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THE STANDARD CR STRUCTURE ON THE UNIT TANGENT BUNDLE

Dedicated to Professor Takesi Kotake on his sixtieth birthday

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Abstract. The unit tangent bundle of a Riemannian manifold is one of popular examples of contact manifolds. It has the standard CR structure which is not integrable in general. We study the recently defined gauge invariant of type (1,3) of the CR structure and show that the invariant vanishes, if and only if the Riemannian manifold is of constant curvature -1.

Introduction. Popular examples of contact manifolds are the odd-dimensional spheres and the unit tangent bundles of Riemannian manifolds. These examples have the standard contact Riemannian structures and their associated CR structures.

A contact Riemannian structure satisfying the integrability condition $Q=0$ corresponds to a strongly pseudo-convex integrable CR structure. There are rich results in the study of strongly pseudo-convex integrable CR structures. If one wants to generalize the Chern-Moser-Tanaka invariant of (1,3)-type of CR structures to a (1,3)-type invariant of gauge transformations of contact Riemannian structures, it seems to be necessary that one fixes a linear connection (Tanno [12]) or a nowhere vanishing m-form on the contact manifold M, where $\dim M=m$ (cf. §3).

In §4 we give the expression for our (1,3)-type invariant B of the standard contact Riemannian structure on the unit tangent bundle of a Riemannian manifold.

THEOREM A. *Let (M, g) be a Riemannian manifold of dimension $m\geq 3$ and (T^1M, η, g^*) be its unit tangent bundle with the standard contact Riemannian structure (η, g^*). Then the gauge invariant B of (1,3)-type of (T^1M, η, g^*) vanishes, if and only if (M, g) is of constant curvature -1.*

It is worth noticing that if (M, g), $m\geq 3$, is of constant curvature -1 then the CR structure associated with the contact Riemannian structure (η, g^*) on the unit tangent bundle T^1M is integrable and yet the natural almost complex structure of the ambient space, i.e., the tangent bundle TM, is not integrable.

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1. A contact form and its associated Riemannian metrics. A 1-form η on a manifold M of dimension $m=2n+1$ is called a contact form if it satisfies $\eta\wedge(d\eta)^n\neq 0$ everywhere on M. The equivalence class of contact forms containing η is denoted by $\{\eta\}$, which is called a contact structure. The pair $(M, \{\eta\})$ is called a contact manifold. By P we denote the subbundle of the tangent bundle TM of M defined by $\eta=0$, and at the same time the $2n$-dimensional distribution on M, which is called the contact distribution associated with the contact structure.

By a contact manifold (M, η) we mean a contact manifold $(M, \{\eta\})$ with a fixed contact form η. Since $d\eta$ is of rank $2n$, there is a unique vector field ξ such that $\eta(\xi)=1$ and $L_\xi\eta=0$, where L_ξ denotes the Lie derivation by ξ. ξ is called the associated vector field. It is known that there exist a Riemannian metric g and a (1,1)-tensor field ϕ satisfying $\phi\xi=0$, $\eta\cdot\phi=0$, $g(X, \xi)=\eta(X)$ for $X\in T_xM$, and

(1) $$\phi\phi=-I+\eta\otimes\xi\,,\qquad d\eta(X, Y)=2g(X, \phi Y)\,,$$

(2) $$g(X, Y)=g(\phi X, \phi Y)+\eta(X)\eta(Y)$$

for X, $Y\in T_xM$, $x\in M$. g is called an associated Riemannian metric. Although g and ϕ are not unique for η, the pair g and ϕ are canonically related. The pair (η, g) is called a contact Riemannian structure. The restriction $\hat\phi=\phi\,|\,P$ of ϕ to P defines an almost complex structure to P. So (η, g) is equivalent to (η, ϕ) or $(\eta, \hat\phi)$, where $\hat\phi$ is an almost complex structure for P such that $\hat g(X, Y)$ $(=-(1/2)d\eta(X, \hat\phi Y))$, X, $Y\in P$, defines an almost Hermitian structure for P. The pair $(\eta, \hat\phi)$ is called the CR structure associated with the contact Riemannian structure (η, g). A contact Riemannian structure (η, g) is a strongly pseudo-convex integrable CR structure, if it satisfies the integrability condition $Q=0$, where Q is a (1,2)-tensor field on M defined by

(3) $$Q(X, Y)=(\nabla_Y\phi)(X)+(\nabla_Y\eta)(\phi X)\xi+\eta(X)\phi\nabla_Y\xi$$

for X, $Y\in T_xM$ (cf. [10]). A natural question is stated as follows:

Which metric is most proper among Riemannian metrics associated with η?

