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A QUALITATIVE THEORY OF SIMILARITY PSEUDOGROUPS: HOLONOMY OF THE ORBITS WITH BUBBLES

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Abstract. We are concerned with the qualitative theory of high codimension foliations. In order to restrict the object of our study, we consider the actions of a pseudogroup of local similarity transformations of a Euclidean space. For an orbit "with bubbles" of such an action, we obtain analogs of the qualitative properties of codimension one foliations.

1. Introduction. Let $\mathscr{G}$ be a codimension q foliation of a manifold. A foliation $\mathscr{G}$ is said to be *transversely similar* if all holonomy transition functions of $\mathscr{G}$ are local similarity transformations of $\boldsymbol{R}^q$. Therefore, if $\mathscr{G}$ is transversely similar, we obtain the holonomy pseudogroup of $\mathscr{G}$ which consists of local similarity transformations of $\boldsymbol{R}^q$. Since there exists a correspondence between the terminology in the qualitative theory of foliations and that of pseudogroups, we treat pseudogroups of local similarity transformations of $\boldsymbol{R}^q$ instead of codimension q transversely similar foliations.

If $q=1$, then transversely similar foliations are well studied. In particular, these foliations is said to be *transversely affine foliations*. Indeed, the qualitative theories of codimension one C^2 foliations are studied by many people. Here, we recall a few interesting theorems. Let $\mathscr{F}$ be a transversely orientable, codimension one foliation of class C^2 on a closed smooth manifold M. A leaf F of $\mathscr{F}$ is *semiproper* if it is asymptotic to itself from at most one side. Let F be a *nonproper* but *semiproper* leaf of $\mathscr{F}$, that is, a leaf which is asymptotic from exactly one side (which is called the *nonproper side*). In this case, F is an exceptional leaf contained in a local minimal set $\mathscr{M}$ of exceptional type (see [1]).

THEOREM A (cf. Sacksteder [9]). *The local minimal set $\mathscr{M} \supset F$ has a leaf with linearly contracting holonomy.*

THEOREM B (cf. Hector [4], Duminy (see Cantwell-Conlon [3])). *F has a germinal contracting holonomy on the nonproper side of F.*

THEOREM C (cf. Cantwell-Conlon [2]). *If the local minimal set $\mathscr{M} \supset F$ is Markov, that is, a local minimal set whose holonomy is modeled on symbolic dynamics, then $\mathscr{M}$ contains at most finitely many semiproper exceptional leaves.*

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THEOREM D (cf. Inaba [5]). *Suppose that a foliation $\mathscr{F}$ is transversely piecewise linear. If $\mathscr{F}$ is topologically conjugate to a C^2 foliation, then the local minimal set $\mathscr{M} \supset F$ contains at most finitely many semiproper exceptional leaves.*

On the other hand, in the case of $q \geq 2$, interesting results have not been obtained yet, because the asymptotic behaviors of leaves are very chaotic. For example, even the types of minimal sets do not seem to have been completely characterized.

In order to restrict the object of the study, Nishimori formulated a concept of "orbits with bubbles", which is a substitute for the concept of "semiproper orbits" in the codimension one case. He obtained an analog of Sacksteder's theorem in [7]. In the preceding paper [6], the first author of the present paper obtained a weak verison of an analog of the theorem of Hector and Duminy. In this paper, we continue to investigate the qualitative properties of the orbits with bubbles. In particular, we prove analogs of Theorems B, C and D for codimension one foliations.

2. Similarity pseudogroups and the statement of the results. In this section, we recall similarity pseudogroups and state our main results. For more information on pseudogroups in our sense, see Nishimori [7] and [8].

Let $\Gamma_{q,+}^{\mathrm{sim},*}$ be the set of all local homeomorphisms $h: U \to V$ of $\boldsymbol{R}^q$ satisfying the following two properties:

(1) The domain U and the range V of h are both non-empty, bounded and convex open subsets of $\boldsymbol{R}^q$. We denote $D(h)=U$ and $R(h)=V$.

(2) There exists an orientation preserving similarity transformation $\bar{h}: \boldsymbol{R}^q \to \boldsymbol{R}^q$ such that $\bar{h}(D(h))=R(h)$ and the restriction $\bar{h}|_{D(h)}=h$. Such $\bar{h}$ is determined uniquely by h, and is called the *extension* of h.

Let $\Gamma_{q,+}^{\mathrm{sim}}=\Gamma_{q,+}^{\mathrm{sim},*} \cup \{\mathrm{id}_{\boldsymbol{R}^q}, \mathrm{id}_{\varnothing}\}$, where $\mathrm{id}_{\varnothing}$ is the unique transformation on the empty set $\varnothing$.

DEFINITION 2.1. A subset Γ of $\Gamma_{q,+}^{\mathrm{sim}}$ is called a *pseudogroup* if it satisfies the following three conditions:

(1) $\mathrm{id}_{\boldsymbol{R}^q} \in \Gamma$.

(2) If $f, g \in \Gamma$, then $f \circ g \in \Gamma$.

(3) If $f \in \Gamma$, then $f^{-1} \in \Gamma$.

DEFINITION 2.2. Let Γ_0 be a subset of $\Gamma_{q,+}^{\mathrm{sim},*}$.

(1) A subset Γ_0 is said to be *symmetric* if $h \in \Gamma_0$ implies $h^{-1} \in \Gamma_0$.

(2) Denote by $\langle \Gamma_0 \rangle$ the intersection of all the pseudogroups $\Gamma \subset \Gamma_{q,+}^{\mathrm{sim}}$ which contain Γ_0. Then $\langle \Gamma_0 \rangle$ is also a pseudogroup and is called the *pseudogroup generated by* Γ_0.

