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SUBMANIFOLDS OF CONSTANT SECTIONAL CURVATURE WITH PARALLEL OR CONSTANT MEAN CURVATURE

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Abstract. We classify isometric immersions with flat normal bundle between spaces of constant sectional curvature if either the mean curvature vector is parallel or if the manifolds are complete Euclidean spaces and the mean curvature is of constant length.

Let $f: M_c^n \to Q_{\tilde{c}}^N$ be an isometric immersion of a connected n-dimensional Riemannian manifold of constant sectional curvature c into a complete and simply connected Riemannian manifold of constant sectional curvature $\tilde{c}$. It was shown in [Mo$_2$] and [Da] that if f is minimal and has flat normal bundle, then $c=0$ and $f(M_0^n)$ is part of a Clifford torus substantial in some $(2n-1)$-dimensional sphere.

First, instead of the minimality condition, we assume that the mean curvature vector is parallel and obtain the following (local) classification.

THEOREM 1. *Let $f: M_c^n \to Q_{\tilde{c}}^N$, $n \geq 2$, be an isometric immersion with parallel mean curvature vector and flat normal bundle. Then one of the following holds:*

(i) *$c=\tilde{c}$ and either f is totally geodesic or $c=0$ and*

$$f(M_0^n) \subset S^1(r_1) \times \cdots \times S^1(r_k) \times \boldsymbol{R}^{n-k} \subset \boldsymbol{R}^{n+k}.$$

(ii) *$\tilde{c} > c = 0$ and*

$$f(M_0^n) \subset S^1(r_1) \times \cdots \times S^1(r_n) \subset S_{\tilde{c}}^{2n-1} \subset \boldsymbol{R}^{2n},$$

where $r_1^2 + \cdots + r_n^2 = 1/\tilde{c}$.

(iii) *$\tilde{c} < c = 0$ and*

$$f(M_0^n) \subset H^1(r_1) \times S^1(r_2) \times \cdots \times S^1(r_n) \subset H_{\tilde{c}}^{2n-1} \subset \boldsymbol{L}_0^{2n},$$

where $-r_1^2 + r_2^2 + \cdots + r_n^2 = 1/\tilde{c}$.

(iv) *$f = i \circ f'$ where f' is of type* (i), (ii) *or* (iii) *and i denotes an umbilical or totally geodesic inclusion.*

When both sectional curvatures c, $\tilde{c}$ are nonnegative, the above theorem follows from a result due to Erbacher [Er].

Now, we suppose the manifold M^n to be complete and flat. In order to obtain a classification, it is sufficient to assume a much weaker restriction on the mean curvature

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vector.

THEOREM 2. *Let $f: \mathbf{R}^n \to \mathbf{R}^N$ be an isometric immersion with flat normal bundle, constant index of relative nullity and mean curvature vector of constant length. Then $f(\mathbf{R}^n)$ is an orthogonal product of curves with constant first Frenet curvature.*

For $n=2$, the above result was obtained by Enomoto [En] without assuming the index of relative nullity to be constant. Theorem 2 has the following immediate consequence.

COROLLARY 3. *Let $f: \mathbf{R}^n \to S_c^N$ be an isometric immersion with flat normal bundle and mean curvature vector of constant length. Then $f(\mathbf{R}^n)$ is an orthogonal product of spherical curves with constant first Frenet curvature.*

The proofs. Given an isometric immersion $f: M^n \to Q_c^N$, we denote by $\nu_f(x)$ the *index of relative nullity* of f at $x \in M$, defined as

$$\nu_f(x) = \dim\{X \in T_xM : \alpha_f(X, Y) = 0,\ \forall Y \in T_xM\}\,,$$

where $\alpha_f: TM \times TM \to TM^\perp$ stands for the vector valued second fundamental form of f. We denote by $N_1^f(x)$ the *first normal space* of f at x given by

$$N_1^f(x) = \mathrm{span}\{\alpha_f(X, Y) : \forall X, Y \in T_xM\}\,.$$

The proof of Theorem 1 makes use of several lemmas. The first result deals with isometric immersions of arbitrary Riemannian manifolds and improves Theorem 10 in [Ya].

