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ON THE CLASS NUMBER ONE PROBLEM FOR NON-NORMAL QUARTIC CM-FIELDS

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Abstract. We give explicit upper bounds for the discriminants of the non-normal quartic CM-fields with class number one, and for the discriminants of the dihedral octic CM-fields with class number one. These upper bounds are too large to enable us to achieve the determination of these number fields. Nevertheless, whenever a real quadratic number field $\boldsymbol{k}$ is fixed, we can explain how to determine the non-normal quartic CM-fields or the dihedral octic CM-fields with class number one and with real quadratic subfield $\boldsymbol{k}$.

1. Introduction. Uchida [17, Theorem 2] proved that there exist only finitely many imaginary abelian number fields with class number one. In fact, the class number one problem for imaginary abelian number fields has lately been settled by Yamamura [20], and it is now known that there are exactly fifty-four imaginary abelian quartic number fields with class number one, that forty-seven of them are bicyclic biquadratic (see [1]) and that seven of them are cyclic quartic (see [14]). Hence, it is time to move on to the determination of the non-abelian or even non-normal CM-fields with class number one since Uchida [17, Remark 1] also proved that there exist only finitely many CM-fields of fixed degree with class number one. Here, we are interested in the lowest degree cases, i.e. in the non-normal quartic and normal dihedral octic cases. The aim of this paper is to get reasonable upper bounds for the discriminants of these number fields and to explain why these upper bounds would eventually make it possible to determine these number fields thanks to a reasonable amount of numerical class number computations.

Theorem A shows that a dihedral octic CM-field has relative class number one if and only if it is a normal closure of a non-normal quartic CM-field with relative class number one. Theorem B provides a lower bound for the relative class numbers $h^*(\boldsymbol{K})$ of non-normal quartic CM-fields $\boldsymbol{K}$. From this lower bound, we get an upper bound for the discriminants of the non-normal quartic CM-fields $\boldsymbol{K}$ with relative class number one, together with an upper bound for the discriminants of their real quadratic subfields $\boldsymbol{k}$. Then, Theorem C provides much better upper bounds for the discriminants of the non-normal quartic CM-fields with relative class number one which are quadratic extensions of a fixed real quadratic number field. Theorem D provides us with a strong necessary condition for a CM-field to have class number one, and enables us to sieve

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the non-normal quartic CM-fields which are quadratic extensions of a fixed real quadratic subfield, with only few number fields remaining. At present, the techniques we have developed here enable us to find the non-normal quartic CM-fields or the dihedral octic CM-fields with class number one which are extensions of a fixed real quadratic number field. For example, in Theorem E we show that there are exactly six non-isomorphic non-normal quartic CM-fields with class number one, and exactly five dihedral octic CM-fields with class number one which are extensions of $\boldsymbol{Q}(\sqrt{5})$.

We would like to thank the referees for careful reading of the previous versions of this paper and for valuable suggestions for improvements. Moreover, we point out that thanks to Theorem B the author and R. Okazaki have lately managed to determine all these non-normal quartic CM-fields with class number one: there are thirty-seven non-isomorphic such number fields (see [11]).

2. Lower bounds for relative class numbers of dihedral octic CM-fields and of non-normal quartic CM-fields. The normal closure of a non-normal quartic CM-field is a dihedral octic number field. To clarify the relationship between these number fields, we first consider the relationship among the subfields of a general dihedral octic number field. Now, let N be a dihedral octic number field, so that its Galois group is the dihedral group $\boldsymbol{D}_4$ of order 8 with the generator-relation presentation $\boldsymbol{D}_4 = \langle r, s ; r^4 = s^2 = \mathrm{Id}, srs = r^3 \rangle$. Then, N has five quartic subfields and three quadratic subfields. Let N_+ be the unique normal quartic subfield of N and let $\boldsymbol{k}_+$ be the quadratic subfield of N_+ such that $N/\boldsymbol{k}_+$ is cyclic. The lattice of subfields is as in the Figure with

$$\mathrm{Gal}(N/N_+) = \{\mathrm{Id}, r^2\}, \qquad \mathrm{Gal}(N/\boldsymbol{k}_+) = \{\mathrm{Id}, r, r^2, r^3\},$$
$$\mathrm{Gal}(N/\boldsymbol{K}_1) = \{\mathrm{Id}, s\}, \qquad \mathrm{Gal}(N/\boldsymbol{k}_1) = \{\mathrm{Id}, s, r^2, sr^2\},$$
$$\mathrm{Gal}(N/\boldsymbol{K}_2) = \{\mathrm{Id}, sr\}, \qquad \mathrm{Gal}(N/\boldsymbol{k}_2) = \{\mathrm{Id}, sr, r^2, sr^3\},$$
$$\mathrm{Gal}(N/\boldsymbol{K}_1') = \{\mathrm{Id}, sr^2\}, \qquad \mathrm{Gal}(N/\boldsymbol{K}_2') = \{\mathrm{Id}, sr^3\}.$$

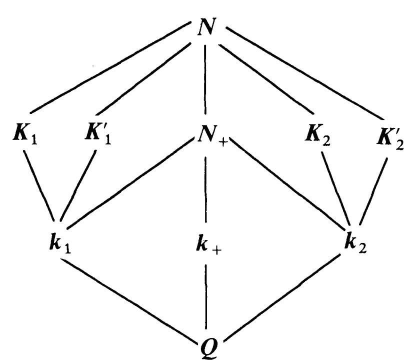

FIGURE.

