Tôhoku Math. J.
42 (1990), 195–203

ASYMMETRY OF MAXIMAL FUNCTIONS ON THE AFFINE GROUP OF THE LINE*

GARTH GAUDRY, SAVERIO GIULINI AND ANNA MARIA MANTERO

(Received February 20, 1989)

1. Introduction. In this paper, we study the weak-type (1, 1), and strong type (p, p) $(1<p<+\infty)$ boundedness of certain Hardy-Littlewood maximal function operators. These are generated by taking the maximal averages over the left- (resp. right-) translates of various families of neighbourhoods of the identity in the affine, or '$ax+b$', group of the line. Investigations of this kind are basic to the study of singular integral operators. Thus, the aim of the paper is to delineate some of the positive and negative results for maximal functions on this group.

Let H be the group of orientation-preserving affine transformations of the real line commonly known as the '$ax+b$' group. We consider H as the upper half-plane

$$H=\{(u, v): u\in \boldsymbol{R}, v>0\}.$$

The product of two elements is given by

$$(u, v)(s, t)=(u+vs, vt).$$

The left-invariant Haar measure element is $dudv/v^2$ and the modular function $\Delta(u, v)$ is v^{-1}, the convention being that

$$\Delta(h^{-1})\int_H f(g)dg=\int_H f(gh)dg.$$

The group H is isomorphic to the AN part of the standard Iwasawa decomposition of $SL_2(\boldsymbol{R})$. It is a connected, simply connected solvable group.

Another perspective on H is that it can be considered as the symmetric space $SL_2(\boldsymbol{R})/SO(2)$. As such, it has a natural geometric structure: $SL_2(\boldsymbol{R})$ acts on it by left translation; there is a canonical Riemannian metric derived from the Killing form on the Cartan component $\mathfrak{p}$ consisting of the set of 2×2 symmetric matrices of trace 0. In our realisation, the Riemannian metric is $ds^2=(du^2+dv^2)/v^2$, and if $r>0$, the Riemannian ball centred at $e=(0, 1)$ of radius r is the Euclidean disc centred at $(0, \cosh r)$ and of radius $\sinh r$.

1980 *Mathematics Subject Classification* (1985 revision). Primary 43A80; secondary 42B25.

Key words and phrases. Maximal functions, solvable Lie groups.

* Research supported by the Australian Research Council, the Flinders University Research Budget and the Italian Consiglio Nazionale delle Ricerche.

With these two perspectives in mind, we consider a number of families $\{B_r\}_{r>0}$ of neighbourhoods of the identity element e.

(i) B_r is the Riemannian ball of radius r.

(ii) [These can be thought of as 'squares'.] H being identified with $\boldsymbol{R}^2_+$,

$$B_r = \{(x, y) \colon -r < x < r,\ \mathrm{e}^{-r} < y < \mathrm{e}^r\}\,.$$

(iii) Inversions of (ii):

$$B_r = \{(x, y)^{-1} \colon -r < x < r,\ \mathrm{e}^{-r} < y < \mathrm{e}^r\} = \{(u, v) \colon |u| < vr,\ |\log v| < r\}\,.$$

(iv) [This subsumes (ii) and (iii).] Let ε satisfy $0 \le \varepsilon \le 1$, and suppose that $\phi \colon \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is an increasing function such that

$$\phi(r) \sim r^Q \qquad \text{as} \quad r \to 0+\,, \tag{1}$$

where Q is a postive number, and $\log(\phi)$ satisfies a Lipschitz condition in the interval $[r_0, +\infty)$ for some $r_0 > 0$. Define

$$B_r^{\varepsilon,\phi} = \{(u, v) \colon |u| < v^\varepsilon \phi(r),\ |\log v| < r\}\,.$$

For each such family $\{B_r\}_{r>0}$, we define maximal functions $\mathscr{M}^L$ and $\mathscr{M}^R$:

$$\mathscr{M}^L f(g) = \sup_{r>0} \frac{1}{|gB_r|} \left| \int_{gB_r} f(u)du \right| \tag{2}$$

and

$$\mathscr{M}^R f(g) = \sup_{r>0} \frac{1}{|B_r g|} \left| \int_{B_r g} f(u)du \right|. \tag{3}$$

The measure in (2) and (3) is left Haar measure.