LEMMA 4. *Let $f: M^n \to Q_c^N$ be an isometric immersion with parallel mean curvature vector and flat normal bundle. Assume* $\dim N_1^f = m$ *everywhere. Then $f(M^n)$ is contained in either a totally geodesic submanifold (if $m<n$) $Q_c^{n+m} \subset Q_c^N$ or a totally umbilical submanifold (if $m=n$) $Q_{\tilde{c}}^{2n-1} \subset Q_c^N$.*

PROOF. By Theorem 1.3 of [Da], we have that $f(M^n)$ is contained in a totally geodesic submanifold Q_c^{n+m} where $m \le n$ because of the flatness of the normal bundle. If $m=n$, consider $g = i \circ f: M^n \to Q_{c'}^{2n+1}$ where $c' < c$ and i denotes an umbilical inclusion of Q_c^{2n} into $Q_{c'}^{2n+1}$. Clearly, g has parallel mean curvature vector, flat normal bundle and $\dim N_1^g = n$. Again from Theorem 1.3 of [Da], we conclude that $g(M^n)$ is contained in the intersection of a totally geodesic hypersurface in $Q_{c'}^{2n+1}$ with Q_c^{2n}, that is, an umbilical hypersurface of Q_c^{2n}. ■

REMARK 5. We take this opportunity to point out that the second statement in Corollary 3.29 in [Da] is not correct.

LEMMA 6. *Let $f: M_c^n \to Q_c^{n+p}$, $p \le n$, be an isometric immersion with parallel mean curvature vector, flat normal bundle and $N_1^f(x) = T_xM^\perp$ everywhere. Then $c=0$ and*

$$f(M_0^n)\subset S^1(r_1)\times\cdots\times S^1(r_p)\times \boldsymbol{R}^{n-p}\subset \boldsymbol{R}^{n+p}.$$

PROOF. We may assume that M^n is simply connected. Then, by Proposition 7 of [D-T], there exist orthonormal frames $X_1,\dots,X_n$ of TM, $\xi_1,\dots,\xi_p$ of $TM^\perp$ and positive functions $\lambda_1,\dots,\lambda_p$ so that

$$\alpha(X_i, X_k)=\lambda_i\delta_{ik}\xi_i\,,\quad \alpha(X_k, X_r)=0\,,$$

where we adopted the following convention for the ranges of the indices: $i, j\in\{1,\dots,p\}$, $k\in\{1,\dots,n\}$ and $r\in\{p+1,\dots,n\}$. The Codazzi equations yield

$$\begin{aligned}&\langle\nabla_{X_i}X_k, X_j\rangle=\lambda_i\delta_{ij}X_k(1/\lambda_i)\,,\qquad i\neq k\,,\quad \nabla_{X_r}X_i=0\,,\\ &\nabla^\perp_{X_i}\xi_j=\lambda_i X_i(1/\lambda_j)\xi_i\,,\qquad i\neq j\,,\quad \nabla^\perp_{X_r}\xi_i=0\,.\end{aligned}\tag{1}$$

The mean curvature vector is $H=\sum_{j=1}^p\lambda_j\xi_j$. Using (1), we obtain

$$X_r(\lambda_i)=X_r\langle H, \xi_i\rangle=\langle H, \nabla^\perp_{X_r}\xi_i\rangle=0\,,\tag{2}$$

$$0=\langle\nabla^\perp_{X_i}H, \xi_j\rangle=(1+\lambda_i^2/\lambda_j^2)X_i(\lambda_j)\,,\qquad i\neq j\,,\tag{3}$$

$$0=\langle\nabla^\perp_{X_i}H, \xi_i\rangle=X_i(\lambda_i)-\sum_{j\neq i}\frac{\lambda_i}{\lambda_j}X_i(\lambda_j)\,.\tag{4}$$