$\boldsymbol{K}_1$, $\boldsymbol{K}_1'$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_2'$ are the four non-normal quartic subfields of N, and $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_+$ are the three quadratic subfields of N. Moreover, $\boldsymbol{K}_1' = r^3(\boldsymbol{K}_1)$ is isomorphic to $\boldsymbol{K}_1$ and $\boldsymbol{K}_2' = r^3(\boldsymbol{K}_2)$ is isomorphic to $\boldsymbol{K}_2$.

THEOREM A (see [19, Corollary]). *Let the notation be as above. We denote by* $\boldsymbol{K}$ *any one of the four non-normal quartic subfields of* N *and by* $\boldsymbol{k}$ *its quadratic subfield. Then we have the following relations for Dedekind zeta functions, discriminants and relative class numbers*:

(a) $\zeta_N(s) = \zeta_{\boldsymbol{K}_1}(s)\zeta_{\boldsymbol{K}_2}(s)\zeta_{\boldsymbol{k}_+}/\zeta_{\boldsymbol{Q}}(s)^2$.

(b) $\zeta_N(s)/\zeta_{N_+}(s) = (\zeta_{\boldsymbol{K}}(s)/\zeta_{\boldsymbol{k}}(s))^2$.

(c) $d(N)/d(N_+) = (d(\boldsymbol{K})/d(\boldsymbol{k}))^2$. *Hence*, $d(\boldsymbol{K}_1)/d(\boldsymbol{k}_1) = d(\boldsymbol{K}_2)/d(\boldsymbol{k}_2)$.

(d) N *is a* CM-*field if and only if* $\boldsymbol{K}$ *is a* CM-*field. Moreover, if* N *is a* CM-*field then* N_+ *is the maximal totally real subfield of* N *and*

$$h^*(N) = Q_N(h^*(\boldsymbol{K}))^2/2 .$$

Here $Q_N \in \{1, 2\}$ *is Hasse's unit index of* N. *Hence*, $h^*(\boldsymbol{K}_1) = h^*(\boldsymbol{K}_2)$. *In particular*, $h^*(N) = 1$ *if and only if* $h^*(\boldsymbol{K}_1) = h^*(\boldsymbol{K}_2) = 1$.

PROOF. The eight elements of $\boldsymbol{D}_4$ fall into five conjugacy classes:

$$C_1 = \{\mathrm{Id}\}, \quad C_2 = \{r^2\}, \quad C_3 = \{r, r^3\}, \quad C_4 = \{s, sr^2\} \quad \text{and} \quad C_5 = \{sr, sr^3\} .$$

Hence, there are five irreducible characters. Let ψ_i, $1 \le i \le 4$ be the four characters of degree one and let ψ_0 be the character of degree 2. As in [13, pp. 52–53], the irreducible characters of $\boldsymbol{D}_4$ are as in Table 1.

Let $\boldsymbol{M}$ be an intermediate number field between N and $\boldsymbol{Q}$. Let $\boldsymbol{H} = \mathrm{Gal}(N/\boldsymbol{M})$. Let χ_0 be the principal character of $\boldsymbol{H}$ and let $\chi_{\boldsymbol{M}} = \chi_0^*$ be its induced character of $\boldsymbol{D}_4$. Hence,

$$(*) \qquad \chi_{\boldsymbol{M}}(g) = \frac{1}{\#\boldsymbol{H}} \#\{x \in \boldsymbol{D}_4 ;\, g \in x^{-1}\boldsymbol{H}x\} , \quad g \in \boldsymbol{D}_4 .$$

TABLE 1.

	C_1	C_2	C_3	C_4	C_5
ψ_0	2	-2	0	0	0
ψ_1	1	1	1	1	1
ψ_2	1	1	1	-1	-1
ψ_3	1	1	-1	1	-1
ψ_4	1	1	-1	-1	1

TABLE 2.

	C_1	C_2	C_3	C_4	C_5
χ_N	8	0	0	0	0
χ_{N_+}	4	4	0	0	0
χ_{k_+}	2	2	2	0	0
χ_{K_1}	4	0	0	2	0
χ_{k_1}	2	2	0	2	0
χ_{K_2}	4	0	0	0	2
χ_{k_2}	2	2	0	0	2
χ_Q	1	1	1	1	1

Then, $\zeta_M(s)=L(s, \chi_0, N/M)=L(s, \chi_M, N/Q)$. Now, from $(*)$ we get Table 2. Hence, we get (a) and (b) from the following factorizations:

$$\chi_N=2\psi_0+\psi_1+\psi_2+\psi_3+\psi_4 \quad \text{and} \quad \zeta_N=L_{\psi_0}^2L_{\psi_1}L_{\psi_2}L_{\psi_3}L_{\psi_4}\,,$$

$$\chi_{N_+}=\psi_1+\psi_2+\psi_3+\psi_4 \quad \text{and} \quad \zeta_{N_+}=L_{\psi_1}L_{\psi_2}L_{\psi_3}L_{\psi_4}\,,$$

$$\chi_{k_+}=\psi_1+\psi_2 \quad \text{and} \quad \zeta_{k_+}=L_{\psi_1}L_{\psi_2}\,,$$