One method of finding conditions of nice Riemannian metrics is to study variational problems and their critical points. Blair [3], [4], Chern and Hamilton [5], and Tanno [10] studied variational problems on contact manifolds. Let (M, η) be a compact contact manifold. Then an associated Reimannian metric g is critical with respect to the Dirichlet energy functional $E(g)=\int_M\|L_\xi g\|^2dM$ if and only if

(4) $$\nabla_\xi L_\xi g=2L_\xi g\cdot\phi\,.$$

An associated Riemannian metric g satisfying (4) is called E-critical. One may consider that a contact Riemannian structure (η, g) belongs to a nice class if g is E-critical.

2. Gauge transformations of contact Riemannian structure. If one changes η to $\tilde\eta=\sigma\eta$ in $\{\eta\}$ for a positive function σ, then one may hope that the choice of an associated

Riemannian metric $\tilde{g}$ should be determined in a natural way. This is done by assuming $\tilde{\phi} \mid P = \phi \mid P$. Namely we have

(5) $\tilde{\xi} = (1/\sigma)(\xi + \zeta)$, $\zeta = (1/2\sigma)\phi \operatorname{grad} \sigma$, $\tilde{\phi} = \phi + (1/2\sigma)\eta \otimes (\operatorname{grad} \sigma - \xi\sigma \cdot \xi)$,

(6) $$\tilde{g} = \sigma g - \sigma(\eta \otimes w + w \otimes \eta) + \sigma(\sigma - 1 + \| \zeta \|^2)\eta \otimes \eta ,$$

where w is dual to ζ with respect to g. $\tilde{g}$ and g are conformal with respect to P. We call $(\eta, g) \to (\tilde{\eta}, \tilde{g})$ a gauge transformation of contact Riemannian structure. A natural problem is stated as follows:

What are gauge invariants and what are their properties in contact geometry?

Using the generalized Tanaka-Webster scalar curvature S^T of contact Riemannian structure, a scalar gauge invariant $K_{(\eta,g)}$ of contact Riemannian structure (η, g) is defined on a compact contact manifold (Tanno [10]). This invariant is a natural generalization of the Jerison-Lee invariant for strongly pseudo-convex integrable CR manifolds ([7]).

3. A gauge invariant of (1,3)-type. Let R and ρ denote the Riemannian curvature tensor and the Ricci tensor of a contact Riemannian manifold (M, η, g). Let p be a (0,2)-tensor field defined by $2p = L_\xi g$. Define $\rho^\#$ and $p^\#$ by $g(X, \rho^\# Y) = \rho(X, Y)$ and $g(X, p^\# Y) = p(X, Y)$. Since the action of ϕ, Q, $p^\#$ and $\nabla_\xi p^\#$ to ξ is trivial, by the same notations ϕ, Q, $p^\#$ and $\nabla_\xi p^\#$, we mean also their restrictions to P. Now we denote the restriction of R and $\rho^\#$ to P with respect to g by $\hat{R}$ and $\hat{\rho}^\#$, i.e., $\hat{R} \in \Gamma(P \otimes P^{*3})$ and $\hat{\rho}^\# \in \Gamma(P \otimes P^*)$.

By a P-related frame we mean a frame $\{e_j\} = \{e_u, e_0 = \xi; 1 \le u \le 2n\}$ such that $e_u \in P$. The indices u, v, w, x, y, and z run from 1 to $2n$. We define $U \in \Gamma(P^2 \otimes P^{*3})$ by

(7) $$\begin{aligned} U^{vu}_{zxy} = 2[(2/(m+3))[&-\delta^u_x(Q^v_{yw} + Q^v_{wy})\phi^w_z - \phi^u_x(Q^v_{zy} + Q^v_{yz}) + g_{xz}(Q^v_{yw} + Q^v_{wy})\phi^{wu} \\ &- \phi_{xz}(Q^v_{yw} + Q^v_{wy})g^{wu}] + Q^v_{zy}\phi^u_x + \phi_{xz}Q^v_{wy}g^{uw} + Q^v_{xy}\phi^u_z \\ &+ (1/2)(Q^v_{wz} - Q^v_{zw})g^{wu}\phi_{xy} + \phi^v_z Q^u_{xy} + \phi^v_x Q^u_{zy} - (1/2)\phi^{uv}Q^w_{yz}g_{xw}]_{xy} , \end{aligned}$$

where $[\cdots]_{xy}$ denotes the skew symmetric part of $[\cdots]$ with respect to x, y.