To conclude the proof, first observe that by the equations (2), (3) and (4), the λ_j's are constant. Using (1), it follows that the distributions $L_1=\operatorname{span}\{X_1\},\dots,L_p=\operatorname{span}\{X_p\}$ and $L_{p+1}=\operatorname{span}\{X_{p+1},\dots,X_n\}$ are parallel. The remaining of the argument is straightforward. ∎

Following Moore [Mo$_3$], we say that a point $x\in M_c^n$ is a *weak-umbilic* for an isometric immersion $f\colon M_c^n\to Q_{\tilde c}^N$, $c>\tilde c$, if there exists a unit vector $\delta\in T_x^f M^\perp$ such that the tangent valued second fundamental form verifies $A_\delta^f=\sqrt{c-\tilde c}\,\mathrm{Id}$.

LEMMA 7. *Let $f\colon M_c^n\to Q_{\tilde c}^{n+p}$, $c>\tilde c$, $2\le p\le n-1$, be an isometric immersion with parallel mean curvature vector, flat normal bundle and $N_1^f(x)=T_xM^\perp$ everywhere. If all points are weak-umbilics, then $f(M^n)$ is contained in an umbilical hypersurface Q_c^{n+p-1} of $Q_{\tilde c}^{n+p}$.*

PROOF. From Lemma 8 of [O'N] or Proposition 9 in [D-T], we have that α_f splits orthogonally and smoothly as

$$\alpha_f=\sqrt{c-\tilde c}\langle\ ,\ \rangle\eta\otimes\gamma\,,$$

where η is a unit normal vector field and, by the Gauss equations, the bilinear form γ is flat, that is,

$$\langle\gamma(X, Y), \gamma(Z, W)\rangle-\langle\gamma(X, W), \gamma(Z, Y)\rangle=0$$

for all $X, Y, Z, W\in TM$. Therefore, by Proposition 7 of [D-T], there exist smooth orthonormal frames $X_1, \dots, X_n$ of TM, $\eta, \xi_1, \dots, \xi_{p-1}$ of $TM^{\perp}$ and smooth positive functions $\mu_1, \dots, \mu_{p-1}$ such that

$$\gamma(X_i, X_j)=\mu_i\delta_{ij}\xi_i, \quad 1\le i\le p-1, \quad \text{and} \quad \gamma(X_i, X_j)=0, \quad p\le i\le n.$$

The Codazzi equation for A_η yields

$$A_{\nabla^{\perp}_{Y}\eta}Z=A_{\nabla^{\perp}_{Z}\eta}Y, \qquad \text{for all} \quad Y, Z\in TM,$$

from which we easily obtain that

$$\langle\nabla^{\perp}_{X_i}\eta, \xi_j\rangle=0, \qquad i\neq j, \quad 1\le i\le n, \quad 1\le j\le p-1. \tag{5}$$

On the other hand, since H is parallel, we have

$$\begin{aligned}0&=\langle\nabla^{\perp}_{X_i}H, \eta\rangle=X_i\langle H, \eta\rangle-\langle H, \nabla^{\perp}_{X_i}\eta\rangle\\&=X_i(\text{trace } A_\eta)-\mu_i\langle\nabla^{\perp}_{X_i}\eta, \xi_i\rangle.\end{aligned}$$

From trace $A_\eta=n\sqrt{c-\tilde{c}}$, we get

$$\langle\nabla^{\perp}_{X_i}\eta, \xi_i\rangle=0, \qquad 1\le i\le p-1. \tag{6}$$

By (5) and (6) we have that η is parallel in the normal connection and the statement follows. ∎

LEMMA 8. *Let $f\colon M^n_c\to Q^{2n-1}_{\tilde{c}}$, $c>\tilde{c}$, be an isometric immersion with parallel mean curvature vector such that no point is a weak-umbilic. Then $c=0$ and*

$$f(M^n_0)\subset H^1(r_1)\times S^1(r_2)\times\cdots\times S^1(r_n)\subset H^{2n-1}_{\tilde{c}}\subset \mathbf{L}^{2n}_0,$$

where $-r_1^2+r_2^2+\cdots+r_n^2=1/\tilde{c}$.

