_{g_j}|^2(x_{a,j}) \to \infty \qquad \text{as} \quad j \to \infty$$

and

$$\int_{B(x_a, r_\infty)} |R_{g_\infty}|^{n/2} \le \frac{\varepsilon}{2}$$

with a positive number $\varepsilon \le \varepsilon_4/2$ to be determined later. From now on we fix an arbitrary singular point x_a and look at the blowing up process. Since (X_j, g_j) converges to (X_∞, g_∞) in C^∞-topology except at the singular points, for sufficiently large j we can find a positive number $r_0 = r_{0,j}$ such that

$$\int_{D(r_0, r_\infty)} |R_{g_j}|^{n/2} = \varepsilon\,,$$

where we denote the subset $B(x_{a,j}; r_2) \setminus B(x_{a,j}; r_1)$ in X_j by $D(r_1, r_2)$. Then we get

$$r_0 \to 0 \qquad \text{as} \quad j \to \infty\,.$$

PROPOSITION 2. *There is a subsequence $\{k\} \subset \{j\}$ such that the sequence of pointed Einstein manifolds $((X_k, r_0^{-2} g_k), x_{a,k})$ converges to $((Y, h), y_\infty)$ in the pointed Hausdorff distance, where (Y, h) is a complete, non-compact, Ricci-flat, non-flat n-orbifold only with finitely many isolated singular points. (Y, h) is* ALE *of order $n-1$ in general, and of order n if $n=4$ or (Y, h) is Kähler. The convergence is actually in C^∞-topology except at the singular points.*

The proof is the same as that of Theorem A. We refer to [1], [8] and [4].

Thus we know that for large $1 < K_1 < K_2$ the two subsets $D(K_1 r_0, K_2 r_0)$ and $D(K_2^{-1} r_\infty, K_1^{-1} r_\infty)$ in X_k are very close to portions of flat cones $\boldsymbol{R}^n/\Gamma_0$ and $\boldsymbol{R}^n/\Gamma_\infty$, respectively. To show that $\Gamma_0 = \Gamma_\infty$ and $D(K_1 r_0, K_2^{-1} r_\infty)$ is also close to a portion of the flat cone, we need the following curvature estimate.

PROPOSITION 3. *There exist positive constants C_7 and ε_5 such that for $4r_0 \le r < 4r \le r_\infty$ it holds that*

$$r^2 |R_{g_j}| \le C_7 \max\left\{ \left(\frac{r_0}{r}\right)^{\varepsilon_5}, \left(\frac{r}{r_\infty}\right)^{\varepsilon_5} \right\}.$$

PROOF. First apply Lemma 1 to the equation $\Delta |R| \ge -C_4 |R|^2$ on $R = R_{g_j}$, assuming $C_4^{n/2} \varepsilon \le \varepsilon_3$. Then we get that for $2r_0 \le r < 2r \le r_\infty$

$$\int_{D(r, 2r)} |R|^{n/2} \le A r^{-(2q-n)}$$

with a constant A and $q=\gamma n/2$. Next we apply Lemmas 2 and 3 to the equation $\Delta|R|^{1-\delta}\geq -C_5|R|^{2-\delta}$ with $p=(1-\delta)^{-1}n/2>\gamma$. If $C_5^{n/2}\varepsilon\leq\varepsilon_2$, we get that for $4r_0\leq r<4r\leq r_\infty$

$$r^2|R_{g_j}|\leq C_3^{2/n}\left\{\int_{D(r/2,4r)}|R|^{n/2}\right\}^{2/n}\leq C_3^{2/n}(3C_2 2^{\varepsilon_3})^{2/n}\max\left\{\left(\frac{r_0}{r}\right)^{\varepsilon_5},\left(\frac{r}{r_\infty}\right)^{\varepsilon_5}\right\}$$

with $\varepsilon_5=2\varepsilon_3/n$. We choose $\varepsilon=\min\{\varepsilon_3 C_4^{-n/2}, \varepsilon_2 C_5^{-n/2}, \varepsilon_4/2\}$, and the proof is complete.

Once we get the curvature estimate, we can construct coordinates as in the proof of the existence theorem of coordinates at infinity [4]. We need only minor changes, so we omit the proof of the following proposition.

