ON TOWERS OF LIFTINGS AND HYPERCUSPIDALITY FOR UNITARY GROUPS

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Abstract. Given a cuspidal automorphic representation on $U(2, 3)$, then its theta lift to $U(i, i)$ is cuspidal if and only if its theta lift to $U(i-1, i-1)$ is zero. Also, the theta lift of a cuspidal generic representation from $U(2, 3)$ to $U(3, 3)$ is generic. The theta lift of a cuspidal representation from $U(2, 3)$ to $U(4, 4)$ or to $U(5, 5)$ is hypercuspidal.

In this paper we study the cuspidality and hypercuspidality of some automorphic forms on the group $U(i, i)$ for $i=1$ to 5. These automorphic forms are “theta lifted” from automorphic forms belonging to the space of a cuspidal representation $\pi$ for the group $U(2, 3)$.

In Chapter 1 we study the tower of liftings $\theta^i(\pi, s)$ for $i=1$ to 5 to find conditions for the cuspidality of the lift $\theta^i(\pi, s)$ in terms of the lift $\theta^{i-1}(\pi, s)$. More explicitly, Theorem 1.1 states that $\theta^i(\pi, s)$ is cuspidal if and only if $\theta^{i-1}(\pi, s)$ is zero. Moreover, $\theta^5(\pi, s)$ is nonzero, so higher theta lifts cannot be cuspidal. Therefore we stop the tower at $i=5$. These are well-known results for split groups (cf. [Ra]).

In Chapter 2 we generalize some results of [Wa], concerning the hypercuspidality of such lifts. Theorem 2.1 states that the lift $\theta^3(\pi, s)$ is already nonzero for generic representations $\pi$ on $U(2, 3)$. Moreover, a Whittaker function of the lift can be expressed in terms of a Whittaker function of $\pi$. Theorem 2.2 states that $\theta^4(\pi, s)$ and $\theta^5(\pi, s)$, if cuspidal, are also hypercuspidal in the sense that all Whittaker functions disappear.

In the proof of Theorem 2.1 we use the Witt decomposition for the space of $U(2, 3)$, i.e., the existence of a maximal isotropic subspace of dimension two, and an anisotropic subspace of dimension one. In general if $\pi$ is a cuspidal generic representation of $U(n, n+1)$, then the $n+1$ lift should also be generic. All theta lifts above this level should be hypercuspidal.

We conclude by remarking that the “simpler” tower, $U(1, 2)$ to $U(i, i)$, for $i=1$ to 3 is computed in [Wa]. If $\pi$ on $U(1, 2)$ is generic, then the lift to $U(2, 2)$ is also generic [Wa, Theorem 4.3]. Using Theorem 2.2, it is easy to show that the theta lift of $U(1, 2)$ to $U(3, 3)$ is hypercuspidal.
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Notation. Let $K$ be a global field, and $L$ a quadratic Galois extension of $K$. We write $L = K(i)$, and $i$ for the Galois involution on $i \in L$. If $U$ is an algebraic group defined over $K$. We write $U_K$ for the group of its $K$-rational points, and $U_A$ or $U_A^*$ for the adele group.

Let $W$ be a 5-dimensional vector space over $L$ equipped with a Hermitian form $\langle , \rangle_W$ having the matrix

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

in the basis $\{w_1, w_2, w_0, w_{-2}, w_{-1}\}$.

Let $V_i$ for $i = 1 \text{ to } 5$ be a $2i$-dimensional vector space equipped with a skew-Hermitian form $\langle , \rangle_{V_i}$ having the matrix

$$J_i = \begin{pmatrix} I_i \\ -I_i \end{pmatrix}$$

in the basis $\{e_1, \ldots, e_i, \hat{e}_1, \ldots, \hat{e}_i\}$.