Let ω be a nowhere vanishing m-form on M and fix it. Let $dM(g)$ denote the volume element of (M, η, g), i.e., $dM(g) = ((-1)^n/2^n n!)\eta \wedge (d\eta)^n$. We define λ by $dM(g) = \pm e^\lambda \omega$ and $\theta \in \Gamma(P^*)$ by $\theta(X) = X\lambda$ for $X \in P$. By $X \wedge Y$ we denote the operator defined by $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. Now we define $B \in \Gamma(P \otimes P^{*3})$ by

(8) $$\begin{aligned} (m+3)B(X, Y)Z = {} & (m+3)\hat{R}(X, Y)Z - (X \wedge Y)\hat{\rho}^\# Z - \hat{\rho}^\#(X \wedge Y)Z + \phi(X \wedge Y)\hat{\rho}^\#\phi Z \\ & - [\rho(X, \phi Y) - \rho(Y, \phi X)]\phi Z + \phi\hat{\rho}^\#(X \wedge Y)\phi Z \\ & - (1/2)d\eta(X, Y)(\phi\hat{\rho}^\# + \hat{\rho}^\#\phi)Z + [S^T/(m+1) - 4](X \wedge Y)Z \\ & + [S^T/(m+1) + m - 1][-\phi(X \wedge Y)\phi Z + d\eta(X, Y)\phi Z] \\ & + (m-3)[p^\#(X \wedge Y)\phi Z - \phi(X \wedge Y)p^\# Z] \\ & + 6[(X \wedge Y)p^\#\phi Z - \phi p^\#(X \wedge Y)Z] + (m+3)p^\#(X \wedge Y)p^\# Z \\ & - (X \wedge Y)(\nabla_\xi p^\#)Z - (\nabla_\xi p^\#)(X \wedge Y)Z + \phi(\nabla_\xi p^\#)(X \wedge Y)\phi Z \end{aligned}$$

$$+\phi(X\wedge Y)(\nabla_{\xi}p^{\#})\phi Z-((m+3)/(m+1))U(X, Y, Z; \theta)$$

for $X, Y, Z\in P$, where $U(X, Y, Z; \theta)=(\theta_v U^{vu}_{zxy}X^x Y^y Z^z)$. This B is essentially the same as the one defined in [12]. The only difference is θ, where this θ satisfies the same relation $\tilde{\theta}_u-\theta_u=2(n+1)\alpha_u$ for $\sigma=e^{2\alpha}$ as in [12]. We restate Theorem A of [12].

THEOREM. Let (M, η, g) *be a contact Riemannian manifold of dimension* $m=2n+1$ *and let* ω *be a fixed nowhere vanishing m-form on* M. *Then* $B\in\Gamma(P\otimes P^{*3})$ *defined by* (8) *is a gauge invariant of type* (1,3) *of the contact Riemannian structure. Furthermore*:

(i) *If* $B=0$, *then the* CR *structure* $(\eta, \hat{\phi})$ *associated with* (η, g) *is integrable.*

(ii) *If* $Q=0$, *then* B *reduces to the Chern-Moser-Tanaka invariant.*

Note that the condition (4) is written as $\nabla_{\xi}p^{\#}=2p^{\#}\phi$. So, if (4) is satisfied on (M, η, g), then B of (M, η, g) takes somewhat simpler form.

We call B' defined by $B'^{u}_{zxy}=B^{u}_{zxy}+(1/(m+1))\theta_v U^{vu}_{zxy}$ the main part of the invariant B. Now we put $B'_{uzxy}=g_{uw}B'^{w}_{zxy}$. Then we have the following:

$$B'_{uzxy}=-B'_{uzyx}, \qquad B'_{uzxy}=-B'_{zuxy}, \tag{9}$$

$$(m+3)(B'_{uzxy}-B'_{xyuz})=\phi_{xz}Z_{yu}-\phi_{yz}Z_{xu}-\phi_{xu}Z_{yz}+\phi_{yu}Z_{xz}, \tag{10}$$

where we have put

$$Z_{xy}=\rho_{xw}\phi^{w}_{y}+\rho_{yw}\phi^{w}_{x}-4(n-1)p_{xy}+2\nabla_{\xi}p_{xw}\phi^{w}_{y}. \tag{11}$$

Z_{xy} is symmetric with respect to x, y. By Remark (i) of [12] we see that if $Q=0$ then $Z_{xy}=0$, and in particular $B'_{uzxy}=B'_{xyuz}$ holds.

REMARK (i). If $m=3$, then we have $B=0$. In fact, $m=3$ implies that $Q=0$ holds (and hence $U=0$) and that R is expressed in terms of ρ, g and S. In simplifying B' we use the following, (i): $S^T=S-\rho(\xi, \xi)+4n$ (cf. (8.2) of [10]), (ii): the trace of $p^{\#}$ (also $\nabla_{\xi}p^{\#}$) vanishes (cf. [10]) and (iii): $\rho(\xi, \xi)=2n-\| p\|^2$ (cf. Blair [2]), where $\| p\|^2=\|(1/2)L_{\xi}\phi\|^2$. By calculating the component B'^{1}_{212} with respect to an orthonormal basis $\{e_1, e_2=\phi e_1, \xi\}$, we have $B'=0$ for $n=1$.