PROPOSITION 4. *If one takes* $1<K_1<K_2$ *sufficiently large, then the subset* $D(K_1 r_0, K_2^{-1}r_\infty)$ *is close to a portion of a flat cone* $\mathbf{R}^n/\Gamma$ *for large* j.

Thus if (Y, h) has no singularity, then the ball $B(x_{a,k}; r_\infty)$ is diffeomorphic to the smooth manifold Y which bubbles out of X_k.

If (Y, h) has a singular point y_s, then we choose a sufficiently small number r'_∞ and the corresponding point $x_{s,k}$ in X_k so that

$$\int_{B(y_s, r'_\infty)}|R_h|^{n/2}\leq\frac{\varepsilon}{2}$$

$$\sup_{B(x_{s,k}; r_0 r'_\infty)}|R_{g_k}|^2=|R_{g_k}|^2(x_{s,k})\to\infty\ .$$

Choose $r'_0=r'_{0,k}$ so that

$$\int_{D'(r_0 r'_0, r_0 r'_\infty)}|R_{g_k}|^{n/2}=\varepsilon\ ,$$

with $D'(r_1, r_2)=B(x_{s,k}; r_2)\setminus B(x_{s,k}; r_1)$, and consider a sequence of pointed Einstein manifolds $((X_k, (r_0 r'_0)^{-2}g_k), x_{s,k})$. Then we have the same situation as before, and we get a complete, non-compact, Ricci-flat, non-flat, ALE n-orbifold (Y', h') only with finitely many isolated singular points. In the same way we can show that the neck is diffeomorphic to a flat cone. If (Y', h') again has a singular point, we repeat the argument. We also apply the same process at every singular point which appears at each repeated step. Since each singular point contributes at least ε to the curvature integral $\int|R|^{n/2}$, the process terminates in finite steps. In this way we get a picture of the small ball $B(x_{a,j}; r_\infty)$.

DEFINITION. Let X and Y be complete, non-compact, ALE n-orbifolds such that X has a point x which has a neighborhood diffeomorphic to $B(0; R)/\Gamma$ and Y has an end which is diffeomorphic to $(\mathbf{R}^n\setminus B(0; R))/\Gamma$ with the same $\Gamma\subset O(n)$. Since $\partial B(0; R)/\Gamma$ is diffeomorphic to $\partial(\mathbf{R}^n\setminus B(0; R))/\Gamma$, we can glue X and Y along them and get a new ALE n-orbifold $X\# Y$, which we call an IS-connected sum of X and Y

around x and the end of Y.

THEOREM 1. *The small ball* $B(x_{a,j}; r_\infty)$ *in* X_j *corresponding to a singular point* x_a *of the limit orbifold* X_∞ *is diffeomorphic to an* IS-*connected sum of finite number of complete, non-compact, Ricci-flat, non-flat,* ALE *n-orbifolds only with finitely many isolated singular points, where all singular points are glued to the ends and we end up with an* ALE *manifold.*

REMARK. We may also use the following gap theorem to show that the process terminates in finite steps.

THEOREM 2. *Let* (X, g) *be an n-dimensional* $(n \geq 4)$, *complete, non-compact, Ricci-flat Riemannian orbifold which satisfies*

$$\left\{\int v^{2\gamma}\right\}^{1/\gamma} \leq S\int |\nabla v|^2 \qquad \text{for all} \quad v \in C_c^1(X)$$

with a constant $S > 0$. *Then there exists a constant* $\varepsilon_6 = \varepsilon_6(n, S) > 0$ *such that the inequality*

$$\int_X |R|^{n/2} \leq \varepsilon_6$$

implies that (X, g) *is the Euclidean space.*

PROOF. Apply Lemma 1.

3. Einstein Kähler surfaces. In this section we assume that all manifolds (X_j, g_j) are Einstein-Kähler surfaces. Since the limit space X_∞ is an orbifold, there is a neiborhood U of the singular point x_a which is biholomorphic to a quotient B/Γ of the unit ball $B \subset C^2$ with a finite subgroup $\Gamma \subset U(2)$ acting freely on $C^2 \setminus \{0\}$. Let det: $U(2) \to S^1$ be the group homomorphism defined by the determinant. Then the image $\det(\Gamma)$ is a finite cyclic group, say, Z_m. Then U has a branched Z_m-covering: $\tilde{U} \to U$ with a branch point x_a such that $\tilde{U}$ has trivial canonical line bundle $K_{\tilde{U}}$. Namely, set $\tilde{\Gamma} = (\ker\det) \cap \Gamma \subset SU(2)$. Then we have a natural projection $\tilde{U} = B/\tilde{\Gamma} \to U$ and the non-vanishing holomorphic 2-form $\omega = dz^1 \wedge dz^2$ descends to $\tilde{U}$, where (z^1, z^2) is the standard coordinates in C^2. We have the corresponding result on $x_{a,j} \in X_j$ for large j.