Let $U(2, 3)$ (resp. $U(i, i)$) be the group of transformations in $GL(5)_L$ (resp. $GL(2i)_L$) preserving the form $\langle , \rangle_W$ (resp. $\langle , \rangle_{V_i}$). Then $U(2, 3)$ and $U(i, i)$ are the groups of $K$-rational points of quasi-split algebraic groups defined over $K$, split over $L$. Also

$$H = U(2, 3) = \{g \in GL(5)_L ; g^tQg = Q\}$$
$$U(i, i) = \{g \in GL(2i)_L ; g^tJ_ig = J_i\} .$$

(a) Description of parabolic subgroups of $U(2, 3)$. Let $P_1$ be the maximal parabolic subgroup of $U(2, 3)$ with Levi component $L(P_1) = \text{RES}_K^{GL(1)} \times U(2, 1)$, having the form

$$P_1 = \begin{pmatrix} \begin{pmatrix} d \\ U(2, 1) \end{pmatrix} \\ \bar{d}^{-1} \end{pmatrix} \begin{pmatrix} 1 & a & b & c & z \\ \cdot & 1 & \cdot & \cdot & -\bar{c} \\ \cdot & \cdot & 1 & \cdot & -\bar{b} \\ \cdot & \cdot & \cdot & 1 & -\bar{a} \\ \cdot & \cdot & \cdot & \cdot & 1 \end{pmatrix} ,$$
where \( z = -(a\bar{c} + b\bar{b} + c\bar{a})/2 + is \) and \( s \) in \( K \). Let \( N_1 \) be the unipotent radical of \( P_1 \).

Recall \( \text{RES}_K^L \text{GL}(n) \) is the disconnected quasi-split algebraic group of type \( A_{n-1} \times A_{n-1} \) formed by the restriction of scalars from \( L \) to \( K \). Note that \( \text{RES}_K^L \text{GL}(n)_K = \text{GL}(n)_L \) and \( \text{RES}_K^L \text{GL}(n)_L = \text{GL}(n)_L \times \text{GL}(n)_L \) (cf. [Ta]).

Let \( P_2 \) be the maximal parabolic subgroup of \( U(2, 3) \) with Levi component \( L(P_2) = \text{RES}_K^L \text{GL}(2) \times U(1) \), having the form

\[
P_2 = \begin{bmatrix}
g & \cdot & b & z_1 & z_2 \\
\cdot & 1 & c & z_3 & -\bar{d} \\
\cdot & \cdot & 1 & -\bar{c} & -\bar{b} \\
\cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
\]

where

\[
z_1 = -\bar{c}b + d , \quad z_2 = -(\bar{b}b)/2 + is , \quad z_3 = -(\bar{c}c)/2 + it , \quad J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

for \( s, t \) in \( K \). Let \( N_2 \) be the unipotent radical of \( P_2 \) (\( N_2 \) fixes \( w_1 \) and \( w_2 \)).

Let \( N \) be the maximal unipotent subgroup of \( U(2, 3) \). We can write \( N \) as \( Z_3, \bar{U} \), i.e.,

\[
\begin{bmatrix}
1 & a & b & z_1 & z_2 \\
\cdot & 1 & c & z_3 & -\bar{d} \\
\cdot & \cdot & 1 & -\bar{c} & -\bar{b}+\bar{a}\bar{c} \\
\cdot & \cdot & \cdot & 1 & -\bar{a} \\
\cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & a & b & a_1 & a_2 \\
\cdot & 1 & c & -\bar{c}c)/2 & 0 \\
\cdot & \cdot & 1 & -\bar{c} & -\bar{b}+\bar{a}\bar{c} \\
\cdot & \cdot & \cdot & 1 & -\bar{a} \\
\cdot & \cdot & \cdot & \cdot & 1
\end{bmatrix}
\]

where

\[
z_1 = d + ait - \bar{c}b + a(\bar{c}c)/2 , \quad z_2 = -[(a\bar{d} + \bar{a}\bar{d}) + (-b + ac)(\bar{b} + \bar{a}c)]/2 + is \, ,
\]

\[
z_3 = -(\bar{c}c)/2 + it , \quad a_1 = -\bar{c}b + a(\bar{c}c)/2 , \quad a_2 = -(-b + ac)(\bar{b} + \bar{a}c)/2 \, ,
\]

for \( s, t \) in \( K \).
(b) Description of maximal parabolic subgroups of $U(i, i)$. Let $P^j_i$ denote the maximal parabolic subgroup of $U(i, i)$ having a Levi component $L(P^j_i) = \text{RES}_{k}^{GL(j)} \times U(i-j, i-j)$, unipotent radical $U^{i,j}$, and of the form

$$
\begin{array}{c}
\begin{bmatrix}
  * & * \\
  * & * \\
  \hline
  I_{i-j} & I_i \\
\end{bmatrix}
\end{array}
\quad \text{RES}_{k}^{GL(j)} \times U(i-j, i-j)
$$

where $g \in GL(j)_k$ and $\bar{Y}^t = Y$. We decompose $U^{i,j}$ as $U^{i,j}_1 \cdot U^{i,j}_2$.