4. The tangent bundle and unit tangent bundle. Let (M, g) be an m-dimensional Riemannian manifold and let TM denote its tangent bundle with projection π. For a local coordinate neighborhood (U, x^i) in M, $(\pi^{-1}U, x^i, v^i)$ is a local coordinate neighborhood in TM, where $v=(x, v)=(v^i\partial/\partial x^i)\in\pi^{-1}U$. A vector field W on TM or tangent vector W at a point of TM is denoted by $W=(W^i, W^{m+i})$, where $W=W^i\partial/\partial x^i+W^{m+i}\partial/\partial v^i$. For a vector field X on M or a tangent vector X at a point of M its horizontal lift X^H is defined by $X^H=(X^i, -\Gamma^i_{jk}v^k X^j)$, where (Γ^i_{jk}) denotes the coefficients of the Riemannian connection ∇. The vertical lift X^V of X is given by $X^V=(0, X^i)$. Then we have the following: $[X^V, Y^V]=0$, $[X^H, Y^V]=(\nabla_X Y)^V$,

$$[X^H, Y^H]=[X, Y]^H-(R(X, Y)v)^V. \tag{12}$$

A natural almost complex structure tensor J of TM is defined by $JX^H = X^V$ and $JX^V = -X^H$. J is integrable if and only if (M, g) is locally flat (Dombrowski [6]).

The Sasaki metric $\tilde{g}$ of TM is defined by

$$\tilde{g}(X^H, Y^H) = g(X, Y)\cdot\pi\,, \quad \tilde{g}(X^H, Y^V) = 0\,, \quad \tilde{g}(X^V, Y^V) = g(X, Y)\cdot\pi\,.$$

$\tilde{g}(JZ, JW) = \tilde{g}(Z, W)$ is easily verified, and $(TM, \tilde{g}, J)$ is an almost Hermitian manifold. The Riemannian connection $\tilde{\nabla}$ of $\tilde{g}$ is given by the following (cf. Sasaki [9]):

$$\begin{aligned} \tilde{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H - (1/2)(R(X, Y)v)^V\,, \\ \tilde{\nabla}_{X^H} Y^V &= (\nabla_X Y)^V - (1/2)(R(Y, v)X)^H\,, \\ \tilde{\nabla}_{X^V} Y^H &= -(1/2)(R(X, v)Y)^H\,, \quad \tilde{\nabla}_{X^V} Y^V = 0\,. \end{aligned} \tag{13}$$

The geodesic flow vector $\xi_0 = v^i(\partial/\partial x^i)^H$ on TM satisfies $\tilde{\nabla}_{\xi_0}\xi_0 = 0$. TM admits a natural 1-form $\eta = (1/2)\eta_0 = (1/2)g_{ij}v^j dx^i$, which defines the standard contact structure on T^1M. We have $2d\eta(Z, W) = \tilde{g}(Z, JW)$ for any $Z, W \in T_vTM$. This means that $(TM, \tilde{g}, J)$ is an almost Kählerian manifold, which is Kählerian if and only if (M, g) is locally flat.

The unit tangent bundle $T^1 M$ of M is a submanifold of TM defined by $g_{ij}v^iv^j = 1$ and so $n = v^i\partial/\partial v^i = v^i(\partial/\partial x^i)^V$ is a unit normal. By (13), etc., we obtain

$$\tilde{\nabla}_{X^H} n = \tilde{\nabla}_{X^H}\left(v^i\left(\frac{\partial}{\partial x^i}\right)^V\right) = 0\,, \quad \tilde{\nabla}_{X^V} n = X^V\,, \tag{14}$$

and hence the Weingarten map A is characterized by $AX^H = 0$, $A\xi_0 = 0$ and $AX^V = -X^V$, where $v \perp X \in T_xM$.

By Z'^T we denote the tangent part of Z' to T^1M for $Z' \in T_vTM$, and by g^* we denote the (1/4)-times the induced metric on T^1M form $\tilde{g}$ on TM. We define ϕ and ξ by

$$\xi = 2\xi_0 = -2Jn\,, \quad \phi W = (JW)^T\,, \quad W \in T_vT^1M\,. \tag{15}$$

Then $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, $\eta(Z) = g^*(\xi, Z)$ and $2g^*(Z, \phi W) = d\eta(Z, W)$ hold. So, $(\eta, g^*) = (\phi, \xi, \eta, g^*)$ defines a contact Riemannian structure on T^1M, which we call the standard contact Reimannian structure on T^1M.

REMARK (ii). The second fundamental form h of T^1M in TM is degenerate. However, the hybrid part $(1/2)[h(X, Y) + h(JX, JY)]$ of h restricted to P is non-degenerate.