PROPOSITION 5. *There exists a positive constant* δ *such that for j large there is a smooth* Z_m-*covering*: $\tilde{U}_j \to U_j \supset B(x_{a,j}; \delta)$ *with* $\tilde{U}_j$ *having topologically trivial canonical line bundle* $K_{\tilde{U}_j}$.

PROOF. We may assume that the domain $U \subset X_\infty$ has smooth boundary ∂U. Then there exists a sequence of neighborhoods $U_j \subset X_j$ of $x_{a,j}$ which have smooth boundaries $\partial U_j = F_j(\partial U)$. We take δ so small that $B(x_{a,j}; \delta) \subset U_j$. Then it is sufficient to show that for large j there are sections θ_j of $K_{U_j}^{\otimes m}$ on U_j such that

$$C_8^{-1} \le |\theta_j| \le C_8\,, \quad \text{and} \quad |\nabla\theta_j| \le C_9$$

with positive constants C_8, C_9.

Define an operator $\square = \square_j$ acting on the space of sections of $K_{X_j}^{\otimes m}$ by

$$\square = -\bar\partial^*\bar\partial = \operatorname{tr} \nabla'\nabla''\,,$$

where we decompose the covariant differentiation $\nabla = \nabla' + \nabla''$ into (1, 0)- and (0, 1)-parts. Let ψ be the local holomorphic uniformization $\psi\colon B \to B/\Gamma \stackrel{\simeq}{\to} U$ and η be a radial cut-off function on B such that $\eta = 0$ on $B(0; 1/3)$ and $\eta = 1$ on $B \setminus B(0; 2/3)$. Through ψ the section $\eta\omega^{\otimes m}$ of $K_B^{\otimes m}$ defines a section of $K_U^{\otimes m}$, which we still denote by $\eta\omega^{\otimes m}$. For j large we define sections $\theta_0 = \theta_{0,j}$ of $K_{X_j}^{\otimes m}$ on U_j by $\theta_{0,j} = \operatorname{proj}(F_j^{-1})^*\eta\omega^{\otimes m}$, where $\operatorname{proj} = \operatorname{proj}_j$ is the projection map for tensors to $K_{X_j}^{\otimes m}$. (Note that the maps $F_j\colon X_\infty \setminus S \to X_j$ need not be holomorphic, but become closer and closer to be holomorphic as j tends to ∞). We solve the following equation for a section $\theta = \theta_j$ of $K_{X_j}^{\otimes m}$ on U_j:

$$\square\theta = 0 \quad \text{and} \quad \theta|_{\partial U_j} = \theta_0|_{\partial U_j}\,.$$

Then θ satisfies

$$\Delta\theta = \operatorname{tr} \nabla\nabla\theta = -2me_j\theta\,.$$

Set $\theta' = \theta - \theta_0$. Then θ' has vanishing boundary value and satisfies

$$\Delta\theta' = -2me_j\theta' + \zeta$$

with $\zeta = \zeta_j$ on which we have good control. We have

$$\lambda \int |\theta'|^2 \le \int |\nabla|\theta'||^2 \le \int |\nabla\theta'|^2 = -\int (\theta', \Delta\theta') = 2me_j \int |\theta'|^2 - \int (\theta', \zeta)$$
$$\le 2me_j \int |\theta'|^2 + \left(\int |\theta'|^2\right)^{1/2} \left(\int |\zeta|^2\right)^{1/2}$$

with the first eigenvalue $\lambda = \lambda_j$ of the Laplacian acting on functions on U_j with the Dirichlet condition. If we choose U, hence U_j, sufficiently small so that $\lambda \ge 2m|e_j| + 1$, we get L^2-estimates for θ' and $\nabla\theta'$, hence, for θ and $\nabla\theta$. We apply Lemma 3 to the inequality $\Delta|\theta| \ge -2m|e_j||\theta|$, and get C^0-estimate for θ.