$$\chi_{K_1}=\psi_0+\psi_1+\psi_3 \quad \text{and} \quad \zeta_{K_1}=L_{\psi_0}L_{\psi_1}L_{\psi_3}\,,$$

$$\chi_{k_1}=\psi_1+\psi_3 \quad \text{and} \quad \zeta_{k_1}=L_{\psi_1}L_{\psi_3}\,,$$

$$\chi_{K_2}=\psi_0+\psi_1+\psi_4 \quad \text{and} \quad \zeta_{K_2}=L_{\psi_0}L_{\psi_1}L_{\psi_4}\,,$$

$$\chi_{k_2}=\psi_1+\psi_4 \quad \text{and} \quad \zeta_{k_2}=L_{\psi_1}L_{\psi_4}\,,$$

$$\chi_Q=\psi_1 \quad \text{and} \quad \zeta_Q=L_{\psi_1}\,.$$

(c) follows from (b) and the functional equations satisfied by the four Dedekind zeta functions ζ_N, ζ_{N_+}, ζ_K and ζ_k. Indeed, set

$$A=\frac{d(N)d(k)^2}{d(N_+)d(K)^2}\,.$$

Then, there exist integers m and n (such that $m+2n=0$) which depend on the numbers of real and complex embeddings of k, K, N_+ and N such that

$$f(s):=A^s\Gamma(s/2)^m(\Gamma(s)/2)^n\frac{\zeta_N\zeta_k(s)^2}{\zeta_{N_+}(s)\zeta_K(s)^2}$$

$$=A^2\Gamma(s/2)^m(\Gamma(s)/2)^n \quad \text{(thanks to Theorem A(b))}$$

satisfies $f(s)=f(1-s)$, hence is holomorphic in the whole plane. Looking at the value at $s=-1$ we get $n=0$. Looking at the value at $s=0$ we then get $m=0$. Hence, $f(s)=A^s=A^{1-s}=f(1-s)$ implies $A=1$. Let $r_1(\boldsymbol{M})$ denote the number of real embeddings of any number field $\boldsymbol{M}$. We note that we have

$$(*) \qquad 0=m=r_1(\boldsymbol{N})-r_1(\boldsymbol{N}_+)+2(r_1(\boldsymbol{k})-r_1(\boldsymbol{K}))\,.$$

Now we prove the first part of (d). Assume that $\boldsymbol{N}$ is a CM-field. Then, the complex conjugation c commutes with any element of the Galois group $\boldsymbol{D}_4$ of $\boldsymbol{N}$. Hence, $c=r^2$ and $\boldsymbol{N}_+$ is totally real. Thus, $\boldsymbol{k}$ is a real quadratic number field and $(*)$ yields $0=-2r_1(\boldsymbol{K})$. Hence, $\boldsymbol{K}$ is a totally imaginary quartic number field that is a quadratic extension of the real quadratic number field $\boldsymbol{k}$. Thus, $\boldsymbol{K}$ is a CM-field. Conversely, assume that $\boldsymbol{K}$ is a CM-field. Then $\boldsymbol{K}$ is totally imaginary and $\boldsymbol{k}$ is a real quadratic number field. Hence, $\boldsymbol{N}$ also is totally imaginary. Thus, $r_1(\boldsymbol{N})=0$, $r_1(\boldsymbol{k})=2$ and $r_1(\boldsymbol{K})=0$. Then $(*)$ yields $r_1(\boldsymbol{N}_+)=4$. Thus, $\boldsymbol{N}$ is a CM-field.

The second part of (d) follows from (b), (c) and the analytic class number formulas, since ± 1 are the only roots of unity in $\boldsymbol{N}$ and since Lemma 1 below provides $Q_{\boldsymbol{K}}=1$.

LEMMA 1. *Let $\boldsymbol{K}$ be a non-normal quartic* CM-*field which is a quadratic extension of a real quadratic number field $\boldsymbol{k}$. Then, the roots of unity in $\boldsymbol{K}$ are -1 and $+1$, and the fundamental unit $\varepsilon_0(\boldsymbol{K})$ of $\boldsymbol{K}$ may be taken to be to the one $\varepsilon_0(\boldsymbol{k})>1$ of $\boldsymbol{k}$, i.e. $Q_{\boldsymbol{K}}=1$.*

PROOF. There exist non-zero integers n and n' such that $\varepsilon_0(\boldsymbol{k})=\pm(\varepsilon_0(\boldsymbol{K}))^n$ and $N_{\boldsymbol{N}/\boldsymbol{k}}(\varepsilon_0(\boldsymbol{K}))=(\varepsilon_0(\boldsymbol{k}))^{n'}$, and we may choose $\varepsilon_0(\boldsymbol{K})$ so that n is positive. Hence, $(\varepsilon_0(\boldsymbol{k}))^2=N_{\boldsymbol{K}/\boldsymbol{k}}(\varepsilon_0(\boldsymbol{k}))=(N_{\boldsymbol{K}/\boldsymbol{k}}(\varepsilon_0(\boldsymbol{K})))^n=(\varepsilon_0(\boldsymbol{k}))^{nn'}$. Hence, $n=1$ or $n=2$. If we had $n=2$ then, as $\sqrt{\pm\varepsilon_0(\boldsymbol{k})}\in\boldsymbol{k}$, we would have $\boldsymbol{K}=\boldsymbol{k}(\sqrt{\pm\varepsilon_0(\boldsymbol{k})})$ and $\boldsymbol{K}$ would not be totally imaginary if we had $\pm=+$, or if we had $\pm=-$ and $N_{\boldsymbol{k}/\boldsymbol{Q}}(\varepsilon_0(\boldsymbol{k}))=-1$, while it would be normal if we had $\pm=-$ and $N_{\boldsymbol{k}/\boldsymbol{Q}}(\varepsilon_0(\boldsymbol{k}))=+1$.