As for $\nabla^*_Z\xi$ and $\nabla^*_Z\phi$ we have the following (cf. Tashiro [13], Blair [1]).

$$\nabla^*_{X^H}\xi = \tilde{\nabla}_{X^H}\left(2v^i\left(\frac{\partial}{\partial x^i}\right)^H\right) = -(R(X, v)v)^V\,, \tag{16}$$

$$\nabla^*_{X^V}\xi = -2\phi X^V - (R(X, v)v)^H\,, \quad X \perp v\,, \tag{17}$$

$$(\nabla^*_{X^H}\phi)Y^H = \tilde{\nabla}_{X^H} Y^V - J\tilde{\nabla}_{X^H} Y^H = -(1/2)(R(X, v)Y)^H\,, \tag{18}$$

$$(19)\qquad (\nabla^*_{X^H}\phi)Y^V=(1/2)[(R(X,v)Y)^V]^T\,,\qquad Y\perp v\,,$$

$$(20)\qquad (\nabla^*_{X^V}\phi)Y^H=(1/2)[(R(X,v)Y)^V]^T-2\eta(Y^H)X^V\,,\qquad X\perp v\,,$$

$$(21)\qquad (\nabla^*_{X^V}\phi)Y^V=(1/2)(R(X,v)Y)^H+(1/2)\tilde{g}(X^V,Y^V)\xi\,,\qquad X,Y\perp v\,.$$

PROPOSITION 4.1. *For* $\dim M=m\geq 3$, *the* CR *structure* $(\eta,\hat{\phi})$ *on* T^1M *associated with the standard contact Riemannian structure* (η, g^*) *is integrable, if and only if* (M,g) *is of constant curvature.*

PROOF. For $X, Y\in T_xM$ such that $X, Y\perp v$, by the definition (3) of Q and (16)–(21) we obtain the following:

$$Q(X^H,Y^H)=-Q(X^V,Y^V)=-(1/2)[R(Y,v)X-g(R(Y,v)X,v)v]^H\,,$$
$$Q(X^H,Y^V)=Q(X^V,Y^H)=(1/2)[R(Y,v)X-g(R(Y,v)X,v)v]^V\,.$$

Therefore, $Q=0$ is equivalent to the relation $R(X,Y)Z\perp v$ for any X, Y, $Z\perp v$. This means that $R(X,Y)Z$ is a linear combination of X, Y and Z, and hence (M,g) is of constant curvature, if the dimension $m\geq 3$ (cf. Ogiue [8]). q.e.d.

REMARK (iii). Proposition 4.1 shows that the unit tangent bundle T^1M of a Riemannian manifold (M,g), $m\geq 3$, of non-zero constant curvature is an interesting example of integrable CR manifolds, in the sense that the almost complex structure J of the ambient space TM is not integrable.

REMARK (iv). The associated vector field ξ of the standard contact Riemannian structure (η, g^*) on T^1M is a Killing vector field if and only if (M,g) is of constant curvature 1 (Tashiro [13]).

In the following we denote by $m'=n+1$ the dimension of M. It is convenient to use the following basis of $T_{(x,v)}T^1M$ at a point (x,v) of T^1M:

$$\{\dot{e}_u, \dot{e}_0;\ 1\leq u\leq 2n\}=\{\dot{e}_\alpha=e^H_\alpha,\ \dot{e}_{n+\alpha}=e^V_\alpha,\ \dot{e}_0=\xi_0=v^H;\ 1\leq\alpha\leq n\}\,,$$

where $\{e_\alpha, e_0=e_{n+1}=v\}$ is a basis of T_xM at x such that $g(e_\alpha,v)=0$. We use the following ranges of indices:

$$1\leq\alpha,\beta,\lambda,\ldots\leq n\,,\quad 1\leq a,b,r,s,\ldots\leq n+1\,,\quad 1\leq u,v,\ldots\leq 2n\,,\quad \bar{\alpha}=n+\alpha\,.$$

PROPOSITION 4.2 (cf. Yampol'skií [14]). *The Riemannian curvature tensor* R^*, *the Ricci tensor* ρ^* *and the scalar curvature* S^* *of* $(T^1M, g^*=(1/4)\dot{g})$ *are given by*

$$R^{*d}{}_{cab}=R^d_{cab}+(1/4)(R^d_{a\sigma 0}R^\sigma_{0bc}-R^d_{b\sigma 0}R^\sigma_{0ac}-2R^d_{c\sigma 0}R^\sigma_{0ab})\,,$$
$$R^{*\bar{\lambda}}{}_{cab}=(1/2)\nabla_c R^\lambda_{0ab}\,,$$
$$R^{*d}{}_{\bar{\gamma}ab}=-(1/2)(\nabla_a R^b_{b\gamma 0}-\nabla_b R^d_{a\gamma 0})\,,$$
$$R^{*\bar{\lambda}}{}_{\bar{\gamma}ab}=R^\lambda_{\gamma ab}+(1/4)(R^\lambda_{0as}R^s_{b\gamma 0}-R^\lambda_{0bs}R^s_{a\gamma 0})\,,$$
$$R^{*d}{}_{ca\bar{\beta}}=-(1/2)\nabla_a R^d_{c\beta 0}\,,\qquad R^{*\bar{\lambda}}{}_{ca\bar{\beta}}=(1/2)R^\lambda_{\beta ac}+(1/4)R^\lambda_{0as}R^s_{c\beta 0}\,,$$