As for C^1-estimate we differentiate the equation on θ and get the following equations:

$$\Delta|\nabla'\theta|^2 = 2|\nabla\nabla'\theta|^2 + 2e_j|\nabla'\theta|^2\,,$$
$$\Delta|\nabla''\theta|^2 = 2|\nabla\nabla''\theta|^2 + 2(-4m+1)e_j|\nabla''\theta|^2\,,$$
$$\Delta|\nabla\theta|^2 \ge 2|\nabla\nabla\theta|^2 - 2(4m-1)|e_j||\nabla\theta|^2\,,$$
$$\Delta|\nabla\theta| \ge -(4m-1)|e_j||\nabla\theta|\,.$$

Then again applying Lemma 3, we get C^1-estimate away from boundaries. As for one near boundaries, we have good control on the smoothness of the boundaries, the boundary values and the equations. We also have C^0-estimate for θ. So there is no trouble in getting C^∞-estimate on $U_j \setminus B(x_{a,j}; r)$ for any fixed $r>0$.

Now consider the sequence $\{\text{proj}\, \psi^* F_j^* \theta_j\}$ on $B\setminus\{0\}$ which has uniform C^∞-estimate away from the origin 0. Thus it has a convergent subsequence with limit, say, $\tilde{\theta}$ defined on $B\setminus\{0\}$. $\tilde{\theta}$ satisfies the equation $\square\tilde{\theta}=0$ and has C^1-estimate, so it extends to a smooth solution of the equation across the origin. It must coincide with the unique solution $\omega^{\otimes n}$. Hence the sequence $\{\text{proj}\, \psi^* F_j^* \theta_j\}$ itself converges to $\omega^{\otimes m}$, and there is a positive constant C_{10} such that for fixed $r>0$ we have that for large $j\geq j(r)$

$$|\theta_j|\geq C_{10} \qquad \text{on} \quad U_j\setminus B(x_{a,j}; r)\,.$$

By Theorem 1 there exists a constant C_{11} such that every point in $B(x_{a,j}; r)$ can be connected to the boundary $\partial B(x_{a,j}; r)$ with a curve of length at most $C_{11}r$. Thus for $j\geq j(r)$ with $r=C_{10}/2C_9C_{11}$ we have $|\theta_j|\geq C_{10}/2$ on U_j.

REMARK. One can also show that $K_{U_j}^{\otimes m}$ is complex analytically trivial for large j.

From now on we work on the covering space $\tilde{U}_j$, and denote it simply by U_j. Then we have $m=1$. We made a trivialization θ of K_{U_j} with uniform C^1-estimate. Thus if we conformally change it, the triviality is preserved in the process of bubbling out of complete, Ricci-flat, ALE orbifold Kähler surfaces. Hence the local fundamental groups of the singular points and the fundamental group at infinity of each bubble are contained in $SU(2)$.

PROPOSITION 6. *Let (X, g) be a complete, Ricci-flat,* ALE *orbifold Kähler surface. If its canonical line bundle K_X is topologically trivial, then (X, g) is hyper-Kähler.*

PROOF. By assumption K_X is flat and defines an element in $H^1(X; S^1)$. The exact sequence

$$H^1(X; \boldsymbol{R})\rightarrow H^1(X; S^1)\rightarrow H^2(X; \boldsymbol{Z})$$

and Lemma 5 imply that the topologically trivial K_X has a trivial connection.

Thus our bubbles are all hyper-Kähler. Hence if there is only one bubble coming out, the proof of the main theorem is complete.

THEOREM 3. *If we take $\delta>0$ sufficiently small, then for sufficiently large j, the geodesic ball $B(x_{a,j}; \delta)$ in X_j is diffeomorphic to a cyclic quotient of an* ALE *gravitational instanton.*

REMARK. It is likely that $B(x_{a,j}; \delta)$ is biholomorphic to a domain of a cyclic quotient of an ALE gravitational instanton.

We can prove Theorem 3 by applying the following theorem inductively.