LEMMA 2 (see [5, Lemma 2] and [15, Lemma 3]). *Set $c=(3+2\sqrt{2})/2$. Let $\boldsymbol{M}\neq\boldsymbol{Q}$ be an algebraic number field. Then, $\zeta_{\boldsymbol{M}}$ has at most one real zero in the interval $[1-(1/c\log(|d(\boldsymbol{M})|)), 1[$; if such a zero exists, it is simple.*

COROLLARY 3. *Let the notation be as in Theorem* A. *Set $A_{\boldsymbol{K}/\boldsymbol{k}}=d(\boldsymbol{K})/d(\boldsymbol{k})$, which does not depend on the choice of $\boldsymbol{K}$ by Theorem* A(c). *Set $\boldsymbol{I}_c=[1-(1/4c\log(A_{\boldsymbol{K}/\boldsymbol{k}})), 1[$. Note that $\zeta_{\boldsymbol{K}'_1}=\zeta_{\boldsymbol{K}_1}$ and $\zeta_{\boldsymbol{K}'_2}=\zeta_{\boldsymbol{K}_2}$.*

(a) *At least one of $\zeta_{\boldsymbol{K}_1}(s)$ and $\zeta_{\boldsymbol{K}_2}(s)$ has no real zero in the interval $\boldsymbol{I}_c$.*

(b) *If $\zeta_{\boldsymbol{k}}(s)$ has no real zero in the interval $\boldsymbol{I}_c$, then $\zeta_{\boldsymbol{K}}(s)$ also has no real zero in this interval.*

PROOF. The first assertion follows from Lemma 2 (with $M=N$) and Theorem A(a), and the second assertion follows from Lemma 2 (with $M=N$) and Theorem A(b), thanks to Theorem A(c) which provides

$$d(N)\leq\left(\frac{d(N)}{d(N_+)}\right)^2=(A_{K/k})^4\,.$$

LEMMA 4 (see [8, Proposition A(a)]). *Let K be a totally imaginary number field of degree $2N$ such that $\zeta_K(s_0)\leq 0$ for some $s_0\in[(1/2),1[$. Then,*

$$\mathrm{Res}_{s=1}(\zeta_K)\geq(1-s_0)d(K)^{(s_0-1)/2}\left(1-\frac{2\pi N}{d(K)^{s_0/2N}}\right).$$

THEOREM B. *Let K be a non-normal quartic CM-field with real quadratic subfield k. Set $A_{K/k}=d(K)/d(k)$. Then, we have:*

$$h^*(K)\geq\frac{1}{200}\frac{\sqrt{A_{K/k}}}{\log^2(A_{K/k})}\qquad \textit{if}\quad A_{K/k}\geq 3\cdot 10^5$$

and $h^*(K)>1$ if $A_{K/k}>5\cdot 10^9$. Hence, $h^*(K)>1$ if $d(k)\geq 25\cdot 10^8$ or if $d(K)\geq 13\cdot 10^{18}$.

PROOF. The analytic class number formula and Lemma 1 show

$$h^*(K)=\frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_k)}\frac{Q_K w(K)}{(2\pi)^2}\sqrt{A_{K/k}}=\frac{\mathrm{Res}_{s=1}(\zeta_K)}{\mathrm{Res}_{s=1}(\zeta_k)}\frac{\sqrt{A_{K/k}}}{2\pi^2}\,.$$

We now estimate $\mathrm{Res}_{s=1}(\zeta_K)$ and $\mathrm{Res}_{s=1}(\zeta_k)$ from below and above, respectively.

Let N be the normal closure of K. Then N is a dihedral octic CM-field and therefore by Theorem A(c) and Corollary 3(a) we may assume that $\zeta_K(s)$ has no real zero in the interval $[1-(1/4c\log(A_{K/k})),1[$. Hence we can apply Lemma 4 to K with $s_0=1-(1/4c\log(A_{K/k}))$. Thus, thanks to $5A_{K/k}\leq d(K)\leq(A_{K/k})^2$, we get

$$\begin{aligned}\mathrm{Res}_{s=1}(\zeta_K)&\geq(1-s_0)(A_{K/k})^{s_0-1}\left(1-\frac{4\pi}{(5A_{K/k})^{s_0/4}}\right)\\&=\frac{1}{4ce^{1/4c}\log(A_{K/k})}\left(1-\frac{4\pi}{(5A_{K/k})^{s_0/4}}\right).\end{aligned}$$

On the other hand, if we denote by χ the primitive character associated to k, then we have (see [10])

$$\mathrm{Res}_{s=1}(\zeta_k)=L(1,\chi)\leq\frac{1}{2}\log(d(k))+\frac{2+\gamma-\log(4\pi)}{2}\leq\frac{1}{2}\log(A_{K/k})\,.$$