Remarks on the families (i)–(iv). (a) The family (ii) (resp. (iii)) clearly corresponds to the case $\phi(r) = r$, $\varepsilon = 0$ (resp. 1) of (iv). Moreover, $(B_r^{\varepsilon,\phi})^{-1} = B_r^{1-\varepsilon,\phi}$; in particular, the sets $B_r^{1/2,\phi}$ are symmetric.

(b) When the group H is thought of as arising from the standard Iwasawa decomposition of $SL_2(\boldsymbol{R})$, it is conventional to regard it as the semi-direct product $\boldsymbol{R} \rtimes \boldsymbol{R}$, in which the first factor corresponds to the Cartan sub-algebra and the second to the root space of a designated positive root. In this setting, it is very natural to consider the family of squares centred at (0, 0), and the corresponding maximal function. Transforming these ideas to the realization of H as $\boldsymbol{R}^2_+$, we have the family (ii).

(c) If $\varepsilon = 1/2$, and $\phi(r) = 2\sinh(r/2)$, the family $\{B_r^{1/2,\phi}\}_{r>0}$ is equivalent to the family (i), since

$$B_r \subseteq B_r^{1/2,\phi} \subseteq B_R\,,$$

where $\cosh R = 2\cosh r - 1$ and $|B_R| = 2|B_r|$.

We study the weak-type (1, 1), and strong type (p, p) $(1<p<+\infty)$ boundedness of the two maximal functions defined with respect to the various families. We note that, in case (i), $\mathscr{M}^L$ is known to be of weak type (1, 1) [5], as it is also in case (ii) [3]. Actually, the results in [3] and [5] are considerably more general, and deal with arbitrary noncompact Riemannian symmetric spaces, and general AN groups, respectively.

Our main results are that

If $0\leq\varepsilon<1/2$, the maximal operator $\mathscr{M}^L$ is of weak type (1, 1) for every ϕ; therefore, by interpolation, it is bounded on L^p for every $p\in(1, +\infty]$. If $1/2<\varepsilon\leq 1$, then $\mathscr{M}^L$ is unbounded on L^p for p close to 1; à fortiori, it is not of weak type (1, 1).

If $0\leq\varepsilon<1$, $\mathscr{M}^R$ is unbounded on L^p for every $p\in[1, +\infty)$. A fortiori it is not of weak type (1, 1). On the other hand, if $\varepsilon=1$, $\mathscr{M}^R$ is bounded on L^p for every $p\in(1, +\infty]$.

We shall see that, if $\varepsilon=1$, $\mathscr{M}^R$ can be interpreted equivalently as the maximal function on $\boldsymbol{R}^2$ taken with respect to a non-homogenoeus family of rectangles. Giulini and Sjögren [4] have shown that the weak type (1, 1) behaviour of such maximal functions depends on the function ϕ. Most notably, the maximal function $\mathscr{M}^R$ turns out to be not of weak type (1, 1) if $\phi(r)=r$; this is the case (iii).

We remark that boundedness results for the right- (resp. left-) invariant maximal functions would be useful for controlling convolution operators defined by convolution on the left (resp. right) by appropriate kinds of kernels. Indeed, the right-invariant convolution operators are, in many contexts, the 'natural' operators: for instance, they include homogeneous differential operators generated by right-invariant vector fields. Such operators are (depending on the degree) either skew-adjoint or self-adjoint, with respect to left Haar measure.

A general discussion of singular integrals and maximal functions on solvable groups can be found in [2], where some of the problems treated in the present paper were first raised.

2. Left-invariant maximal functions. Simple computations show that

$$|B_r^{\varepsilon,\phi}|=\begin{cases}4(1-\varepsilon)^{-1}\sinh[(1-\varepsilon)r]\phi(r)\,, & 0\leq\varepsilon<1\\ 4r\phi(r)\,, & \varepsilon=1\,.\end{cases} \tag{4}$$

THEOREM 1. *If* $0\leq\varepsilon<1/2$, *the maximal operator* $\mathscr{M}^L$ *is of weak type* (1, 1).