$$R^{*d}{}_{\bar\gamma a\bar\beta}=(1/2)R^{d}_{a\gamma\beta}-(1/4)R^{d}_{s\beta0}R^{s}_{a\gamma0}\,,\qquad R^{*\bar\lambda}{}_{\bar\gamma a\bar\beta}=0\,,$$
$$R^{*d}{}_{c\bar\alpha\bar\beta}=R^{d}_{c\alpha\beta}+(1/4)(R^{d}_{s\alpha0}R^{s}_{c\beta0}-R^{d}_{s\beta0}R^{s}_{c\alpha0})\,,\qquad R^{*\bar\lambda}{}_{c\bar\alpha\bar\beta}=0\,,$$
$$R^{*d}{}_{\bar\gamma\bar\alpha\bar\beta}=0\,,\qquad R^{*\bar\lambda}{}_{\bar\gamma\bar\alpha\bar\beta}=\delta^{\lambda}_{\alpha}g_{\beta\gamma}-\delta^{\lambda}_{\beta}g_{\alpha\gamma}\,,$$
$$\rho^{*}_{bc}=\rho_{bc}-(1/2)R^{\sigma}{}_{0}{}^{r}{}_{b}R_{\sigma0rc}\,,$$
$$\rho^{*}_{b\bar\gamma}=(1/2)(\nabla_{0}\rho_{b\gamma}-\nabla_{\gamma}\rho_{b0})\,,$$
$$\rho^{*}_{\bar\beta\bar\gamma}=(m'-2)g_{\beta\gamma}+(1/4)R^{sr}{}_{0\beta}R_{sr0\gamma}\,,$$
$$S^{*}=4S+4(m'-1)(m'-2)-R^{sr\sigma}{}_{0}R_{sr\sigma0}\,.$$

REMARK (v). Proposition 4.2 implies that if (M, g) is a 2-dimensional flat Riemannian manifold then its unit tangent bundle (T^1M, η, g^*) is a 3-dimensional flat contact Riemannian manifold.

PROPOSITION 4.3. *$p=(1/2)L_\xi g^*$ and $\nabla^*_\xi p$ on (T^1M, η, g^*) are given by*

$$p_{ab}=p_{\bar\alpha\bar\beta}=p_{0\bar\beta}=0\,,\qquad p_{\alpha\bar\beta}=(1/4)(g_{\alpha\beta}-R^{0}_{\alpha0\beta})\,,$$
$$\nabla^*_\xi p_{0a}=\nabla^*_\xi p_{0\bar\alpha}=0\,,\qquad \nabla^*_\xi p_{\alpha\bar\beta}=-(1/2)\nabla_0 R^{0}_{\alpha0\beta}\,,$$
$$\nabla^*_\xi p_{\alpha\beta}=-\nabla^*_\xi p_{\bar\alpha\bar\beta}=(1/2)(-R^{0}_{\alpha0\beta}+R^{\sigma}_{0\alpha0}R_{\sigma0\beta0})\,.$$

PROOF. The first two follow from (16) and (17). For example, the second is

$$\begin{aligned}2p(X^H, Y^V)&=g^*(\nabla^*_{X^H}\xi, Y^V)+g^*(X^H, \nabla^*_{Y^V}\xi)\\&=-g^*((R(X, v)v)^V, Y^V)+g^*(X^H, 2Y^H-(R(Y, v)v)^H)\\&=-(1/4)g(R(X, v)v, Y)+(1/4)g(X, 2Y-R(Y, v)v)\,.\end{aligned}$$

To verify the other, first we use (13), etc., to get

$$\nabla^*_\xi X^H=2\tilde{\nabla}_{v^H}X^H=2(\nabla_v X)^H+(R(X, v)v)^V\,,$$
$$\nabla^*_\xi X^V=2\tilde{\nabla}_{v^H}X^V=2(\nabla_v X)^V-(R(X, v)v)^H$$

and, for example, we use the following:

$$\begin{aligned}\nabla^*_\xi(p(X^H, Y^V))&=(1/2)\xi_0[g(X, Y)-v^a v^b g(X, R(Y, \partial_a)\partial_b)]\\&=(1/2)[\nabla_v(g(X, Y))-v^a v^b\nabla_v(g(X, R(Y, \partial_a)\partial_b))]\\&\quad-(1/2)(-\Gamma^t_{rs}v^r v^s(\partial/\partial v^t))(v^a v^b)\cdot g(X, R(Y, \partial_a)\partial_b)\,.\end{aligned}$$

The remaining part consists of similar direct calculations. q.e.d.