Let Γ_0 be a symmetric subset of $\Gamma_{q,+}^{\mathrm{sim},*}$, and $\Gamma=\langle \Gamma_0 \rangle$. Denote by $W(\Gamma_0)$ the set of all words with Γ_0 as alphabet, that is, $W(\Gamma_0)=\coprod_{n=0}^{\infty}(\Gamma_0)^n$, where $(\Gamma_0)^n$ means the product of n copies of Γ_0 and $(\Gamma_0)^0$ the set consisting only of the empty word ( ). This set $W(\Gamma_0)$ is useful in treating the pseudogroup Γ, because the maps $\Phi\colon W(\Gamma_0)\to\Gamma$ defined by $\Phi((\ ))=\mathrm{id}_{\boldsymbol{R}^q}$ for the empty word $(\ )$ and by $\Phi(w)=h_m\circ\cdots\circ h_1$ for a word $w=(h_m,\dots,h_1)$ is surjective.

NOTATION 2.3. (1) For a word $w\in W(\Gamma_0)$, we put $g_w=\Phi(w)$.

(2) For words $w=(h_m,\dots,h_1)$, $w'=(k_n,\dots,k_1)\in W(\Gamma_0)$, we denote the product of w and w', and the inverse of w by

$$ww'=(h_m,\dots,h_1,k_n,\dots,k_1)\,,\quad w^{-1}=(h_1^{-1},\dots,h_m^{-1})\,.$$

Note that $g_{ww'}=g_w\circ g_{w'}$ and $g_w^{-1}=g_{w^{-1}}=\Phi(w^{-1})=h_1^{-1}\circ\cdots\circ h_m^{-1}$.

DEFINITION 2.4. Let $x\in\boldsymbol{R}^q$. The *Γ-orbit of x* is the set $\Gamma(x)=\{g(x)\mid g\in\Gamma,\ x\in D(g)\}$.

DEFINITION 2.5. Let $x\in\boldsymbol{R}^q$. The *stabilizer pseudogroup of x* is the set $\mathrm{Stab}(x)=\{g\in\Gamma\mid x\in D(g)\text{ and }g(x)=x\}$.

Here, we give some examples of similarity pseudogroups. For $x\in\boldsymbol{R}^q$ and $r>0$, we denote by $U(x;r)$ the r-neighbourhood of x.

EXAMPLE 2.6. Consider the case $q=2$ and let $x_1=(0,0)$, $x_2=(1,0)$.

(1) Let $U_1=U((1/2,0);1/2+\varepsilon)$ for some $1/2>\varepsilon>0$. Define similarity transformations $\bar{h}_1$, $\bar{h}_2$ of $\boldsymbol{R}^2$ by $\bar{h}_1(x,y)=(x/3,y/3)$, $\bar{h}_2(x,y)=((x+2)/3,y/3)$ and let $h_i=\bar{h}_i|_{U_1}$. Denote by Γ the pseudogroup generated by $\Gamma_0=\{h_1,h_2,h_1^{-1},h_2^{-1}\}\subset\Gamma_{2,+}^{\mathrm{sim},*}$. Then $\overline{\Gamma(x_1)}=\overline{\Gamma(x_2)}$ is the standard Cantor set in $[0,1]\times\{0\}\subset\boldsymbol{R}^2$. Note that h_i for $i=1,2$ is a contraction to x_i which is the unique fixed point of h_i.

(2) Let $U_2=U((0,0);1+\varepsilon)$ for some small $\varepsilon>0$. Take $\bar{h}$ to be the rotation around $(0,0)\in\boldsymbol{R}^2$ by angle θ, and define $h=\bar{h}|_{U_2}$. Denote by Γ the pseudogroup generated by $\Gamma_0=\{h,h^{-1}\}$. If θ/π is irrational, then $\overline{\Gamma(x_2)}=S^1\subset\boldsymbol{R}^2$ and no element of Γ is a contraction to x_2.

(3) We modify the first example. Let U_1 be as in (1). Define similarity transformations $\bar{h}'_1$, $\bar{h}'_2$ of $\boldsymbol{R}^2$ as follows: $\bar{h}'_1$ is the rotation around $(1/2,0)$ by angle π while $\bar{h}'_2$ is the composite $\bar{h}'_1\circ\bar{h}_1$ of $\bar{h}_1$ in Example (1) and $\bar{h}'_1$. Let $h'_i=\bar{h}'_i|_{U_1}$. Denote by Γ' the pseudogroup generated by $\Gamma'_0=\{h'_1,h'_2,h_1'^{-1},h_2'^{-1}\}\subset\Gamma_{2,+}^{\mathrm{sim},*}$. Then $\Gamma'(x_1)=\Gamma'(x_2)$ and $\overline{\Gamma'(x_1)}$ is the standard Cantor set in $[0,1]\times\{0\}\subset\boldsymbol{R}^2$. Note that h'_1 (resp. h'_2) has a unique fixed point $(1/2,0)$ (resp. $(3/4,0)$), which are not contained in $\overline{\Gamma'(x_1)}$.

DEFINITION 2.7. The Γ-orbit $\Gamma(x)$ of $x\in\boldsymbol{R}^q$ is said to be *proper* if for every $y\in\Gamma(x)$, $\overline{\Gamma(x)\setminus\{y\}}$ does not contain y. Otherwise, $\Gamma(x)$ is said to be *nonproper*.

In order to consider analogs of the theorems in the codimension one case, we have to introduce a substitute for the concept of "semiproper Γ-orbits". As an attempt, Nishimori introduced the concept of "Γ-orbits with bubbles".

DEFINITION 2.8 (cf. [7, Definition 3.2]). Let $x_\star\in\boldsymbol{R}^q$. We say that the Γ-orbit $\Gamma(x_\star)$ of $x_\star$ is *with bubbles* if for each $x\in\Gamma(x_\star)$, there exists a non-empty, bounded and convex open subset B_x (called a *bubble at* x) of $\boldsymbol{R}^q$ satisfying the following three properties:

(a) $x \in \partial B_x$, where ∂B_x denotes the boundary of B_x.

(b) If $x_1, x_2 \in \Gamma(x_\bigstar)$ and $x_1 \neq x_2$, then $B_{x_1} \cap B_{x_2} = \emptyset$.