PROOF. Set $g=i\circ f\colon M^n_c\to \mathbf{L}^{2n}_c$, where $\mathbf{L}^{2n}_c$ stands for the geodesically complete and simply connected Lorentzian manifold of constant sectional curvature c. Then

$$\alpha_g=\alpha_f\oplus\sqrt{c-\tilde{c}}\langle\ ,\ \rangle e, \tag{7}$$

where $\langle e, e\rangle=-1$. Since no point is a weak-umbilic for f, we have that N^g_1 is a nondegenerate subspace everywhere. In fact, if $N^g_1(x)$ is degenerate, there exists a unit vector $\delta\in T^f_xM^{\perp}$ such that

$$0=\langle\alpha_g(X, Y), \delta+e\rangle=\langle\alpha_f(X, Y), \delta\rangle-\sqrt{c-\tilde{c}}\langle X, Y\rangle$$

for all $X, Y\in T_xM$. This is a contradiction. Moreover, it follows from (7) that $\nu_g=0$ everywhere. We have from Corollaries 1 and 2 of [Mo$_3$] that $\dim N^g_1=n$, and by Theorem 2 part b) of [Mo$_3$] that α_g splits at each point as an orthogonal sum of one-dimensional flat forms

$$\alpha_g(x)=\beta_1\oplus\cdots\oplus\beta_n.$$

Similarly to [Mo$_3$, p. 468] we argue that there exists a basis of unit vectors $X_1, \dots, X_n$

of T_xM so that

$$\alpha_g(X_i, X_j)=0 \qquad \text{if} \quad i\neq j,$$

and we conclude using (7) that the X_j's are, in fact, orthogonal. By Proposition 7 of [D-T], there exist smooth orthonormal frames $X_1, \ldots, X_n$ of TM, $\xi_1, \ldots, \xi_n$ of $T_gM^\perp$ with $\langle \xi_1, \xi_1\rangle=-1$, and smooth positive functions $\lambda_1, \ldots, \lambda_n$ verifying

$$\alpha_g(X_i, X_j)=\lambda_i\delta_{ij}\xi_i, \qquad 1\leq i\leq n. \tag{8}$$

From the Codazzi equations, we have

$$\nabla_{X_i}X_j=\lambda_iX_j(1/\lambda_i)X_i, \quad \nabla^\perp_{X_i}\xi_j=\lambda_iX_i(1/\lambda_j)\xi_i, \qquad i\neq j. \tag{9}$$

Using (9), we get

$$0=\langle\nabla^\perp_{X_i}H^g, \xi_j\rangle=(-\varepsilon_j\lambda_j^2-\varepsilon_i\lambda_i^2)X_i(1/\lambda_j), \qquad i\neq j, \tag{10}$$

$$0=\langle\nabla^\perp_{X_i}H^g, \xi_i\rangle=\varepsilon_i\left(X_i(\lambda_i)-\sum_{j\neq i}\frac{\lambda_i}{\lambda_j}X_i(\lambda_j)\right), \tag{11}$$

where $\varepsilon_1=-1$ and $\varepsilon_j=1$ if $2\leq j\leq n$. On the other hand, the equation (8) implies that

$$A^g_\delta=\mathrm{Id}, \qquad \delta=\sum_{j=1}^n\frac{1}{\lambda_j}\xi_j$$

and, conversely, any umbilical normal vector field must be a multiple of δ. Therefore, from $A^g_e=-\sqrt{c-\tilde{c}}\,\mathrm{Id}$, it follows that

$$1/\lambda_1^2-1/\lambda_2^2-\cdots-1/\lambda_n^2=1/(c-\tilde{c}).$$

In particular, $\lambda_j\neq\lambda_1$ for all $2\leq j\leq n$, and we conclude from (10) and (11) that the λ_j's are all constant. By (9), the distributions $L_i=\mathrm{span}\{X_i\}$ are parallel and the remaining of the proof is straightforward. ■

Proof of Theorem 1. *Case* $c<\tilde{c}$. Set $g=i\circ f\colon M^n_c\to Q^{N+1}_{\tilde{c}}$. Then, $\dim N^g_1=n$ everywhere, since $\nu_g=0$. The result follows from Lemmas 4 and 6.