Here, γ is the Euler constant, $(2+\gamma-\log(4\pi))/2=0.023\cdots\leq\log(2)/2$ and $d(k)\leq A_{K/k}/2$ for K/k is ramified at least one finite place (see [11]). Hence, we get the desired lower bound from the following lower bound:

$$h^*(\boldsymbol{K}) \geq \frac{1}{4\pi^2 c e^{1/4c}}\left(1-\frac{4\pi}{(5A_{\boldsymbol{K}/\boldsymbol{k}})^{s_0/4}}\right)\frac{\sqrt{A_{\boldsymbol{K}/\boldsymbol{k}}}}{\log^2(A_{\boldsymbol{K}/\boldsymbol{k}})}.$$

Finally, thanks to $A_{\boldsymbol{K}/\boldsymbol{k}} \geq 2d(\boldsymbol{k})$ and $A_{\boldsymbol{K}/\boldsymbol{k}} \geq \sqrt{2d(\boldsymbol{K})}$, we get the last assertion.

Even though this upper bound for the discriminants of the non-normal quartic CM-fields with class number one seems too large to enable us to find all such number fields, it is reasonable to expect to achieve the determination. Indeed, thanks to Theorems C and D below, we will show that whenever $\boldsymbol{k}$ with class number one is fixed, only very few number fields $\boldsymbol{K}$ can have class number one, and the determination of these number fields does not require a great amount of computation. Let us note that this upper bound for $d(\boldsymbol{K})$ is more satisfactory than the ones given in [16].

THEOREM C. *Let $\boldsymbol{K}$ be a non-normal quartic* CM*-field with real quadratic subfield $\boldsymbol{k}$. Whenever $h^*(\boldsymbol{K})=1$ and $d(\boldsymbol{K}) \geq 16 d(\boldsymbol{k})^2$, we have*

$$h(\boldsymbol{k}) \log(\varepsilon_0(\boldsymbol{k})) \geq \frac{1}{481}\left(1-\frac{13}{d(\boldsymbol{K})^{1/4}}\right)\frac{\sqrt{d(\boldsymbol{K})}}{\log(d(\boldsymbol{K}))},$$

where $\varepsilon_0(\boldsymbol{k})>1$ and $h(\boldsymbol{k})$ are the fundamental unit and class number of $\boldsymbol{k}$, respectively.

In particular, the non-normal quartic CM*-fields $\boldsymbol{K}$ which are quadratic extensions of $\boldsymbol{Q}(\sqrt{5})$ and which have class number one satisfy $d(\boldsymbol{K}) \leq 3 \cdot 10^7$.*

PROOF. From [8, Theorem 1] we get that $\zeta_{\boldsymbol{k}}$ has no real zero on $]0,1[$. By Lemma 1, the regulator $\mathrm{Reg}(\boldsymbol{K})$ of $\boldsymbol{K}$ equals $2\log(\varepsilon_0(\boldsymbol{k}))$. By Corollary 3(b), we can apply Lemma 4 with $s_0 = 1-(1/4c\log(d(\boldsymbol{K})))$, so that we get the desired result from

$$\mathrm{Res}_{s=1}(\zeta_{\boldsymbol{K}}) = \frac{(2\pi)^2 h(\boldsymbol{K})\,\mathrm{Reg}(\boldsymbol{K})}{w(\boldsymbol{K})\sqrt{d(\boldsymbol{K})}} = \frac{4\pi^2 h(\boldsymbol{k})\log(\varepsilon_0(\boldsymbol{k}))}{\sqrt{d(\boldsymbol{K})}}$$
$$\geq \frac{1}{4ce^{1/8c}\log(d(\boldsymbol{K}))}\left(1-\frac{4\pi e^{1/16c}}{d(\boldsymbol{K})^{1/4}}\right).$$

3. The non-normal quartic CM-fields with class number one which are quadratic extensions of a given real quadratic number field.

THEOREM D. *Let $\boldsymbol{K}$ be a* CM*-field of degree $2N$ with maximal totally real subfield $\boldsymbol{k}$. Let $d(\boldsymbol{K})$ and $d(\boldsymbol{k})$ be the absolute values of the discriminants of $\boldsymbol{K}$ and $\boldsymbol{k}$. If $\boldsymbol{K}$ has class number one, then for any prime ideal $\boldsymbol{P}$ in $\boldsymbol{K}$ which is not inert in $\boldsymbol{K}/\boldsymbol{k}$ we have*

$$N_{\boldsymbol{K}/\boldsymbol{Q}}(\boldsymbol{P}) \geq \frac{d(\boldsymbol{K})}{4^N d(\boldsymbol{k})^2}.$$

PROOF. Let $\boldsymbol{R}_{\boldsymbol{k}}$ and $\boldsymbol{R}_{\boldsymbol{K}}$ be the rings of algebraic integers of the number fields $\boldsymbol{k}$ and $\boldsymbol{K}$. Since $\boldsymbol{K}$ has class number one, $\boldsymbol{k}$ also has class number one, so that there exists

α in $\boldsymbol{R_k}$ which is square-free in $\boldsymbol{R_k}$ such that $\boldsymbol{K}=\boldsymbol{k}(\sqrt{-\alpha})$. Here, α is totally, positive. Moreover, there exist a and b in $\boldsymbol{R_k}$ such that $\boldsymbol{R_K}=\boldsymbol{R_k}[\omega]$ with $\omega=(a+b\sqrt{-\alpha})/2$.