(c) If $h \in \Gamma_0$ and $x \in D(h) \cap \Gamma(x_\bigstar)$ satisfy $h(x) \neq x$, then $\bar{h}(B_x) = B_{h(x)}$, where $\bar{h}$ is the extension of h.

EXAMPLE 2.9. In Example 2.6, (1), the Γ-orbits $\Gamma(x_1)$ and $\Gamma(x_2)$ are with bubbles. For example, take $V^0 = U((-1/2, 0); 1/2)$ as B_{x_1}, $h_2(V^0) = U((1/2, 0); 1/6)$ as $B_{h_2(x_1)}$, $h_1 \circ h_2(V^0) = U((1/6, 0); 1/18)$ as $B_{h_1 \circ h_2(x_1)}$, $h_2^2(V^0) = U((5/6, 0); 1/18)$ as $B_{h_2^2(x_1)}$ and so on. By the same construction, we can find bubbles of $\Gamma(x_2)$.

On the contrary, as we see later, $\Gamma(x_2)$ in Example 2.6, (2) and $\Gamma'(x_1)$ in (3) cannot be with bubbles.

From now on, a finite, symmetric subset $\Gamma_0 \subset \Gamma_{q,+}^{\mathrm{sim},*}$, $\Gamma = \langle \Gamma_0 \rangle$ and $x_\bigstar \in \boldsymbol{R}^q$ are supposed to satisfy the following two properties:

(S1) There exists a constant $\varepsilon > 0$ such that the distance $\mathrm{dist}(\overline{\Gamma(x_\bigstar)}, \bigcup_{h \in \Gamma_0} \partial D(h))$ is greater than ε.

(S2) The Γ-orbit $\Gamma(x_\bigstar)$ of $x_\bigstar$ is nonproper and with bubbles $\{B_x\}_{x \in \Gamma(x_\bigstar)}$.

In this situation, Nishimori obtained an analog of Sacksteder's theorem.

THEOREM 2.10 (cf. Nishimori [7, Theorem 3.3]). *Assume that the pseudogroup* Γ *generated by a finite, symmetric subset* Γ_0 *of* $\Gamma_{q,+}^{\mathrm{sim},*}$ *and* $x_\bigstar \in \boldsymbol{R}^q$ *satisfy* (S1) *and* (S2). *Then there exist* $g \in \Gamma$ *and* $z \in \overline{\Gamma(x_\bigstar)}$ *such that* $z \in D(g)$, $g(z) = z$ *and that* g *is a contraction, that is, the similitude ratio of* g *is less than* 1.

The organization of the rest of this paper is as follows. In the next section, we continue to list terminology and notation and find a common domain of generators of holonomy of a Γ-orbit with bubbles. In the preceding paper [6], the first author proved the existence of nontrivial holonomy for a Γ-orbit with bubbles, but could not specify the existence of contracting holonomy. In Section 4, we prove the following theorems, the first of which is a complete analog of the theorem of Hector-Duminy in this sense.

THEOREM 2.11. *Assume that the pseudogroup* Γ *generated by a finite, symmetric subset* Γ_0 *of* $\Gamma_{q,+}^{\mathrm{sim},*}$ *and* $x_\bigstar \in \boldsymbol{R}^q$ *satisfy* (S1) *and* (S2). *Then there exists* $g \in \Gamma$ *such that* $x_\bigstar \in D(g)$, $g(x_\bigstar) = x_\bigstar$ *and that the similitude ratio of* g *is less than* 1, *that is,* g *is a contraction to* $x_\bigstar$.

THEOREM 2.12. *The closure* $\overline{\Gamma(x_\bigstar)}$ *of* $\Gamma(x_\bigstar)$ *contains at most finitely many nonproper* Γ*-orbits with bubbles.*

In the final section, we treat the case of $q = 2$ and prove the following:

THEOREM 2.13. *Suppose that* $q = 2$. *Let* $x \in \overline{\Gamma(x_\bigstar)}$ *and let* $g \in \mathrm{Stab}(x)$ *be a rotation at* x. *Then there exists* $n \in \boldsymbol{N}$ *such that* $\bar{g}^n = \mathrm{id}_{\boldsymbol{R}^2}$.

THEOREM 2.14. *Suppose that $q=2$ and $x\in\overline{\Gamma(x_\star)}$ so that $\Gamma(x)$ is a nonproper Γ-orbit with bubbles. Then the Γ-orbit $\Gamma(x)$ of x has a compactly supported holonomy, that is, there exists a compact subset K of $\Gamma(x)$ such that $\bar{g}_w=\mathrm{id}_{\boldsymbol{R}^2}$ for every $z\in\Gamma(x)\setminus K$ and every loop $w=(h_m,\dots,h_1)\in W(\Gamma_0)$ at z satisfying $h_i\circ\cdots\circ h_1(z)\in\Gamma(x)\setminus K$ $(1\le i\le m)$.*

We refer the reader to Definition 3.2 below for the definition of loops.

3. Domains of generators of holonomy of Γ-orbit with bubbles. Assume that the pseudogroup Γ generated by a finite, symmetric subset Γ_0 of $\Gamma_{q,+}^{\mathrm{sim},*}$ and $x_\star\in\boldsymbol{R}^q$ satisfy (S1) and (S2). Let $\{B_x\}_{x\in\Gamma(x_\star)}$ be bubbles of $\Gamma(x_\star)$. We continue to recall more notions from Nishimori [7] and Matsuda [6].

DEFINITION 3.1. (1) For a word $w\in W(\Gamma_0)$, we denote by $|w|$ the word length of w, that is, $|w|=0$ for the empty word $w=(\ )$ and $|w|=m$ for $w=(h_m,\dots,h_1)$.

(2) For $x, y\in\boldsymbol{R}^q$ with $y\in\Gamma(x)$, put

$$d_{\Gamma_0}(x,y)=\min\{|w|\mid w\in W(\Gamma_0),\ x\in D(g_w)\text{ and }g_w(x)=y\}\,.$$

Then d_{Γ_0} is a natural distance on the orbit $\Gamma(x)$.