PROPOSITION 4.4. *The generalized Tanaka-Webster scalar curvature S^T of the unit tangent bundle (T^1M, η, g^*) is given by $S^T=4S+4n^2-4\rho_{00}-R^{\rho\mu\sigma}{}_0R_{\rho\mu\sigma0}$.*

PROOF. By (8.2) of [10] and $\rho^*(\xi, \xi)=4\rho^*(\xi_0, \xi_0)=4\rho^*_{00}$ we have

$$S^T=S^*-\rho^*(\xi, \xi)+4n=4S+4n(n-1)-R^{sr\sigma}{}_0R_{sr\sigma0}-4\rho_{00}+2R^{\sigma}{}_0{}^{\tau}{}_0R_{\sigma0\tau0}+4n\,,$$

from which the above expression of S^T follows. q.e.d.

PROPOSITION 4.5. *Let (T^1M, η, g^*) be the unit tangent bundle of a Riemannian manifold. Then the main part of the gauge invariant B of (1,3)-type is given by the following*:

$$
\begin{aligned}
(22)\quad 2(n+2){B'}^{\lambda}_{\gamma\alpha\beta} &= 2(n+2)R^{\lambda}_{\gamma\alpha\beta} + [S^T/8(n+1)+2](\delta^{\lambda}_{\alpha}g_{\beta\gamma} - \delta^{\lambda}_{\beta}g_{\alpha\gamma}) \\
&\quad + ((n+2)/2)(R^{\lambda}_{\alpha\sigma 0}R^{\sigma}_{0\beta\gamma} - R^{\lambda}_{\beta\sigma 0}R^{\sigma}_{0\alpha\gamma} - 2R^{\lambda}_{\gamma\sigma 0}R^{\sigma}_{0\alpha\beta}) \\
&\quad + \delta^{\lambda}_{\beta}F_{\alpha\gamma} - \delta^{\lambda}_{\alpha}F_{\beta\gamma} + g_{\alpha\gamma}F^{\lambda}_{\beta} - g_{\beta\gamma}F^{\lambda}_{\alpha}\,,
\end{aligned}
$$

$$
(23)\quad 4(n+2){B'}^{\bar\lambda}_{\gamma\alpha\beta} = 2(n+2)\nabla_{\gamma}R^{\lambda}_{0\alpha\beta} + \delta^{\lambda}_{\alpha}G_{\gamma\beta} - \delta^{\lambda}_{\beta}G_{\gamma\alpha} + g_{\alpha\gamma}G^{\lambda}{}_{\beta} - g_{\beta\gamma}G^{\lambda}{}_{\alpha} + \delta^{\lambda}_{\gamma}(G_{\alpha\beta} - G_{\beta\alpha})\,,
$$

$$
\begin{aligned}
(24)\quad 2(n+2){B'}^{\bar\lambda}_{\bar\gamma\alpha\beta} &= 2(n+2)R^{\lambda}_{\gamma\alpha\beta} + ((n+2)/2)(R^{\lambda}_{0\alpha\sigma}R^{\sigma}_{\beta\gamma 0} - R^{\lambda}_{0\beta\sigma}R^{\sigma}_{\alpha\gamma 0}) \\
&\quad + [S^T/8(n+1)+2](\delta^{\lambda}_{\alpha}g_{\beta\gamma} - \delta^{\lambda}_{\beta}g_{\alpha\gamma}) + \delta^{\lambda}_{\beta}F_{\alpha\gamma} - \delta^{\lambda}_{\alpha}F_{\beta\gamma} + g_{\alpha\gamma}F^{\lambda}_{\beta} - g_{\beta\gamma}F^{\lambda}_{\alpha}\,,
\end{aligned}
$$

$$
\begin{aligned}
(25)\quad 2(n+2){B'}^{\lambda}_{\bar\gamma\alpha\bar\beta} &= (n+2)[R^{\lambda}_{\alpha\gamma\beta} - (1/2)R^{\lambda}_{\sigma\beta 0}R^{\sigma}_{\alpha\gamma 0} + \delta^{\lambda}_{\beta}R^{0}_{\alpha 0\gamma} + \delta^{\lambda}_{\gamma}g_{\alpha\beta}] \\
&\quad + [S^T/8(n+1)-n](\delta^{\lambda}_{\alpha}g_{\beta\gamma} + g_{\alpha\gamma}\delta^{\lambda}_{\beta} + 2g_{\alpha\beta}\delta^{\lambda}_{\gamma}) \\
&\quad + \delta^{\lambda}_{\alpha}H_{\beta\gamma} + \delta^{\lambda}_{\gamma}H_{\alpha\beta} + g_{\alpha\gamma}H^{\lambda}_{\beta} + g_{\alpha\beta}H^{\lambda}_{\gamma} - \delta^{\lambda}_{\beta}F_{\alpha\gamma} - \delta^{\lambda}_{\gamma}F_{\alpha\beta} - g_{\alpha\beta}F^{\lambda}_{\gamma} - g_{\beta\gamma}F^{\lambda}_{\alpha}\,,
\end{aligned}
$$