PROOF. This follows the lines of Clerc and Stein [1]. See also [3]. We split $\mathscr{M}^L$ into two maximal operators, $\mathscr{M}^L=\mathscr{M}_0^L+\mathscr{M}_\infty^L$, where

$$\mathscr{M}_0^L f(g)=\sup_{0<r<1}|B_r^{\varepsilon,\phi}|^{-1}\left|\int_{gB_r^{\varepsilon,\phi}}f(u)du\right|,$$

and

$$\mathcal{M}_\infty^L f(g) = \sup_{r \geq 1} |B_r^{\varepsilon,\phi}|^{-1} \left| \int_{gB_r^{\varepsilon,\phi}} f(u)du \right| .$$

The operator $\mathcal{M}_0^L$ takes into account the 'local' behaviour of f. Now

$$B_r^{0,e^{-\varepsilon\phi}} \subseteq B_r^{\varepsilon,\phi} \subseteq B_r^{0,e^{\varepsilon\phi}} \qquad (0 < r < 1) . \tag{5}$$

The ratios of the measures of the sets in (5) remain bounded as $r \to 0$; see (4). Consequently, it is enough to prove that $\mathcal{M}_0^L$ is of weak type (1, 1) when $\varepsilon = 0$. Since $\phi(r) \sim r^Q$ as $r \to 0+$, standard Vitali arguments apply to coverings of a set by translates of the neighbourhoods $\{B_r^{0,\phi}\}_{0<r<1}$. So the operator $\mathcal{M}_0^L$ is of weak type (1, 1), by well-worn arguments.

As for the operator $\mathcal{M}_\infty^L$, it turns out to be *norm bounded on* L^1, as we now show.

For $(u, v) \in H$, let

$$R(u, v) = \inf\{r \geq 1 : (u, v) \in B_r^{\varepsilon,\phi}\}$$

and

$$\tau(u, v) = |B_{R(u,v)}^{\varepsilon,\phi}|^{-1} .$$

Then

$$|gB_r^{\varepsilon,\phi}|^{-1} \left| \int_{gB_r^{\varepsilon,\phi}} f(z)dz \right| \leq |B_r^{\varepsilon,\phi}|^{-1} \int \chi_{B_r^{\varepsilon,\phi}}(z) |f(gz)| dz ,$$

and so

$$\mathcal{M}_\infty^L f(g) \leq \int_H \tau(z) |f(gz)| dz = |f| * \check{\tau}(z) .$$

It suffices to show that $\check{\tau} \in L^1(H)$, equivalently, that τ is integrable with respect to right Haar measure $dudv/v$. Making the successive changes of variables $u = xv^\varepsilon$, $v = e^{y'}$, $y = y'(1-\varepsilon)$, we see that

$$\|\check{\tau}\|_1 \leq C \int_{\mathbf{R}} dy \exp\left(\frac{\varepsilon y}{1-\varepsilon}\right) \int_{\mathbf{R}_+} \frac{dx}{1 + \phi[|y|/(1-\varepsilon)] \sinh|y| + x \sinh[(1-\varepsilon)\phi^{-1}(x)]}$$
$$< C(I_1 + I_2) ,$$

say, where

$$I_1 = \int_{\mathbf{R}} dy \exp\left(\frac{\varepsilon y}{1-\varepsilon}\right) \int_0^{\phi(|y|/(1-\varepsilon))} \frac{dx}{1 + \phi[|y|/(1-\varepsilon)] \sinh|y|} , \tag{6}$$

and

(7) $$I_2=\int_{\mathbf{R}} dy \exp\left(\frac{\varepsilon y}{1-\varepsilon}\right)\int_{\phi(|y|/(1-\varepsilon))}^{+\infty} \frac{dx}{1+x\sinh[(1-\varepsilon)\phi^{-1}(x)]}.$$

Now

$$I_1\leq C\int_0^{+\infty} dy \exp\left[\left(\frac{2\varepsilon-1}{1-\varepsilon}\right)y\right],$$

which is finite if $0\leq\varepsilon<1/2$. Inverting the order of integration in (7), we see that

$$I_2\leq C\int_0^{+\infty} dx\frac{\exp(\varepsilon\phi^{-1}(x))}{1+x\sinh[(1-\varepsilon)\phi^{-1}(x)]}=C\int_0^{+\infty} d\phi(z)\frac{\exp(\varepsilon z)}{1+\phi(z)\sinh(1-\varepsilon)z}.$$

Keeping in mind that $\log\phi(z)$ satisfies a Lipschitz condition, over $z\geq r_0$, we have that

$$I_2\leq D+C\int_{r_0}^{+\infty}\exp[(2\varepsilon-1)z]\frac{d\phi(z)}{\phi(z)},$$

which is finite if $0\leq\varepsilon<1/2$.