DEFINITION 3.2. Let $x, y\in\boldsymbol{R}^q$. A word $w\in W(\Gamma_0)$ is called a *chain at x to y* if $x\in D(g_w)$ and $g_w(x)=y$. Furthermore, if $g_w(x)=x$, then w is called a *loops at x*.

NOTATION 3.3. For a chain $w=(h_m,\dots,h_1)\in W(\Gamma_0)$ at $x\in\boldsymbol{R}^q$, denote $g_k=h_k\circ\cdots\circ h_1\in\Gamma$, $g_0=\mathrm{id}_{\boldsymbol{R}^q}$ and $x_k=g_k(x)$ for every $k=0,1,\dots,m$. Note that $g_w=g_m$, $x_0=x$ and $x_k\in D(h_{k+1})$ for $k=0,1,\dots,m-1$.

DEFINITION 3.4. A word $w=(h_m,\dots,h_1)\in W(\Gamma_0)$ is called a *simple chain* (resp. *simple loop*) *at $x\in\boldsymbol{R}^q$* if

(1) w is a chain at x,

(2) $x_i\neq x_j$ for every $0\le i<j\le m$ (resp. $x_i\neq x_j$ for every $0\le i<j\le m-1$ and $g_w(x)=x$).

Note that if $w\in W(\Gamma_0)$ is a simple chain at x to y, then w^{-1} is a simple chain at y to x.

DEFINITION 3.5. Let $w=(h_m,\dots,h_1)\in W(\Gamma_0)$ $(m\ge 1)$ be a chain at $x\in\boldsymbol{R}^q$.

(1) We define a *sub-chain at x_{j-1}* by $w_{ij}=(h_i,\dots,h_j)$ $(1\le j\le i\le m)$.

(2) A sub-chain w_{ij} is a *sub-loop at x_{j-1}* if $x_i=x_{j-1}$.

(3) A sub-chain (resp. sub-loop) w_{ij} at x_{j-1} is called a *proper sub-chain* (resp. *proper sub-loop*) of w if $w_{ij}\neq w$.

DEFINITION 3.6. Let $w=(h_m,\dots,h_1)\in W(\Gamma_0)$ $(m\ge 1)$ be a chain at $x\in\boldsymbol{R}^q$ and w_{ij} a sub-loop at x_{j-1}. Then we can define a new chain $w\setminus w_{ij}$ at x by $(h_m,\dots,h_{i+1},h_{j-1},\dots,h_1)$.

NOTATION 3.7. For $g\in\Gamma_{q,+}^{\mathrm{sim}}$, we denote the *similitude ratio of g* by $\mathrm{SR}(g)$.

The following lemma is obvious.

LEMMA 3.8. *Let* $g \in \Gamma^{\mathrm{sim}}_{q,+}$. *If there exists* $x \in D(g)$ *and* $r > 0$ *so that* $U(x; r) \subset D(g)$, *then* $g(U(x; r)) = U(g(x); r \cdot \mathrm{SR}(g))$.

The next lemma is an easy consequence of the definition of bubbles and the assumption (S1).

LEMMA 3.9. *Let* $h \in \Gamma_0$ *and* $x \in D(h) \cap \Gamma(x_\star)$.
(1) $U(x; \varepsilon) \subset D(h)$ *and* $h(U(x; \varepsilon)) = U(h(x); \varepsilon \cdot \mathrm{SR}(h))$.
(2) *If* $h(x) \neq x$, *then* $\bar{h}(B_x) = B_{h(x)}$, *hence* $\mathrm{SR}(h) = \mathrm{diam}(B_{h(x)})/\mathrm{diam}(B_x)$.

Nishimori [7, Lemma 4.5] proved that the total volume and the diameters of all bubbles are bounded.

Let $\delta = \sup\{\mathrm{diam}(B_y) \mid y \in \Gamma(x_\star)\}$.

LEMMA 3.10. *Let* $w = (h_m, \dots, h_1) \in W(\Gamma_0)$ *be a simple chain at* $x \in D(g_w) \cap \Gamma(x_\star)$. *Then* g_w *is defined on* $U(x; \varepsilon \cdot \mathrm{diam}(B_x)/\delta)$ and $\bar{g}_w(B_x) = B_{g_w(x)}$, *hence* $\mathrm{SR}(g_w) = \mathrm{diam}(B_{g_w(x)})/\mathrm{diam}(B_x)$.

PROOF. Nishimori proved similar lemmas in [7, Lemmas 4.3, 4.7] for a short-cut w at x, which is a simple chain at x with some auxiliary conditions. But in these proofs, he used only the fact that w is a simple chain at x. So the same argument is applicable to the proof of this lemma. ∎

LEMMA 3.11. *Let* $w = (h_m, \dots, h_1) \in W(\Gamma_0)$ *be a simple loop at* $x \in D(g_w) \cap \Gamma(x_\star)$ *with* $|w| \geq 2$, *that is,* $m \geq 2$. *Then* g_w *is defined on* $U(x; \varepsilon \cdot \mathrm{diam}(B_x)/\delta)$ *and* $\bar{g}_w(B_x) = B_{g_w(x)} (= B_x)$, *hence* $\mathrm{SR}(g_w) = 1$.

PROOF. Put $w' = (h_{m-1}, \dots, h_1)$ and $w'' = (h_m)$. Then $w = w''w'$ and w' is a simple chain at x to $g_{w'}(x)$ and w'' is a simple chain at $g_{w'}(x)$ to $g_w(x) = x$. By Lemma 3.10, $\bar{g}_{w'}(B_x) = B_{g_{w'}(x)}$, $\bar{g}_{w''}(B_{g_{w'}(x)}) = B_{g_{w''}(x)} = B_x$ and

$$g_{w'}\left(U\left(x; \varepsilon \cdot \frac{\mathrm{diam}(B_x)}{\delta}\right)\right) = U\left(g_{w'}(x); \varepsilon \cdot \frac{\mathrm{diam}(B_{g_{w'}(x)})}{\delta}\right) \subset D(g_{w''}) .$$

Hence $\bar{g}_w(B_x) = \bar{g}_{w''} \circ \bar{g}_{w'}(B_x) = B_x$, and

$$U\left(x; \varepsilon \cdot \frac{\mathrm{diam}(B_x)}{\delta}\right) \subset D(g_{w''} \circ g_{w'}) = D(g_w) .$$

∎

Put $M = \sup_{h \in \Gamma_0} \mathrm{SR}(h) \in [1, \infty)$.