Let $\pi$ be an automorphic cuspidal representation of $U(2, 3)$, with $V_\pi$ as its vector space. We use the Schrödinger realization of $\omega_\psi$ in $S((X_1)_A)$, the Schwartz-Bruhat space of $(X_1)_A$ (cf. [Ra], [Ro], and [P.S.]). Then for $f$ in $V_\pi$, $\Phi$ in $S((X_1)_A)$ we write

$$
\theta^{(\psi)}(\pi, s)f(g) = \int_{U(2,3) \backslash U(2,3)_A} \sum_{x \in (X_1)_k} \omega_\psi(s(g, h)) \Phi(x) f(h) dh
$$

for the theta lift of $f$ to $U(i, i)$. We denote by $\theta^{(\pi, s)}$ the space of such functions for all $f$ in $V_\pi$, $\Phi$ in $S((X_1)_A)$. It is well known that $\theta^{(\pi, s)}$ generates an irreducible representation for $U(i, i)$ if it is cuspidal (cf. [Ge-Ro-So]).

Suppose now $V_i \otimes W = U_1 \oplus U_2$ (orthogonal sum), and $G = G_1 \times G_2$ is a subgroup of $U(V_i \otimes W)$, the unitary group of $V_i \otimes W$. Suppose also $G_1$ (resp. $G_2$) acts on $U_1$ (resp. $U_2$). Then any splitting $s$ of $U(V_i \otimes W)$ determines two splittings $s_i$ of $G_i$ into $MP^{\psi}(U_i)$ for $i = 1, 2$. Also

$$
\omega_\psi(s_1(g_1), s_2(g_2)) = \omega_{\psi}^{1}(s_1(g_1)) \otimes \omega_{\psi}^{2}(s_2(g_2)),
$$

where $\omega_{\psi}^{i}$ is the Weil representation of $MP^{\psi}(U_i)$ for $i = 1, 2$.

Picking any one of the standard maximal parabolic subgroups $P^j_i$ of $U(i, i)$, let $U_1$...
be $W_i \oplus \cdots \oplus W_{-j} \oplus \cdots \oplus \hat{W}_i \oplus \cdots \oplus \hat{W}_i$. Then $U(i-j, i-j)$ and $H$ act on $U_1$, where $H$ (and $GL(j)$) act on $U_2$. Therefore there exists a splitting $s_i$ of $U(i-j, i-j) \times H$ into $MP^*(U_1)$, and a splitting $s_2$ of $H$ into $MP^*(U_2)$. Also $\omega_\phi = \omega_\phi^1 \otimes \omega_\phi^2$.

**Theorem 1.1.** (a) $\theta'(\pi, s)$ is cuspidal if and only if $\theta^{i-1}((\pi, s))^1$ is zero, where $\gamma'$ is a character of $H_K \backslash H_A$ defined by

$$
\gamma'(h) = \{\omega_\phi^1(s_{\frac{1}{2}}(h))\Phi\}(0),
$$

and $\theta^{i-1}(\pi', s'_i)$ is the lift defined by $\omega_\phi^i$.

(b) $\theta^2(\pi, s)$ is nonzero.

To prove the theorem we prove:

**Proposition 1.1.** For $j = 1$ to $i-1$

$$
\int_{U_{i-j}^{1} \backslash U_{i-j}^{1}} \{\theta_{\phi}(\pi, s)f\}(ng)dn = \int_{H_K \backslash H_A} \sum_{(w_{1}, \ldots, w_{i-j})} \omega_\phi(s(g, h))\Phi(x_1, \ldots, x_{i-j}, 0, \ldots, 0)f(h)dh.
$$

**Proof** (cf. [Ra] or [Wa]). Write $U_{i-j}^{1}$ as $U_{1}^{1} \times U_{2}^{1}$. First taking integration over $(U_{1}^{i-j})_A \backslash (U_{1}^{i-j})_A$, we have

$$
\int_{(U_{1}^{i-j})_K \backslash (U_{1}^{i-j})_A} \{\theta_{\phi}(\pi, s)f\}(u_1 g)du_1 = \int_{H_K \backslash H_A} \sum_{(x_1, \ldots, x_{i-j})} h \omega_\phi(s(u_1 g, h))\Phi(x_1, \ldots, x_{i-j})f(h)du_1 dh.
$$