THEOREM 2. *If* $\varepsilon=1$, *the operator* $\mathcal{M}^L$ *is unbounded on* L^p *for every finite* p.

PROOF. If $\varepsilon=1$, then

$$B_r^{\varepsilon,\phi}=\{(u,v)\colon -\phi(r)<v^{-1}u<\phi(r),\ \mathrm{e}^{-r}<v<\mathrm{e}^r\},$$
$$(x,y)B_r^{\varepsilon,\phi}=\{(s,t)\colon x-\phi(r)t<s<x+\phi(r)t,\ y\mathrm{e}^{-r}<t<y\mathrm{e}^r\},$$

and

(8) $$|(x,y)B_r^{\varepsilon,\phi}|=4r\phi(r).$$

For a given point (x, y) and $r>0$, $(x,y)B_r^{1,\phi}$ is a trapezium about (x, y), with base line at level $y\mathrm{e}^{-r}$, top at level $y\mathrm{e}^r$, and side slopes equal to $-1/\phi(r)$, $1/\phi(r)$. Take a test function of the form

$$f=\chi_{[2,+\infty)\times[1/2,1]}(x,y)x^{-\delta},$$

δ being such that $p\delta>1$. Then $f\in L^p(H)$.

If $0<y<1/2$, choose r so that $y\mathrm{e}^r=1$. Then

(9) $$y\mathrm{e}^{-r}=y^2<1/2.$$

If we impose the condition

(10) $$x>2+\phi(|\log y|),$$

then, in view of (8), (9) and (10),

$$\mathscr{M}^L f(x,y) = \sup_{r>0} \frac{1}{4r\phi(r)} \int_{ye^{-r}}^{ye^r} t^{-2}dt \int_{|s-x|<t\phi(r)} f(s,t)ds$$
$$\geq \frac{C}{|\log y|\phi(|\log y|)} \int_{1/2}^{1} t^{-2}dt \int_{|s-x|<t\phi(|\log y|)} s^{-\delta}ds \tag{11}$$
$$\geq C|\log y|^{-1}[x+\phi(|\log y|)]^{-\delta}\,.$$

Taking the condition (10) into account, we get from (11)

$$\|\mathscr{M}^L f\|_p^p \geq C \int_0^{1/2} y^{-2}|\log y|^{-p}[2+2\phi(|\log y|)]^{1-p\delta}dy\,. \tag{12}$$

From the Lipschitz condition on ϕ, it follows that

$$\phi(|\log y|) \leq Cy^{-K} \qquad \text{if} \quad y \in (0, y_0)\,,$$

for suitable C, K and y_0. We can therefore choose δ so that the integral in (12) diverges, without violating the restriction $p\delta > 1$.

THEOREM 3. *If* $1/2 < \varepsilon < 1$, *then* $\mathscr{M}^L$ *is unbounded on* $L^p(H)$ *for* p *sufficiently close to* 1 (*depending on* ε). *Consequently,* $\mathscr{M}^L$ *is not of weak type* (1, 1).

PROOF. The proof uses similar ideas to the one just given for Theorem 2. We sketch the necessary modifications.

In this instance,

$$(x,y)B_r^{\varepsilon,\phi} = \{(s,t) \colon |s-x| < \phi(r)y^{1-\varepsilon}t^{\varepsilon},\ ye^{-r} < t < ye^r\}\,.$$

Take the same test function f, and, for $0 < y < 1/2$, choose the corresponding r as before. Then, (compare (10)), provided

$$x > 2 + \phi(|\log y|)y^{1-\varepsilon}\,, \tag{13}$$

we have, in view of (4) and (9),

(14)
$$\mathscr{M}^L f(x,y) \geq C \sinh [(1-\varepsilon)|\log y|]^{-1}\phi(|\log y|)^{-1}\phi(|\log y|)y^{1-\varepsilon}(x+y^{1-\varepsilon}\phi(|\log y|)^{-\delta}$$
$$\geq Cy^{2(1-\varepsilon)}(x+y^{1-\varepsilon}\phi(|\log y|))^{-\delta}\,.$$

Compare (11). From (14), it can be seen that

$$\|\mathscr{M}^L f\|_p^p \geq C \int_0^{1/2} y^{2[p(1-\varepsilon)-1]}(2+2y^{1-\varepsilon}\phi(|\log y|))^{1-p\delta}\,dy\,. \tag{15}$$

Compare (12). If $2[p(1-\varepsilon)-1] < -1$, then the right side of (15) is infinite if $p\delta - 1$ is sufficiently small.