LEMMA 3.12. *Let* $w = (h) \in W(\Gamma_0)$ *be a simple loop at* $x \in D(g_w) \cap \Gamma(x_\star)$ *that is,* $|w| = 1$. *Then* g_w *is defined on* $U(x; \varepsilon \cdot \mathrm{diam}(B_x)/(M \cdot \delta))$ *and*

$$g_w\left(U\left(x; \frac{\varepsilon}{M}\cdot\frac{\operatorname{diam}(B_x)}{\delta}\right)\right)=U\left(x; \varepsilon\cdot\frac{\operatorname{diam}(B_x)}{\delta}\cdot\frac{\operatorname{SR}(g_w)}{M}\right)\subset U\left(x; \varepsilon\cdot\frac{\operatorname{diam}(B_x)}{\delta}\right).$$

PROOF. Since $g_w\in\Gamma_0$, $x\in D(g_w)\cap\Gamma(x_\bigstar)$ and $\operatorname{diam}(B_x)/(M\cdot\delta)<1$, this is an easy consequence of Lemma 3.9. ■

DEFINITION 3.13. A word $w\in W(\Gamma_0)$ is called a *basic loop at* $x\in \boldsymbol{R}^q$ if there exist $\xi, \eta\in W(\Gamma_0)$ such that ξ is a simple chain at x, η is a simple loop at $g_\xi(x)$ and $w=\xi^{-1}\eta\xi$. We call ξ the *simple chain part of* w and η *the simple loop part of* w.

NOTATION 3.14. Unless otherwise stated, for a basic loop w at $x\in\Gamma(x_\bigstar)$, we denote the simple chain part of w by ξ and the simple loop part by η.

Note that for every basic loop $w=\xi^{-1}\eta\xi$ at x, we have $\operatorname{SR}(g_w)=\operatorname{SR}(g_\xi^{-1})\cdot\operatorname{SR}(g_\eta)\cdot \operatorname{SR}(g_\xi)=\operatorname{SR}(g_\eta)$.

LEMMA 3.15. *Let* $w=\xi^{-1}\eta\xi\in W(\Gamma_0)$ *be a basic loop at* $x\in D(g_w)\cap\Gamma(x_\bigstar)$ *with* $|\eta|\geq 2$. *Then* g_w *is defined on* $U(x; \varepsilon\cdot\operatorname{diam}(B_x)/\delta)$ *and* $\bar{g}_w(B_x)=B_{g_w(x)}(=B_x)$, *hence* $\operatorname{SR}(g_w)=1$.

PROOF. Note that ξ is a simple chain at x to $g_\xi(x)$, η is a simple loop at $g_\xi(x)$ with $|\eta|\geq 2$ and ξ^{-1} is a simple chain at $g_\xi(x)$ to x. Therefore this lemma follows from Lemmas 3.10 and 3.11. ■

Similarly, using Lemmas 3.10 and 3.12, we obtain the following:

LEMMA 3.16. *Let* $w=\xi^{-1}\eta\xi\in W(\Gamma_0)$ *be a basic loop at* $x\in D(g_w)\cap\Gamma(x_\bigstar)$ *with* $|\eta|=1$. *Then* g_w *is defined on* $U(x; \varepsilon\cdot\operatorname{diam}(B_x)/(M\cdot\delta))$ *and*

$$g_w\left(U\left(x; \frac{\varepsilon}{M}\cdot\frac{\operatorname{diam}(B_x)}{\delta}\right)\right)=U\left(x; \frac{\varepsilon}{M}\cdot\frac{\operatorname{diam}(B_x)}{\delta}\cdot\operatorname{SR}(g_\eta)\right)\subset U\left(x; \varepsilon\cdot\frac{\operatorname{diam}(B_x)}{\delta}\right).$$

The next lemma follows from Lemma 3.15:

LEMMA 3.17. *Let* $w_i=\xi_i^{-1}\eta_i\xi_i\in W(\Gamma_0)$ $(i=1,\dots,m)$ *be a basic loop at* $x\in\Gamma(x_\bigstar)$ *with* $|\eta_i|\geq 2$ *for every* $i=1,\dots,m$ *and* $w=w_m\cdots w_1$. *Then* g_w *is defined on* $U(x; \varepsilon\cdot\operatorname{diam}(B_x)/\delta)$ *and* $\bar{g}_w(B_x)=B_{g_w(x)}$ $(=B_x)$, *hence* $\operatorname{SR}(g_w)=1$.

By the above observations, we can find the domains of the generators of $\operatorname{Stab}(x_\bigstar)$.

PROPOSITION 3.18. *There exists a subset* Ω_H *of* $W(\Gamma_0)$ *such that*

(1) *every* $w\in\Omega_H$ *is a basic loop at* $x_\bigstar$,

(2) *for every loop* $\zeta\in W(\Gamma_0)$ *at* $x_\bigstar$ *(hence* $g_\zeta\in\operatorname{Stab}(x_\bigstar)$*), there exist* $w_1,\dots,w_m\in\Omega_H$ *such that* $g_\zeta=g_{w_m\cdots w_1}$ *on* $D(g_{w_m\cdots w_1})$,

(3) *for every* $w\in\Omega_H$,

$$U\left(x_{\bigstar};\frac{\varepsilon}{M}\cdot\frac{\operatorname{diam}(B_{x_\bigstar})}{\delta}\right)\subset D(g_w)\,.$$

PROOF. By a standard argument (for example, by the basic loop theorem [8, Theorem 8.10]), there exists a subset Ω_H of $W(\Gamma_0)$ satisfying the condition (1) and (2).