Case $c=\tilde{c}$. Let $V\subset M^n_c$ be a connected component of the open and dense subset $\mathcal{U}$ of points where N^f_1 has locally constant dimension. If $\dim N^f_1=k$ on V, we conclude from Lemmas 4 and 6 that $f|_V$ satisfies our statement. It is now clear that $V=M^n_c$.

Case $c>\tilde{c}$. From Lemma 4 the immage of each connected component of $\mathcal{U}$ is contained in either a totally geodesic or an umbilical submanifold $Q^{n+s}_{c'}$ of $Q^N_{\tilde{c}}$ with $s\leq n-1$. It is sufficient to consider the case $c>c'$. Assume that $x\in\mathcal{U}$ is not a weak-umbilic. Then the same holds in a neighborhood $W\subset\mathcal{U}$ where Lemma 8 applies. Clearly $W=M^n_c$. If all points are weak-umbilics the result follows from Lemma 7 and the preceding case. ■

Proof of Theorem 2. By a result due to Hartman [Ha] we may assume $\nu_f=0$. As before, consider an orthonormal frame $X_1, \ldots, X_n$ of eigenvectors and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. We have

$$X_i(\lambda_j)=\lambda_j\langle\nabla_{X_j}X_j, X_i\rangle \qquad \text{if} \quad 1\leq i\neq j\leq n\,.$$

Set

$$Y=\varepsilon_1\lambda_1X_1+\cdots+\varepsilon_n\lambda_nX_n\,,$$

where $\varepsilon_j=\pm 1$, $1\leq j\leq n$. We claim that $\nabla_Y Y=0$ for any set $\varepsilon_1, \ldots, \varepsilon_n$. In fact,

$$\begin{aligned}\nabla_Y Y&=\sum_{j=1}^{n}\varepsilon_j\lambda_j\left(\sum_{h=1}^{n}\varepsilon_hX_j(\lambda_h)X_h+\sum_{h=1}^{n}\varepsilon_h\lambda_h\nabla_{X_j}X_h\right)\\&=\sum_{j=1}^{n}\lambda_jX_j(\lambda_j)X_j+\sum_{j\neq h}\varepsilon_j\varepsilon_h\lambda_jX_j(\lambda_h)X_h+\sum_{j=1}^{n}\lambda_j^2\nabla_{X_j}X_j+\sum_{j\neq h}\varepsilon_j\varepsilon_h\lambda_j\lambda_h\nabla_{X_j}X_h\\&=\sum_{j=1}^{n}X_j(\lambda_j^2/2)X_j+\sum_{j\neq h}\varepsilon_h\varepsilon_j\lambda_jX_j(\lambda_h)X_h+\sum_{j\neq h}\lambda_j^2\langle\nabla_{X_j}X_j, X_h\rangle X_h\\&\quad+\sum_{j\neq h}\varepsilon_j\varepsilon_h\lambda_j\lambda_h\langle\nabla_{X_j}X_h, X_j\rangle X_j\\&=\frac{1}{2}\,\mathrm{grad}(\|H\|^2)=0\,.\end{aligned}$$

Then, for any set $\varepsilon_1, \ldots, \varepsilon_n$, the vector fields Y are complete, and thus tangent to parallel lines. We easily obtain that the λ_j's must be constant and that the distributions $L_i=\mathrm{span}\{X_i\}$ are parallel. One can now conclude the proof from the Main Lemma in [Mo_1]. ■

REMARK 9. By a result of E. Cartan (see Theorem 1 in [Mo_2], any isometric immersion $f\colon M_c^n \hookrightarrow Q_{\tilde{c}}^{2n-1}$, $c<\tilde{c}$, has automatically flat normal bundle.

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