$\boldsymbol{P}$ being principal, we have $\boldsymbol{P}=(z)$ for $z=x+y\omega\in\boldsymbol{R_K}$ with x and y in $\boldsymbol{R_k}$. Moreover, $y\neq 0$, for otherwise $\boldsymbol{P}$ would be inert in $\boldsymbol{K}/\boldsymbol{k}$. Now, we have

$$N_{\boldsymbol{K}/\boldsymbol{Q}}(\boldsymbol{P})=N_{\boldsymbol{k}/\boldsymbol{Q}}(N_{\boldsymbol{K}/\boldsymbol{k}}(z))=\frac{1}{4^N}N_{\boldsymbol{k}/\boldsymbol{Q}}(X^2+b^2Y^2\alpha)$$

with $X=2x+ay$ and $Y=y$. Hence, $\boldsymbol{k}$ being totally real and α being totally positive, we get

$$N_{\boldsymbol{K}/\boldsymbol{Q}}(\boldsymbol{P})\geq\frac{1}{4^N}(N_{\boldsymbol{k}/\boldsymbol{Q}}(X)^2+N_{\boldsymbol{k}/\boldsymbol{Q}}(Y)^2N_{\boldsymbol{k}/\boldsymbol{Q}}(b^2\alpha))\geq\frac{1}{4^N}N_{\boldsymbol{k}/\boldsymbol{Q}}(b^2\alpha)\,.$$

On the other hand,

$$d(\boldsymbol{K})=d(\boldsymbol{k})^2N_{\boldsymbol{k}/\boldsymbol{Q}}((\omega-\omega')^2)=d(\boldsymbol{k})^2N_{\boldsymbol{k}/\boldsymbol{Q}}(b^2\alpha)\,,$$

where ω' is the complex conjugate of ω, so that we get the desired result.

REMARK. The inequality in Theorem D can be written as

$$N_{\boldsymbol{K}/\boldsymbol{Q}}(2\boldsymbol{P})\geq N_{\boldsymbol{k}/\boldsymbol{Q}}(d(\boldsymbol{K}/\boldsymbol{k}))\,,$$

where $d(\boldsymbol{K}/\boldsymbol{k})$ denotes the relative discriminant of $\boldsymbol{K}/\boldsymbol{k}$. Thus, this inequality probably does not hold for $d(\boldsymbol{K}/\boldsymbol{k})$ large, hence might be useful for the class number one problem.

For example, let $\boldsymbol{K}$ be an imaginary cyclic quartic number field with class number one associated to a quartic odd Dirichlet character χ_f. In 1972 Uchida proved in [18, Proposition 6] that their conductors are less than 50000. At that time, he did not wish to carry out the required computations of the possible number fields up to such a large upper bound for these conductors, until he had a much better upper bound. It was Setzer who did these computations eight years later and showed that there are seven such number fields. We would like to show that if Uchida had come up with this Theorem D, then he could have obtained the seven such number fields with only little numerical computation, computations which could have easily been done on the computers at this time.

Indeed, we may suppose that the conductor f of $\boldsymbol{K}$ is a prime p such that $p\equiv 5 \pmod 8$, so that $d(\boldsymbol{K})=p^3$ and $d(\boldsymbol{k})=p$. If q is a prime such that $\chi_p(q)=+1$, then q splits in $\boldsymbol{K}/\boldsymbol{Q}$. Hence, Theorem C provides us with the lower bound $q\geq p/16$. Now, numerical computations show that there are only 10 primes $p\equiv 5 \pmod 8$ less than 50000 such that $\chi_p(q)\neq +1$ for any prime q with $2\leq q<p/16$, namely

$$p\in\{5, 13, 29, 37, 53, 61, 157, 173, 197, 373\}\,.$$

Let us note that $\chi_p(q)=+1$ if and only if $q^{(p-1)/4}\equiv 1 \pmod p$. Hence, the computation of the relative class numbers of these ten number fields would have provided him with the desired result.

Now, we do not want to dwell at length on the characterization of the non-normal quartic CM-fields $\boldsymbol{K}$ with odd class numbers which contain $\boldsymbol{Q}(\sqrt{5})$ (see [3] and [11]). One could show that any such number field $\boldsymbol{K}$ is isomorphic to some

$$\boldsymbol{K}_{5,q} := \boldsymbol{Q}(\sqrt{-\alpha_q}) \qquad \text{with} \qquad \alpha_q = \frac{x_q + y_q\sqrt{5}}{2}$$

where $q \equiv 1 \pmod 4$ is a prime which splits in $\boldsymbol{Q}(\sqrt{5})$ such that the class number of the real quadratic number field $\boldsymbol{Q}(\sqrt{5q})$ is 2 modulo 4, and where x_q and y_q are any positive integers such that $4q = x_q^2 - 5y_q^2$. Moreover, $d(\boldsymbol{K}_{5,q}) = 5^2 q$, and $\boldsymbol{K}_{5,q}$ and $\boldsymbol{K}_{5,q'}$ are isomorphic if and only if they are equal, and they are equal if and only if $q = q'$. Finally, using a straightforward generalization of [9, Theorem 5] to compute the quadratic symbols $\left[\frac{-\alpha_q}{\boldsymbol{P}}\right]$ which govern the decomposition in the quadratic extension $\boldsymbol{K}/\boldsymbol{k} = \boldsymbol{K}/\boldsymbol{Q}(\sqrt{5})$ of the prime ideal $\boldsymbol{P}$ of $\boldsymbol{Q}(\sqrt{5})$ lying above an odd prime l, we get that Theorem D implies:

COROLLARY 5. *Let $q \equiv 1 \pmod 4$ be a prime which splits in $\boldsymbol{Q}(\sqrt{5})$. If $\boldsymbol{K}_{5,q}$ has class number one, then the following two conditions are satisfied*:

(a) *For all odd primes with* $\left(\frac{5}{l}\right) = -1$ *and* $l^2 < q/16$, *we have* $\left(\frac{l}{q}\right) = -1$.

(b) *For all odd primes with* $\left(\frac{5}{l}\right) = 1$ *and* $l < q/16$, *we have* $\left(\frac{l}{q}\right) = 1$.

THEOREM E. *There exist exactly six non-isomorphic non-normal quartic* CM-*fields with class number one which contain* $\boldsymbol{Q}(\sqrt{5})$, *namely*:

$$\boldsymbol{K}_{5,41} = \boldsymbol{Q}(\sqrt{-(13+\sqrt{5})/2}), \qquad \boldsymbol{K}_{5,61} = \boldsymbol{Q}(\sqrt{-(17+3\sqrt{5})/2}),$$
$$\boldsymbol{K}_{5,109} = \boldsymbol{Q}(\sqrt{-(21+\sqrt{5})/2}), \qquad \boldsymbol{K}_{5,149} = \boldsymbol{Q}(\sqrt{-(13+2\sqrt{5})}),$$
$$\boldsymbol{K}_{5,269} = \boldsymbol{Q}(\sqrt{-(17+2\sqrt{5})}), \qquad \boldsymbol{K}_{5,389} = \boldsymbol{Q}(\sqrt{-(41+5\sqrt{5})/2}).$$

There exist exactly five dihedral octic CM-*fields with class number one which contain* $\boldsymbol{Q}(\sqrt{5})$ *and have class number one*; *namely the normal closures of* $\boldsymbol{K}_{5,41}$; $\boldsymbol{K}_{5,61}$; $\boldsymbol{K}_{5,109}$; $\boldsymbol{K}_{5,149}$; *and* $\boldsymbol{K}_{5,389}$. *They are also the narrow Hilbert class fields for the real quadratic number fields* $\boldsymbol{Q}(\sqrt{5\cdot 41})$, $\boldsymbol{Q}(\sqrt{5\cdot 61})$, $\boldsymbol{Q}(\sqrt{5\cdot 109})$, $\boldsymbol{Q}(\sqrt{5\cdot 149})$ *and* $\boldsymbol{Q}(\sqrt{5\cdot 389})$.

REMARK. The normal closure of $\boldsymbol{K}_{5,269}$, which is also the narrow Hilbert 2-class field for the real quadratic number field $\boldsymbol{Q}(\sqrt{5\cdot 269})$, has class number three. This field has relative class number one, while its maximal real subfield $\boldsymbol{Q}(\sqrt{5}, \sqrt{269})$ has class number three.

PROOF. We wrote a program to sieve the primes $q\equiv 1 \pmod 4$ which split in $Q(\sqrt{5})$ such that $d(K_{5,q})=5^2q\leq 3\cdot 10^7$, i.e. such that $q\leq 12\cdot 10^5$ and that (a) and (b) of Corollary 5 are satisfied. We found that there are nine such values of q, namely: $q\in\{29, 41, 61, 89, 101, 109, 149, 269, 389\}$. Now, only six of these values satisfy $h(Q(\sqrt{5q}))\equiv 2 \pmod 4$, namely:

$$q\in\{41, 61, 109, 149, 269, 389\}.$$

It remains to prove that the six corresponding quartic number fields have class number one. We first note that thanks to Minkowski's theorem, it suffices to prove that the prime ideals L of K such that $N_{K/Q}(L)\leq(3/2\pi^2)\sqrt{d(K)}$ are principal, i.e. it suffices to prove that the prime ideals P of $k=Q(\sqrt{5})$ such that $N_{k/Q}(P)\leq(15/2\pi^2)\sqrt{q}$ are inert in K/k, i.e. satisfy $\left[\frac{-\alpha_q}{P}\right]=-1$. Let us note that 2 is inert in k/Q and remains inert in K/k provided that $q\equiv 5 \pmod 8$. Indeed, thanks to [4, Theorem 119], the prime ideal (2) of k splits or remains inert in K/k according as $x^2\equiv-\alpha \pmod{2^3}$ has a solution in R_k or not. Taking norms, we get that if this equation has a solution, then $x^2\equiv q \pmod 8$ also has a solution in Z. Now, the reader would easily verify that this holds whenever $q\in\{109, 149, 269, 389\}$, but does not hold for $q\in\{41, 61\}$ since the prime ideal (2) of $k=Q(\sqrt{5})$ splits in K/k for $q=41$, while the ramified prime ideal $(\sqrt{5})$ of k lying above 5 splits in K/k for $q=61$. Nevertheless, in both cases we get that K has class number one, for if $q=41$, then

$$\alpha_q=\frac{13+\sqrt{5}}{2} \quad\text{and}\quad \beta=\frac{(1-\sqrt{5})/2+\sqrt{-\alpha_q}}{2}\in R_K \quad\text{is such that } N_{K/k}(\beta)=2$$