By Lemmas 3.15 and 3.16, for every basic loop w at $x_{\bigstar}$,

$$U\left(x_{\bigstar};\frac{\varepsilon}{M}\cdot\frac{\operatorname{diam}(B_{x_\bigstar})}{\delta}\right)\subset D(g_w)\,.$$

■

4. Proof of Theorems 2.11 and 2.12.

PROOF OF THEOREM 2.11. Suppose on the contrary that for each $g\in\operatorname{Stab}(x_{\bigstar})$ we have $\operatorname{SR}(g)=1$, that is, g is a rotation which fixes $x_{\bigstar}$.

LEMMA 4.1. *For every* $x\in\Gamma(x_{\bigstar})$ *and* $g\in\operatorname{Stab}(x)$, *we have* $\operatorname{SR}(g)=1$.

LEMMA 4.2. *Let* $g\in\Gamma$ *with* $D(g)\cap\Gamma(x_{\bigstar})\neq\emptyset$. *Then* $\operatorname{SR}(g)=\operatorname{diam}(B_{g(x)})/\operatorname{diam}(B_x)$ *for every* $x\in D(g)\cap\Gamma(x_{\bigstar})$.

PROOF. There exists $w=(h_n,\dots,h_1)\in W(\Gamma_0)$ such that $g_w=g$. We prove the lemma by induction on n.

(I) For $n=1$, $w=(h_1)$ and $g_1=h_1\in\Gamma_0$. If $g_1(x)=x$, then $g_1\in\operatorname{Stab}(x)$. So by Lemma 4.1, $\operatorname{SR}(g_1)=1=\operatorname{diam}(B_{g_1(x)})/\operatorname{diam}(B_x)$. If $g_1(x)\neq x$, by the definition of bubbles, $\bar{g}_1(B_x)=B_{g_1(x)}$. Therefore $\operatorname{SR}(g_1)=\operatorname{diam}(B_{g_1(x)})/\operatorname{diam}(B_x)$.

(II) Assume that the assertion holds for n. Then

$$\begin{aligned}\operatorname{SR}(g_{n+1})&=\operatorname{SR}(h_{n+1})\cdot\operatorname{SR}(h_n\circ\cdots\circ h_1)\\&=\frac{\operatorname{diam}(B_{x_{n+1}})}{\operatorname{diam}(B_{x_n})}\cdot\frac{\operatorname{diam}(B_{x_n})}{\operatorname{diam}(B_{x_0})}=\frac{\operatorname{diam}(B_{x_{n+1}})}{\operatorname{diam}(B_{x_0})}=\frac{\operatorname{diam}(B_{g_{n+1}(x)})}{\operatorname{diam}(B_x)}\,.\end{aligned}$$

■

Without loss of generality, by the boundedness of the total volume of bubbles, we may assume that $\operatorname{diam}(B_{x_\bigstar})\geq\operatorname{diam}(B_y)$ for every $y\in\Gamma(x_{\bigstar})$. Note that $\operatorname{diam}(B_{x_\bigstar})=\operatorname{diam}(B_y)$ for only finitely many $y\in\Gamma(x_{\bigstar})$. Since the orbit $\Gamma(x_{\bigstar})$ is non-proper, we can choose and fix a point $z\in U(x_{\bigstar};\varepsilon/3)\cap\Gamma(x_{\bigstar})$ so that $\operatorname{diam}(B_z)<\operatorname{diam}(B_{x_\bigstar})$.

LEMMA 4.3. *Let* $w\in W(\Gamma_0)$ *be a chain at* $x_{\bigstar}$ *to* z. *Then* $U(x_{\bigstar};\varepsilon/3)\subset D(g_w^{-1})$.

PROOF. Note that w is not the empty word.

We write $w^{-1}=(h_m,\dots,h_1)$ $(|w^{-1}|=m\geqq 1,\ h_i\in\Gamma_0)$, and put $w_k^{-1}=(h_k,\dots,h_1)$ and $g_k=g_{w_k^{-1}}=g_{w_k}^{-1}=h_k\circ\cdots\circ h_1$ for $k=1,2,\dots,m$.

We prove $U(x_\star; \varepsilon/3) \subset D(g_k)$ by induction on k.

(I) For $k=1$, note that $g_1 = h_1 \in \Gamma_0$. Since $z \in D(g_1) \cap \Gamma(x_\star)$, by Lemma 3.9 we have $U(z; \varepsilon) \subset D(g_1)$. By the choice of z, we have $U(x_\star; \varepsilon/3) \subset U(z; \varepsilon) \subset D(g_1)$.

(II) Assume that the assertions hold for k. Then, by Lemma 4.2 and the choice of $x_\star$, $\mathrm{SR}(g_k) = \mathrm{diam}(B_{g_k(x_\star)})/\mathrm{diam}(B_{x_\star}) \leq 1$. Since $z \in U(x_\star; \varepsilon/3) \subset D(g_k)$, it follows that

$$g_k(z) \in g_k\left(U\left(x_\star; \frac{\varepsilon}{3}\right)\right) = U\left(g_k(x_\star); \frac{\varepsilon}{3} \cdot \mathrm{SR}(g_k)\right) \subset U\left(g_k(x_\star); \frac{\varepsilon}{3}\right).$$

Hence $U(g_k(x_\star); \varepsilon/3) \subset U(g_k(z); \varepsilon)$. Since $g_k(z) \in \Gamma(x_\star) \cap D(h_{k+1})$, we get $U(g_k(z); \varepsilon) \subset D(h_{k+1})$ by Lemma 3.9. Therefore $g_k(U(x_\star; \varepsilon/3)) \subset D(h_{k+1})$, that is,

$$U\left(x_\star; \frac{\varepsilon}{3}\right) \subset D(h_{k+1} \circ g_k) = D(g_{k+1}).$$

■

In Lemma 4.3, if we take w to be a simple chain at $x_\star$ to z, then w^{-1} is a simple chain at z to $x_\star$ and by Lemma 3.10, $\bar{g}_w^{-1}(B_z) = B_{g_w^{-1}(z)} = B_{x_\star}$. Furthermore, by the choice of z, $\mathrm{diam}(B_z) < \mathrm{diam}(B_{x_\star})$. Hence

$$\mathrm{SR}(g_w^{-1}) = \frac{\mathrm{diam}(B_{x_\star})}{\mathrm{diam}(B_z)} > 1.$$

On the other hand, by Lemma 4.3 and the choice of $x_\star$,

$$1 \geq \frac{\mathrm{diam}(B_{g_w^{-1}(x_\star)})}{\mathrm{diam}(B_{x_\star})} = \mathrm{SR}(g_w^{-1}),$$

a contradiction.