3. Right-invariant maximal functions. We shall prove that if $0 \leq \varepsilon < 1$, $\mathscr{M}^R$ is unbounded on L^p for every $p \in [1, +\infty)$. Taking into account the results of the previous section, this brings to light the asymmetry of the maximal operators under consideration.

On the other hand, in the limiting case $\varepsilon = 1$, $\mathscr{M}^R$ is bounded on L^p for every $p \in (1, +\infty)$. This maximal operator will be seen to be equivalent to the maximal operator on $\boldsymbol{R}^2$ taken with respect to a certain non-homogeneous family of rectangles. Giulini and Sjögren [4] have shown that the weak type behaviour of such maximal functions depends on the function ϕ; in particular, when $\phi(r) = r$, it is not of weak type (1, 1).

THEOREM 4. *If $0 \leq \varepsilon < 1$, the maximal operator $\mathscr{M}^R$ is unbounded on $L^p(H)$ for every $p \in [1, +\infty)$.*

PROOF. This is similar to the proof of Theorem 3. For a fixed $g = (x, y)$ and $r > 0$,

$$B_r^{\varepsilon,\phi} g = \{(u, v) \colon y e^{-r} < v < y e^r, \, |u - vxy^{-1}| < v^\varepsilon y^{-\varepsilon} \phi(r)\} . \tag{16}$$

It follows from (4) that

$$|B_r^{\varepsilon,\phi} g| = \Delta(g) |B_r^{\varepsilon,\phi}| = 4y^{-1}(1-\varepsilon)^{-1} \sinh[(1-\varepsilon)r]\phi(r) .$$

The geometry of these sets is quite complicated, but if $\varepsilon = 0$, they are parallelograms of height $2r$ whose sides have slope xy^{-1} and whose base and top are parallel to the x-axis.

Consider the test function

$$f(x, y) = \chi_{[2, +\infty) \times [1, 2]}(x, y) h(x) ,$$

where $h(x) = x^{-1/p} \log^{-\delta} x$, and where, in the first instance, δ is restricted so that $p\delta > 1$, making f integrable. If we restrict y so that $y \in [2, +\infty)$, and choose r so that $r = \log y$, then the lower edge of the set (16) lies on the lower edge of the support of f; further,

$$y e^r = y^2 > 2 .$$

If we now restrict x so that

$$x > 2y + 2^\varepsilon y^{1-\varepsilon} \phi(\log y) ,$$

we obtain, from the monotonicity of h in the interval $[2, +\infty)$,

$$\begin{aligned} \mathscr{M}^R f(x, y) &= \sup_{r>0} \frac{(1-\varepsilon)y}{4 \sinh[(1-\varepsilon)r]\phi(r)} \int_{ye^{-r}}^{ye^r} t^{-2} dt \int_{|s - txy^{-1}| < t^\varepsilon y^{-\varepsilon}\phi(r)} f(s, t) ds \\ &\geq C \int_1^2 t^{\varepsilon - 2} h[txy^{-1} + t^\varepsilon y^{-\varepsilon} \phi(\log y)] dt \geq C h[2xy^{-1} + 2^\varepsilon y^{-\varepsilon} \phi(\log y)] . \end{aligned}$$

Now change variables by setting $t = 2xy^{-1} + 2^\varepsilon y^{-\varepsilon} \phi(\log y)$. Then

$$\|\mathscr{M}^R f\|_p^p \geq C \int_2^{+\infty} \log^{1-p\delta}[4 + 4^\varepsilon y^{-\varepsilon} \phi(\log y)] \frac{dy}{y} .$$

Since $\phi(\log y)$ grows at most polynomially, the last integral diverges if we now impose the further condition $p\delta<2$.

The theorem covers two important particular cases: those where $\varepsilon=0$, $\phi(r)=r$, and $\varepsilon=1/2$, $\phi(r)=2\sinh(r/2)$. See Remarks (a)–(c) of the Introduction.