(so that the prime ideals of K lying above the prime ideal (2) of k are principal); and if $q=61$, then

$$\alpha_q=\frac{17+3\sqrt{5}}{2} \quad\text{and}\quad \beta=\frac{(1+\sqrt{5})/2+\sqrt{-\alpha_q}}{2}\in R_K \quad\text{is such that}$$

$$N_{K/k}(\beta)=\frac{1+\sqrt{5}}{2}\sqrt{5}$$

(so that the prime ideals of K lying above the prime ramified ideal $(\sqrt{5})$ of k are principal). Let us note that from [12] we also check that $K_{5,41}$ and $K_{5,61}$ have class number one. Hence, we get the desired first result. The referee pointed out to us that one can find the values of the class numbers of the previous nine $K_{5,q}$'s in [2].

Finally, let N be the normal closure of $K_{5,q}$. Then in the notation of Theorem A we have $N_+=Q(\sqrt{5},\sqrt{q})$. Since

$$h(\boldsymbol{Q}(\sqrt{p}, \sqrt{q})) = \frac{1}{2} h(\boldsymbol{Q}(\sqrt{p}))h(\boldsymbol{Q}(\sqrt{q}))h(\boldsymbol{Q}(\sqrt{pq}))$$

provided that $p \not\equiv 3 \pmod 4$ and $q \not\equiv 3 \pmod 4$ are distinct primes such that the fundamental unit ε_{pq} of the real quadratic number field $\boldsymbol{Q}(\sqrt{pq})$ satisfies $N_{\boldsymbol{Q}(\sqrt{pq})/\boldsymbol{Q}}(\varepsilon_{pq}) = +1$ (see [6]), we get the desired last result.

4. Conclusion. In order to determine all the non-normal quartic CM-fields and the non-abelian normal octic CM-fields with class number one it would be worth developing an efficient analytic method for computing relative class numbers of non-normal quartic CM-fields (see [9] for such an efficient method in the case of non-normal quartic number fields which are quadratic extensions of imaginary quadratic number fields with class number one). We have lately settled this problem (see [7]).

References

[1] E. Brown and J. Parry, The imaginary bicyclic biquadratic fields with class-number 1, J. Reine Angew. Math. 266 (1974), 118–120.

[2] H. Cohn, A computation of some bi-quadratic class numbers, Math. Aids Compt. 12 (1958), 213–217.

[3] P. E. Conner and J. Hurrelbrink, Class number parity, Series in Pure Mathematics 8, World Scientific Publishing Co., Singapore, 1988.

[4] E. Hecke, Lectures on the theory of algebraic numbers, Graduate Texts in Math. 77, Springer-Verlag (1981).

[5] J. Hoffstein, On the Siegel-Tatuzawa theorem, Acta Arith. 38 (1980), 167–174.

[6] T. Kubota, Über die Beziehung der Klassenzahlen der Unterkörper des bizyklischen biquadratischen Zahlkörper, Nagoya Math. J. 6 (1953), 119–127.

[7] S. Louboutin, Calcul des nombres de classes relatifs: application aux corps octiques quaternioniques à multiplication complexe, C. R. Acad. Sci. Paris 317 (1993), 643–646.

[8] S. Louboutin, Lower bounds for relative class numbers of CM-fields, to appear in Proc. Amer. Math. Soc.

[9] S. Louboutin, L-functions and class numbers of imaginary quadratic fields and of quadratic extensions of an imaginary quadratic field, Math. Comp. 59 (1992), 213–230.

[10] S. Louboutin, Majorations explicites de $|L(1, \chi)|$, C. R. Acad. Sci. Paris 316 (1993), 11–14.

[11] S. Louboutin and R. Okazaki, Determination of all non-normal quartic CM-fields and of all non-abelian normal octic CM-fields with class number one, to appear in Acta Arith.

[12] R. Okazaki, On evaluation of L-functions over real quadratic fields, J. Math. Kyoto Univ. 31 (1991), 1125–1153.

[13] J. P. Serre, Représentation linéaire des groupes finis, Hermann, Paris, (1971).

[14] B. Setzer, The determination of all imaginary, quartic abelian number fields with class-number 1, Math. Comp. 35 (1980), 1383–1386.

[15] H. M. Stark, Some effective cases of the Brauer-Siegel theorem, Invent. Math. 23 (1974), 135–152.

[16] J. S. Sunley, Class numbers of totally imaginary quadratic extensions of totally real fields, Trans. Amer. Math. Soc. 175 (1973), 209–232.

[17] K. Uchida, Class numbers of imaginary abelian number fields I, Tôhoku Math. J. 23 (1971), 97–104.

[18] K. Uchida, Imaginary abelian number fields with class number one, Tôhoku Math. J. 24 (1972),

487–499.
[19] K. Uchida, Relative class numbers of normal CM-fields, Tôhoku Math. J. 25 (1973), 347–353.
[20] K. Yamamura, The determination of the imaginary abelian number fields with class number one, to appear in Math. Comp.

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