In writing this we have used a change in the order of integrations justified as in [Ra, Appendix to Section 1]. Suppose

$$
U_{1}^{i-j} = \begin{bmatrix} I_i & [n_{k,i}] \\ \vdots & \vdots \\ I_i & \end{bmatrix}
$$

with $[n_{k,i}]$ a matrix in $M_{i,i}(K \backslash A)$. Then
\[ \omega_\psi(s(u_ig, h))\Phi(x_1, \ldots, x_i) = \prod_{i \geq 1, j > k} \psi(\text{Real}(n_{k,i} \langle x_k, x_i \rangle_w)) \prod_{i \geq k > j - 1} \psi \left( \frac{1}{2} n_{k,i} \langle x_k, x_i \rangle_w \right) \cdot \Phi(x_1, \ldots, x_i). \]

Therefore the integral over \((U^j_1)^K \setminus (U^j_1)^A\) is zero unless

\[
\left[ \langle x_k, x_i \rangle_w \right]_{1 \leq k, l \leq j} = \begin{bmatrix}
* & 0 \\
0 & 0 \\
0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & \cdots & * \\
\end{bmatrix}
\]

Let \((Y_1)_K\) denote the subset of \((X_1)_K\) satisfying the above matrix equation. Then

\[
\int_{(U^j_1)^K \setminus (U^j_1)^A} \{ \theta^\phi_\psi(\pi, s)f(u_ig)du_i \} = \int_{H_K \setminus H_A} \sum_{(Y_1)_K} \omega_\psi(s(g, h))\Phi(x_1, \ldots, x_i)f(h)dh.
\]

Let \(Sp\{w\}\) denote the \(L\)-vector space spanned by \(w\). Then the following lemma is easily proved using Witt’s theorem.

**Lemma 1.1.** There are three types of orbits of \((Y_1)_K\) under the left diagonal action of \(H_K\) and the right action of \((U^j_2)_K\):

(a) \((x, \xi) = (x_1, \ldots, x_{i-j}, \xi_j, \ldots, \xi_1)\)

where \(\xi_k \in Sp\{w_1, w_2\}\) for \(k = 1\) to \(j\) (not all in \(Sp\{w_i\}\), \(i = 1, 2\)), \(x_l \in Sp\{w_0\}\) for \(l = 1\) to \(i-j\), and \((x, \xi)\) runs through a set of representatives of \(L(P_2)\) orbits.

(b) \((x, \xi) = (x_1, \ldots, x_{i-j}, \xi_j, \ldots, \xi_1)\)

where \(\xi_k \in Sp\{w_i\}\) for \(k = 1\) to \(j\), \(x_l \in Sp\{w_2, w_0, w_{-2}\}\) for \(l = 1\) to \(i-j\), and \((x, \xi)\) runs through a set of representatives of \(L(P_1)\) orbits.

(c) \((x, \xi) = (x_1, \ldots, x_{i-j}, 0, \ldots, 0)\).

We shall show now that integration over orbits of type (a) and (b) gives zero. The kernel of the last integrals is:

\[ \sum_{(Y_1)_K} \omega_\psi(s(g, h))\Phi(x_1, \ldots, x_i) = \sum_{(x, \xi)} \sum_{\delta_2 \delta_1} \omega_\psi(s(\delta_1 g, \delta_2 h))\Phi(x, \xi) \]

where \(\delta_1 \in (U^j_2)_K(x, \xi) \setminus (U^j_2)_K \), \(\delta_2 \in H_K(x, \xi) \setminus H_K \), and \((U^j_2)_K(x, \xi)\) (resp. \(H_K(x, \xi)\)) is the stabilizer of \((x, \xi)\) (resp. \((x, \xi)\)) in \((U^j_2)_K\) (resp. \(H_K\)).