COROLLARY. *The right maximal operators $\mathscr{M}^R$ with respect to the families of 'squares' and of hyperbolic spheres are unbounded on $L^p(H)$ for every $p\in[1,+\infty)$.*

THEOREM 5. *If $\varepsilon=1$ and $1<p<+\infty$, the maximal operator $\mathscr{M}^R$ is bounded on $L^p(H)$.*

PROOF. Let

$$A_r f(x,y)=|B_r^{1,\phi}(x,y)|^{-1}\int_{B_r^{1,\phi}(x,y)} t^{-2}f(s,t)dsdt\,.$$

Since $|B_r^{1,\phi}(x,y)|=y^{-1}4r\phi(r)$, if follows from (16) that

$$A_r f(x,y)=\frac{y}{4r\phi(r)}\int_{ye^{-r}}^{ye^r} t^{-2}dt\int_{t(x-\phi(r))/y}^{t(x+\phi(r))/y} f(s,t)ds\,.$$

The mapping J: $(u,v)\mapsto(ue^v,e^v)$ is a measure-preserving mapping of $\boldsymbol{R}^2$ onto H. Now

$$(A_r f)\circ J(u,v)=\frac{e^v}{4r\phi(r)}\int_{v-r}^{v+r}dv'\int_{u-e^{-v}\phi(r)}^{u+e^{-v}\phi(r)} f\circ J(u',v')du'\,, \tag{17}$$

also so

$$\sup_{r>0}|(A_r f)\circ J(u,v)|\leq\sup_{r>0}(2r)^{-1}\int_{v-r}^{v+r}dv'\sup_{q>0}(2q)^{-1}\int_{u-q}^{u+q}|f\circ J(u',v')|du'\,. \tag{18}$$

The right side of (18) is the iterated one-dimensional Hardy-Littlewood maximal operator applied to the function $|f\circ J|$, so

$$\|\sup_{r>0}(A_r f)\circ J\|_{L^p(\boldsymbol{R}^2)}\leq C\|f\circ J\|_{L^p(\boldsymbol{R}^2)}\,,$$

and since J is measure-preserving, we can conclude that

$$\|\mathscr{M}^R f\|_p=\|\sup_{r>0}|A_r f|\|_p\leq C\|f\|_p\,.$$

REMARKS ON THE PROOF. It can be seen from (17) that $\mathscr{M}^R$ can be interpreted as the maximal operator on $\boldsymbol{R}^2$ with respect to a certain family of rectangles which is not translation-invariant. The weak type (1, 1) boundedness of these operators has been investigated in [4]. It is shown there that when $\varepsilon=1$, $\mathscr{M}^R$ is of weak type (1, 1) if $\phi(r)\geq e^{Kr}$ for some $K>1$ (and large r); on the other hand, $\mathscr{M}^R$ is not of weak type (1, 1) if $\phi(r)\leq\psi(r)e^r$ and ψ is an increasing function such that $\log\psi(r)\leq Cr^q$ for some $q<1$.

REFERENCES

[1] J. L. CLERC AND E. M. STEIN, L^p-multipliers for noncompact symmetric spaces, Proc. Nat. Acad. Sci. USA 71 (1974), 3911–3913.

[2] G. I. GAUDRY, Singular integrals and maximal functions on certain Lie groups, Proc. Centre for Math. Analysis (Canberra) 14 (1986), 21–25.

[3] G. I. GAUDRY, S. GIULINI, A. HULANICKI AND A. M. MANTERO, Hardy-Littlewood maximal functions on some solvable Lie groups, J. Austral. Math. Soc. (Series A) 45 (1988), 78–82.

[4] S. GIULINI AND P. SJÖGREN, A note on maximal functions on a solvable Lie group. To appear, Arch. Math. (Basel).

[5] J.-O, STRÖMBERG, Weak type L^1 estimates for maximal functions on noncompact symmetric spaces, Ann. of Math. 114 (1981), 115–126.

SCHOOL OF MATHEMATICAL SCIENCES
THE FLINDERS UNIVERSITY OF SOUTH AUSTRALIA
BEDFORD PARK. S.A. 5042
AUSTRALIA

DIPARTIMENTO DI MATEMATICA "F. ENRIQUES"
UNIVERSITÀ DEGLI STUDI DI MILANO
VIA C. SALDINI 50
MILANO
ITALY

ISTITUTO MATEMATICO DI INGEGNERIA
UNIVERSITÀ DEGLI STUDI DI GENOVA
VIA L.B. ALBERTI 4
GENOVA
ITALY