Therefore there exists $g \in \mathrm{Stab}(x_\star)$ such that $\mathrm{SR}(g) < 1$. This completes the proof of Theorem 2.11.

PROPOSITION 4.4. *There exist $h \in \Gamma_0$ and $x \in \Gamma(x_\star)$ such that h is a contraction to x.*

PROOF. Let w be a loop at some point in $\Gamma(x_\star)$ so that

(1) $\mathrm{SR}(g_w) \neq 1$,

(2) w has a minimal length among such loops.

We denote the base point of w by x.

CLAIM 4.4.1. *w is a simple loop at x.*

PROOF. If w is not a simple loop at x, then w contains a proper sub-loop w'. By the choice of w, we must have $\mathrm{SR}(g_{w'}) = 1$. Then we have $\mathrm{SR}(g_{w \setminus w'}) \neq 1$. But $|w \setminus w'| < |w|$, a contradiction. ■

If $|w| \geq 2$, we have $\mathrm{SR}(g_w) = 1$ by Lemma 3.11. Hence we must have $|w| = 1$, that is, $g_w \in \Gamma_0$, and either g_w or g_w^{-1} is a contraction to x. ■

REMARK. For this reason, $\Gamma(x_2)$ in Example 2.6 (2) and $\Gamma'(x_1)$ in (3) cannot be with bubbles, because Γ_0 in (2) (resp. Γ'_0 in (3)) does not contain a contraction to some point in $\Gamma(x_2)$ (resp. $\Gamma'(x_1)$). Thus the concept of "with bubbles" depends on the choice of the generating set.

PROOF OF THEOREM 2.12. For every nonproper Γ-orbit $\Gamma(x)$ with bubbles, by Proposition 4.4, there exist $c_x \in \Gamma(x)$ and $h_x \in \Gamma_0$ such that h_x is a contraction to c_x, which is a unique fixed point of h_x. We fix such c_x and h_x. Hence we obtain an injective map

$$\Psi: \{\Gamma(x) \mid \Gamma(x) \text{ is a nonproper } \Gamma\text{-orbit with bubbles in } \overline{\Gamma(x_\star)}\} \to \Gamma_0$$

by $\Psi(\Gamma(x)) = h_x$. Since Γ_0 is a finite set, $\overline{\Gamma(x_\star)}$ contains at most finitely many nonproper Γ-orbits with bubbles. ■

5. Two-dimensional case. Throughout this section, we suppose that $q=2$, hence, a finite, symmetric subset $\Gamma_0 \subset \Gamma_{2,+}^{\text{sim},*}$, $\Gamma = \langle \Gamma_0 \rangle$ and $x_\star \in \boldsymbol{R}^2$ satisfy the assumptions (S1) and (S2).

The following two lemmas are elementary.

LEMMA 5.1. *Let $g \in \Gamma$ with $\bar{g} \neq \mathrm{id}_{\boldsymbol{R}^2}$. Then g has at most one fixed point.*

LEMMA 5.2. *Let $g \in \Gamma_{2,+}^{\text{sim}}$ and $x \in D(g)$. If $g(x) = x$ and there exists $z \in D(g) \cap \Gamma(x_\star)$ so that $\bar{g}(B_z) = B_z$, then $\bar{g} = \mathrm{id}_{\boldsymbol{R}^2}$.*

LEMMA 5.3. *Let $x \in \overline{\Gamma(x_\star)}$ and let $w = (h_m, \dots, h_1) \in W(\Gamma_0)$ be a chain at $x \in D(g_w)$. Then there exists $\beta > 0$ such that $U(x; \beta) \subset D(g_w)$ and $\bar{g}_w(B_z) = B_{g_w(z)}$ for every $z \in U(x; \beta) \cap \Gamma(x_\star) \setminus \{x\}$.*

PROOF. By the induction on k, we show the existence of $\beta_k > 0$ such that $U(x; \beta_k) \subset D(g_k)$ and $\bar{g}_k(B_z) = B_{g_k(z)}$ for every $z \in U(x; \beta_k) \cap \Gamma(x_\star) \setminus \{x\}$.

(I) For $k=1$, we have $g_1 = h_1 \in \Gamma_0$.

Case 1. If $g_1(x) = x$, then x is a unique fixed point of g_1. Hence $g_1(z) \neq z$ for every $z \in D(g_1) \setminus \{x\}$. Therefore $\bar{g}_1(B_z) = B_{g_1(z)}$ for every $z \in D(g_1) \cap \Gamma(x_\star) \setminus \{x\}$. Take $\beta_1 > 0$ so that $U(x; \beta_1) \subset D(g_1)$.

Case 2. If $g_1(x) \neq x$, since g_1 has at most one fixed point, there exists $\beta_1 > 0$ such that $U(x; \beta_1) \subset D(g_1)$ and $g_1(z) \neq z$ for every $z \in U(x; \beta_1)$. Then $\bar{g}_1(B_z) = B_{g_1(z)}$ for every $z \in U(x; \beta_1) \cap \Gamma(x_\star)$.

(II) Assume that the assertion holds for k, that is, there exists $\beta_k > 0$ such that $U(x; \beta_k) \subset D(g_k)$ and $\bar{g}_k(B_z) = B_{g_k(z)}$ for every $z \in U(x; \beta_k) \cap \Gamma(x_\star) \setminus \{x\}$. Note that $g_k(x) \in D(h_{k+1})$.