Suppose now \((x, \xi)\) is of type (a). Then \(H_K(x, \xi) = (N_2)_K\). By integrating over \((U^j_2)_K \setminus (U^j_2)^A\) this part of the sum becomes
HYPERCUSPIDALITY FOR UNITARY GROUPS

\[ \sum_{(a)} \sum_{(u_2^t)^{N_2} \setminus H_A} \omega_\psi(s(\delta_1 u_2 g, \delta_2 h)) \Phi(x, \xi) du_2 \]

Next, we integrate this kernel against a cusp form \( f \) on \( H_K \setminus H_A \) to get

\[ \sum_{(a)} \int \int \omega_\psi(s(u_2 g, \delta_2 h)) \Phi(x, \xi) f(h) du_2. \]

Note that the left action of \( N_2 \) on \((x, \xi)\) of type (a) can be written as a right action of \( U_2^t \), so the inner integral is invariant under \((N_2)_K \setminus (N_2)_A\). Then the last integral reads;

\[ \sum_{(a)} \int \int \omega_\psi(s(u g, h)) \Phi((x, \xi)) du \int f(n h) dn d h. \]

Now \( f \) is a cusp form so this integral is zero, and we are done. For orbits of type (b) replace \( N_2 \) by \( N_1 \) and use the same reasoning. This concludes the proof of Proposition 1.1.

We continue the proof of Theorem 1.1. If \( \theta_0^\phi(\pi, s)f \) is cuspidal, then we use Proposition 1.1 for \( P_1^i \). We write then \( \Phi(x_1, \ldots, x_{i-1}, 0) \) as \( \Phi(x_1, \ldots, x_{i-1}) \Phi(0) \) and define \( \gamma'(h) \) to be \( \{\omega_\psi^{1/2}(s_{1/2} h)\Phi\}(0) \). Conversely, we write the lift \( \theta_0^\phi(\pi^\gamma, s_{1/2})f \) as the integral on the right hand side of Proposition 1.1. We consider this integral as a function of \( U(i-1, i-1) \) imbedded in \( P_1^i \). We compute its zero Fourier coefficients in the direction of all standard maximal parabolic subgroups of \( U(i-1, i-1) \). Then these are all integrals

\[ \int_{H_K \setminus H_A} \sum_{W \times \cdots \times W} \omega_\psi(s(g, h)) \Phi(x_1, \ldots, x_l, 0, \ldots, 0) f(h) dh, \]

where \( l = 1 \) to \( i-1 \). But if all these are zero, then \( \theta_0^\phi(\pi, s)f \) is cuspidal. This concludes the proof of Part (a) of Theorem 1.1. The proof of Part (b) is a standard argument (cf. [Ra]).

2. Hypercuspidality. Let \( N \) (resp. \( U \)) denote the standard maximal unipotent subgroup of \( U(2, 3) \) (resp. \( U(3, 3) \)). Let \( \psi_{\xi, \eta} \) (resp. \( \psi_{\xi, \eta, i} \)) denote a nondegenerate character of \( N \) (resp. \( U \)) where \( \xi, \eta \neq 0 \) in \( L \) and \( \xi \neq 0 \) in \( K \) (cf. [Ge-Sh, p. 76]). For an automorphic cuspidal form \( f \) on \( U(2, 3) \) (resp. \( U(3, 3) \)) let \( W_{\xi, \eta}^\phi \) (resp. \( W_{\xi, \eta, i}^\phi \)) denote a \( \psi_{\xi, \eta} \) (resp. \( \psi_{\xi, \eta, i} \)) Fourier coefficient in the direction of \( N \) (resp. \( U \)), i.e., a Whittaker function. We write \( W(\pi, \psi_{\xi, \eta}) \) (resp. \( W(\pi, \psi_{\xi, \eta, i}) \)) for the space of all such Fourier coefficients, where
\( \pi \) is a cuspidal representation of \( U(2, 3) \) (resp. \( U(3, 3) \)). We say that \( \pi \) is \( \psi_{\zeta, \eta} \) (resp. \( \psi_{\zeta, \eta, t} \)) generic if \( W(\pi, \psi_{\zeta, \eta}) \) (resp. \( W(\pi, \psi_{\zeta, \eta, t}) \)) is nonzero. We say \( \pi \) to be generic if \( \pi \) is \( \psi_{\zeta, \eta} \) (resp. \( \psi_{\zeta, \eta, t} \)) generic for some \( \zeta, \eta \neq 0 \) in \( L \) and \( t \neq 0 \) in \( K \). If for all such \( \zeta, \eta \) and \( t \) the space \( W(\pi, \psi_{\zeta, \eta}) \) (resp. \( W(\pi, \psi_{\zeta, \eta, t}) \)) is zero, then we say that \( \pi \) is hypercuspidal.