THEOREM 4. *Let* (X, g) *be a complete, hyper-Kähler,* ALE *orbifold surface which has a singular point* o *with local fundamental group* $\Gamma\subset SU(2)$, *and* (Y, h) *be an* ALE *gravitational instanton which is biholomorphic to the minimal resolution of* C^2/Γ. *Then an* IS-*connected sum* $X\# Y$ *around* o *and the end of* Y *also admits a structure of a complete, hyper-Kähler,* ALE *orbifold surface.*

PROOF. First fix a Kähler structure (X, ω_1) on X, where ω_1 is its Kähler form. We can take a holomorphic local uniformization $\psi_1: B(0;\delta)\subset C^2\to U\ni o$ so that

$$\psi_1^*\omega_1=\sqrt{-1}\partial\bar\partial\phi_1\,,\qquad \phi_1=|z|^2+O(|z|^3)\,,$$
$$\psi_1^*\omega_1^2=2(\sqrt{-1}dz^1\wedge d\bar z^1)(\sqrt{-1}dz^2\wedge d\bar z^2)\,.$$

Let $\psi_2: C^2\setminus B(0;K)\to Y$ be the holomorphic local uniformization of Y at infinity. Then by Kronheimer [6] the Kähler form ω_2 of (Y, h) satisfies the following properties (cf. [3]):

$$\psi_2^*\omega_2=\sqrt{-1}\partial\bar\partial\phi_2\,,\qquad \phi_2=|z|^2+O(|z|^{-2})\,,$$
$$\psi_2^*\omega_2^2=2(\sqrt{-1}dz^1\wedge d\bar z^1)(\sqrt{-1}dz^2\wedge d\bar z^2)\,.$$

For sufficiently small positive numbers δ_1, δ_2, by the map $\psi(z)=z/\delta_1\delta_2$ we identify two subsets $\psi_1(D(\delta_1, 4\delta_1))\subset X$, $\psi_2(D(\delta_2^{-1}, 4\delta_2^{-1}))\subset Y$, and get an orbifold surface $Z=X\# Y$. In this construction the parallel holomorphic 2-forms on X and Y are glued to give a holomorphic 2-form on Z. We define a Kähler metric ω on Z as follows:

$$\omega=\begin{cases}\omega_1 & \text{on}\quad X\setminus\psi_1(B(0;4\delta_1))\\ \sqrt{-1}\partial\bar\partial\{\eta_{4\delta_1}\phi_1+(1-\eta_{4\delta_1})(\delta_1\delta_2)^2\psi^*\phi_2\} & \text{on}\quad \psi_1(D(\delta_1, 4\delta_1))\\ (\delta_1\delta_2)^2\omega_2 & \text{on}\quad Y\setminus\psi_2(C^2\setminus B(0;\delta_2^{-1}))\,,\end{cases}$$

where $\eta_\delta(z)=\eta(z/\delta)$ is a cut-off function. Since $\phi_1-|z|^2$ and $(\delta_1\delta_2)^2\psi^*\phi_2-|z|^2$ are small on $\psi_1(D(\delta_1, 4\delta_2))$, it is easy to see that ω actually defines a Kähler metric on Z.

By assumption there is a coordinate $\psi_\infty: R^4\setminus B(0;K)\to X$ at the infinity of X such that

$$\psi_\infty^* g_{ij}=\delta_{ij}+O(|x|^{-4})\,.$$

Then it is easy to see that $\rho=|x|$ makes a barrier function on X, hence on Z. Thus (Z, ω) satisfies the assumption of Theorem C, that is, Z admits a complete, Ricci-flat, ALE orbifold Kähler metric. Z is hyper-Kähler, since the holomoprhic 2-form on Z is easily seen to be parallel.

Now we prove Theorem 3. Assume that the blowing up process of orbifold singular points terminates in l steps. Then the bubbles coming out at the l-th step are all smooth ALE gravitational instantons. So they are diffeomorphic to the minimal resolutions of

C^2/Γ, $\Gamma \subset SU(2)$. We replace their structrues by those comming from minimal resolutions. Then by Theorem 4 we can glue them to the bubbles of the $(l-1)$-th step, and get smooth ALE gravitational instantons. Repeating this argument we finally get a smooth ALE gravitational instanton which is given by all bubbles glued. This implies Theorem 3.

For examples of bubbling out of ALE gravitational instantons we refer to [7].

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