Case 1. If $h_{k+1}(g_k(x)) = g_k(x)$, then $g_k(x)$ is a unique fixed point of h_{k+1}. Hence $h_{k+1}(z) \neq z$ for every $z \in D(h_{k+1}) \setminus \{g_k(x)\}$. Then $h_{k+1}(B_z) = B_{h_{k+1}(z)}$ for every $z \in D(h_{k+1}) \cap \Gamma(x_\star) \setminus \{g_k(x)\}$. Therefore we can take $\bar{\beta}_{k+1} > 0$ so that $U(g_k(x); \bar{\beta}_{k+1}) \subset$

$D(h_{k+1})\cap g_k(U(x;\beta_k))$. Put $\beta_{k+1}=\bar{\beta}_{k+1}\cdot \mathrm{SR}(g_k^{-1})$.

Case 2. If $h_{k+1}(g_k(x))\neq g_k(x)$, since h_{k+1} has at most one fixed point, we can take $\bar{\beta}_{k+1}>0$ so that $U(g_k(x);\bar{\beta}_{k+1})\subset D(h_{k+1})\cap g_k(U(x;\beta_k))$ and $h_{k+1}(z)\neq z$ for every $z\in U(g_k(x);\bar{\beta}_{k+1})$. Then put $\beta_{k+1}=\bar{\beta}_{k+1}\cdot \mathrm{SR}(g_k^{-1})$. ■

REMARK 5.4. In the above lemma, if $x\in\overline{\Gamma(x_\bigstar)}$ and if each $g_i(x)$ is not a fixed point of h_{i+1} $(i=0, 1, \dots, m-1)$, then $\bar{g}_w(B_x)=B_{g_w(x)}$.

PROOF OF THEOREM 2.13. There exists $w\in W(\Gamma_0)$ such that w is a loop at x and $g_w=g$.

Assume on the contrary that $\bar{g}_w^n=\bar{g}^n\neq \mathrm{id}_{\boldsymbol{R}^2}$ for all $n\in\boldsymbol{N}$.

Since $g_w\in\mathrm{Stab}(x)$, by Lemma 5.3, there exists $\beta>0$ such that $U(x;\beta)\subset D(g_w)$ and $\bar{g}_w(B_z)=B_{g_w(z)}$ for all $z\in U(x;\beta)\cap\Gamma(x_\bigstar)\setminus\{x\}$. Since $x\in\overline{\Gamma(x_\bigstar)}$, we have $U(x;\beta)\cap \Gamma(x_\bigstar)\setminus\{x\}\neq\emptyset$. Furthermore, since g is a rotation at x, we have $g_w(U(x;\beta))=U(x;\beta)$ and g_w^n is defined on $U(x;\beta)$ for all $n\in\boldsymbol{N}$.

Take $z\in U(x;\beta)\cap\Gamma(x_\bigstar)\setminus\{x\}$. Note that $g_w^n(z)\neq g_w^m(z)$ for $n>m\in\boldsymbol{N}$. Indeed, if $g_w^n(z)=g_w^m(z)$, then g_w^{n-m} has two fixed points x and z, but by Lemma 5.1, this contradicts the fact that $\bar{g}_w^{n-m}\neq\mathrm{id}_{\boldsymbol{R}^2}$.

Since $g_w^n(z)\in U(x;\beta)\cap\Gamma(x_\bigstar)$ for all $n\in\boldsymbol{N}$, we have, by the choice of $\beta>0$, $\bar{g}_w(B_{g_w^n(z)})=B_{g_w^{n+1}(z)}$ and $\bar{g}_w(B_{g_w^n(z)})\neq\bar{g}_w(B_{g_w^m(z)})$ for $n\neq m$. Since g_w is a rotation at x, bubbles $B_{g_w^n(z)}$ are similar to each other, that is, there exist infinitely many bubbles on a bounded set which are similar to each other, a contradiction.

Hence there exists $n\in\boldsymbol{N}$ such that $\bar{g}^n=\mathrm{id}_{\boldsymbol{R}^2}$. ■

REMARK 5.5. Let $w_i=\xi_i^{-1}\eta_i\xi_i\in W(\Gamma_0)$ be a basic loop at $x\in\Gamma(x_\bigstar)$ $(i=1,\dots,m)$ and $w=w_m\cdots w_1$. If $|\eta_i|\geq 2$ for every $i=1,\dots,m$, then by Lemma 3.17, $\bar{g}_w(B_x)=B_{g_w(x)}$. Thus by Lemma 5.2, $\bar{g}_w=\mathrm{id}_{\boldsymbol{R}^2}$. Hence if $\bar{g}_w\neq\mathrm{id}_{\boldsymbol{R}^2}$, there exists i such that $|\eta_i|=1$ and that g_{η_i} is either a contraction (i.e., $\mathrm{SR}(g_{\eta_i})<1$) or an expansion (i.e., $\mathrm{SR}(g_{\eta_i})>1$) or a nontrivial rotation (i.e., $\mathrm{SR}(g_{\eta_i})=1$ at the unique fixed point x of $g_{\eta_i}\in\Gamma_0$.

PROOF OF THEOREM 2.14. Since Γ_0 is a finite set,

$$N=\sup\{d_{\Gamma_0}(x,y)\,|\,y \text{ is a nontrivial fixed point of some } h\in\Gamma_0\}$$

is finite. Define a subset $K\subset\Gamma(x)$ by

$$K=\{y\in\Gamma(x)\,|\,d_{\Gamma_0}(x,y)\leq N+1\}\,.$$

Then K satisfies the required property. Indeed, since each h_i has no fixed point in $\boldsymbol{R}^2\setminus K$, we have $\bar{g}_w(B_z)=B_{g_w(z)}=B_z$ by Remark 5.4. Then by Lemma 5.2, $\bar{g}_w=\mathrm{id}_{\boldsymbol{R}^2}$. ■

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