**THEOREM 2.1.** If \( \pi \) is cuspidal generic on \( U(2, 3) \), then its \( \psi \) lift to \( U(3, 3) \) is generic.

Moreover,

\[
W^{\psi_{\zeta, \eta, -t/2}}_{\theta_{\mathbf{a}(\mathbf{n}, \mathbf{w})}} = \begin{cases} 
0 & \text{if } t \text{ is not a norm} \\
\int_{U_2 \cap H_A} \omega_f(s(g, h)) \Phi(xw_0, w_2, w_1) W^{\psi_{\zeta, \eta, -t/2}}_{\mathbf{a}(\mathbf{n}, \mathbf{w})}(h) dh & \text{if } t \text{ is a norm,}
\end{cases}
\]

where \( \mathbf{a} \mathbf{a} = t \).

**PROOF.** Write \( U = U_1 \cdot U_2 \) where

\[
U_1 = \begin{bmatrix}
1 & a_1 & a_2 & a_3 \\
1 & 1 & a_4 & a_5 \\
1 & a_6 & 1 & 1 \\
1 & 1 & 1 & 1
\end{bmatrix},
\]

\( U_2 = U_2(a, b, c) = \begin{bmatrix}
1 & c & 1 \\
b & a & 1 \\
1 & 1 & 1
\end{bmatrix},
\]

with \( a_1, a_4, a_6 \in K \) and \( \bar{z} = ac - b \). Next we integrate the theta lift of \( f \) over \( (U_1)_k \backslash (U_1)_A \) against the character \( \psi_{\zeta, \eta, -t/2}(u_1) = \psi((-t/2)a_1) \). This integral is zero unless

\[
[\langle x_k, x_l \rangle]^w_{U_1}_{1 \leq k, l \leq 3} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

Let \( (Y_1)_K \) denote the subset of \( (X_1)_K \) satisfying the above matrix equation. Then

\[
W^{\psi_{\zeta, \eta, -t/2}}_{\theta_{\mathbf{a}(\mathbf{n}, \mathbf{w})}} = \int_{U_2 \cap H_A} \int_{(Y_1)_K} \omega_f(s(u, h)) \Phi(x_1, x_2, x_3, f(h) \psi_{\zeta, \eta, t}(u)) du dh .
\]

The orbits of \( (Y_1)_K \) under the diagonal action of \( H_K \) can have the following representatives:
if \( t \) is not a norm,
\[
(x, \xi) = (aw_0, w_1, 0)
\]
if \( t \) is a norm,
\[
(x, \xi) = (aw_0, lw_1, w_1)
\]
\[
(x, \xi) = (aw_0, 0, 0)
\]
\[
(x, \xi) = (aw_0, w_2, w_1)
\]
where \( \alpha = t \) and \( l \) in \( L \). This is clear by using the action of \((P_1)_K\) and \((P_2)_K\) on \((Y_1)_K\).

For orbits of type (a), (c), (d) and (f), the action of \((U_2)_K \backslash (U_2)_A\) gives an integration of an additive character over \( L \backslash A_L \). For orbits of type (b) and (e), observe that
\[
\omega \phi(s(u_2(a, b, c)g, h))\Phi(x, lw_1, w_1)\psi_\xi(a)\psi_\zeta(c)dadcdb
\]
and
\[
\omega \phi(s(g, z_3h))\Phi(axw_0 + aw_1, w_2 + aw_1, w_1) \psi_\xi(a)\psi_\zeta(c)dadcdb.
\]
Next we change the variable \( c \) (if \( l \neq 0 \)), so we end with an integration of an additive character over \( L \backslash A_L \). As for orbits of type (g), write \( N \) as \( Z_3 \tilde{U} \), and note that \( \tilde{U}_K \) stabilizes \((aw_0, w_2, w_1)\). Now the integral reads;
\[
\int \int \omega \phi(s(u_2g, h))\Phi(xw_0, w_2, w_1)\psi_\xi(h)dhdu_2.
\]
Observe also that
\[
\omega \phi(s(u_2(a, b, c)g, h))\Phi(axw_0, w_2, w_1) = \Phi(axw_0 + cw_2 + bw_1, w_2 + aw_1, w_1),
\]
and
\[
\omega \phi(s(g, z_3h))\Phi(axw_0 + cxw_2 + abw_1, w_2 + aw_1, w_1).
\]
By using the above equations and a change of variables, the last integral reads;
\[
\int \int \omega \phi(s(g, zh))\Phi(axw_0, w_2, w_1)\psi_\xi\eta(z)\psi_\zeta\eta(z)f(h)dhdy
dh
\]
and
\[
\int \int \omega \phi(s(g, h))\Phi(axw_0, w_2, w_1)\psi_\xi\eta(z)f(z^{-1}h)dzdh.
\]
Note that the last integral cannot be zero for all $\Phi$ in $S((X_1)_{\mathcal{A}})$ (see also [Wa, Theorem 3.1]). So this concludes the proof of Theorem 2.1.