$$
(26)\quad 4(n+2){B'}^{\lambda}_{\bar\gamma\bar\alpha\bar\beta} = \delta^{\lambda}_{\alpha}G_{\beta\gamma} - \delta^{\lambda}_{\beta}G_{\alpha\gamma} + g_{\alpha\gamma}G_{\beta}{}^{\lambda} - g_{\beta\gamma}G_{\alpha}{}^{\lambda} + \delta^{\lambda}_{\gamma}(G_{\alpha\beta} - G_{\beta\alpha})\,,
$$

$$
(27)\quad 2(n+2){B'}^{\bar\lambda}_{\bar\gamma\bar\alpha\bar\beta} = [S^T/8(n+1)+2](\delta^{\lambda}_{\alpha}g_{\beta\gamma} - \delta^{\lambda}_{\beta}g_{\alpha\gamma}) + \delta^{\lambda}_{\alpha}H_{\beta\gamma} - \delta^{\lambda}_{\beta}H_{\alpha\gamma} - g_{\alpha\gamma}H^{\lambda}_{\beta} + g_{\beta\gamma}H^{\lambda}_{\alpha}\,,
$$

where we have put

$$
\begin{aligned}
&F_{\alpha\beta} = \rho_{\alpha\beta} + R^{0}_{\alpha 0\beta} - (1/2)R^{\sigma}{}_{0}{}^{\rho}{}_{\alpha}R_{\sigma 0\rho\beta}\,, \qquad F^{\lambda}_{\beta} = g^{\lambda\theta}F_{\theta\beta}\,, \\
&G_{\alpha\beta} = \nabla_0\rho_{\alpha\beta} - \nabla_{\alpha}\rho_{\beta 0} - \nabla_0 R^{0}_{\alpha 0\beta} = \nabla_{\sigma}R^{\sigma}{}_{\beta 0\alpha}\,, \\
&H_{\alpha\beta} = R^{0}_{\alpha 0\beta} - (1/4)R^{\sigma\tau}{}_{0\alpha}R_{\sigma\tau 0\beta}\,.
\end{aligned}
$$

PROOF. We apply Propositions 4.2 and 4.3 to the expression (8) of B. By (9), (10) and (11) we see that (22)–(27) give the main part of B. q.e.d.

COROLLARY 4.6. *If (M, g) is of constant curvature k, then B of (T^1M, η, g^*) is given by*

$$
\begin{aligned}
&B^{\lambda}_{\gamma\alpha\beta} = B^{\bar\lambda}_{\bar\gamma\alpha\beta} = B^{\bar\lambda}_{\bar\gamma\bar\alpha\bar\beta} = [(k+1)(n+2)/4(n+1)](\delta^{\lambda}_{\alpha}g_{\gamma\beta} - \delta^{\lambda}_{\beta}g_{\gamma\alpha})\,, \\
&B^{\lambda}_{\bar\gamma\alpha\bar\beta} = [(k+1)/4(n+1)][2\delta^{\lambda}_{\gamma}g_{\alpha\beta} - n(\delta^{\lambda}_{\alpha}g_{\gamma\beta} + \delta^{\lambda}_{\beta}g_{\gamma\alpha})]\,, \\
&B^{\bar\lambda}_{\gamma\alpha\beta} = B^{\lambda}_{\bar\gamma\bar\alpha\bar\beta} = 0\,.
\end{aligned}
$$

PROOF. By Proposition 4.1 we have $Q=0$ and $B=B'$. By Proposition 4.4 we have $S^T=4n^2(k+1)$. Furthermore, we have $F_{\alpha\beta}=(n+1)kg_{\alpha\beta}$, $G_{\alpha\beta}=0$ and $H_{\alpha\beta}=kg_{\alpha\beta}$. Then, Corollary 4.6 follows from Propostion 4.5. q.e.d.

PROOF OF THEOREM A. First we assume that $B=0$ holds. Then we have $Q=0$ and $B'=0$. By Proposition 4.1, (M, g) is of constant curvature k. Now Corollary 4.6 shows that $B=0$ implies $k=-1$ if $n \geq 2$. Conversely, if (M, g) is of constant curvature $k=-1$ then Corollary 4.6 implies $B=0$. q.e.d.

REMARK (vi). By Proposition 4.3 the E-critical condition $\nabla^*_{\xi}p = 2p \cdot \phi$ is given by

$R^{\sigma}{}_{0\alpha 0}R_{\sigma 0\beta 0}=g_{\alpha\beta}$ and $\nabla_0 R^0{}_{\alpha 0\beta}=0$. Blair [4] proved that g^* of (T^1M, η, g^*) is E-critical, if and only if (M, g) is of constant curvature 1 or -1.

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