**Corollary 2.1.** Let $\pi$ be a cuspidal representation of $U(2, 3)$. If $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal, then $\pi$ must be hypercuspidal.

**Proof.** If $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal then $\theta^5(\pi_\gamma', s_\gamma') = 0$ (resp. $\theta^3(\pi_\gamma', s_\gamma') = 0$) where $\gamma'(h) = \omega_{\psi}^2(s_{\gamma'}(h))\Phi(0)$ (resp. $\gamma'(h) = \omega_{\psi}^2(s_{\gamma'}(h))\Phi(0, 0)$) (cf. Proposition 1.1). So $\pi_\gamma'$ (resp. $\pi_\delta'$) cannot be generic (cf. Theorem 2.1).

**Theorem 2.2.** Let $\pi$ be a cuspidal representation of $U(2, 3)$. Assume $\theta^4(\pi, s)$ (resp. $\theta^5(\pi, s)$) is cuspidal. Then it is hypercuspidal.

**Proof.** Consider $\theta^4(\pi, s)$. Write the maximal unipotent subgroup of $U(4, 4)$ as $U_1 \cdot U_2$ (as done previously for $U(3, 3)$). The integral of the theta lift over $(U_1)_{\mathcal{K}} \cdot (U_1)_{\mathcal{A}}$ against $\psi_{\xi, \eta, h = \omega_{t/2}}$ is zero unless

$$[\langle x_k, x_l \rangle_w]_{1 \leq k, l \leq 4} = \begin{bmatrix} t & 0 \\ 0 & 0 \\ \vdots & \ddots \\ 0 & 0 \end{bmatrix}.$$ 

Let $(Y_1)_K$ denote the subset of $(X_1)_K$ satisfying the above matrix equation. Then

$$W_{\theta^4(\pi, s)}^{(y, z), -(t/2)}(g) = \int_{(U_2)_{\mathcal{K}}} \int_{(U_2)_{\mathcal{A}}} \sum_{(y, z, h)_{\mathcal{K}, \mathcal{A}}} \omega_{\psi}(s(ug, h))\Phi(x_1, x_2, x_3, x_4)f(h)\psi_{\xi, \eta, h = \omega_{t/2}}(u)du dh .$$

The orbits of $(Y_1)_K$ under the left diagonal action of $H_K$ can have the following representatives:

(a) $(x_1, \xi_1, \xi_2, \xi_3) = (aw_0, aw_1 + bw_2, w_1, w_2)$

(b) $(x_1, \xi_1, \xi_2, \xi_3) = (aw_0, w_1, aw_1, bw_1)$

(c) $(x_1, \xi_1, \xi_2, \xi_3) = (w_{-2} + (t/2)w_2, w_1, aw_1, bw_1)$

(d) $(x_1, \xi_1, \xi_2, \xi_3) = (w_{-2} + (t/2)w_2, 0, 0, 0),$ 

where $a, b$ in $L$, $\omega = t$, and $\xi_1, \xi_2, \xi_3$ can appear in any order. It is now left to show that the integral corresponding to each of the above orbits vanishes. If $\xi_i = 0$ for some $i$, then the conclusion is easy. Otherwise we end as we do in Theorem 2.1 for orbits of type (b) and (e).

A similar argument holds for the theta lift $\theta^5(\pi, s)$. 

REFERENCES


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